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On a Compositeness Test for $(2^p + 1)/3$

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Abstract
In this note, we give a necessary condition for the primality of $(2^p + 1)/3$.

1 Introduction

Let p be an odd prime and $M_p := 2^p - 1$. For $n \geq 0$ define the sequence $\{S_n\}_{n\geq 0}$ by

$$\begin{aligned} S_0 &= 4, \\ S_{k+1} &= S_k^2 - 2, \qquad k \geq 0. \end{aligned}$$

The celebrated Lucas-Lehmer test states:

Theorem 1. *M_p is prime if and only if $S_{p-2} \equiv 0 \pmod{M_p}$.*

The numbers M_p have interested experts and non-experts throughout history. See [7] for an interesting mathematical and historical account. These numbers have been a popular focus among those searching for large primes because of their unique set of convenient properties for primality testing, the most important of these being the Lucas-Lehmer test, given in Theorem 1. Indeed, via Lucas-Lehmer test, the determination of the primality of M_p is achieved through the calculation of $p-2$ ($< \log M_p$) squares modulo M_p. Furthermore, the reduction of a $2p$-bit integer modulo M_p is very fast compared to the reduction modulo any other number of a similar size.

Observe that $M_p = \phi_p(2)$, where $\phi_p(X)$ is the p-th cyclotomic polynomial. In this paper, we look at primes of the form

$$N_p := \phi_p(-2) = \frac{2^p+1}{3}.$$

For p a prime, the family of numbers $\{N_p\}_{p\geq 3}$ shares some of the properties that make the numbers $\{M_p\}_{p\geq 3}$ interesting to searchers of large primes. For instance, if N_p is prime, then p must be a prime. Additionaly, divisors of N_p are congruent to 1 modulo $2p$, an observation that helps in the search for small prime divisors of N_p. Furthermore, Melham proved the following theorem (see Theorem 7 in [5]), to which we will refer as Melham's probable prime test for N_p.

Theorem 2. *Let p be an odd prime. Define the sequence $\{S_n\}_{n\geq 0}$ by*

$$\begin{aligned} S_0 &= 6, \\ S_{k+1} &= S_k^2 - 2, \qquad k \geq 0. \end{aligned}$$

If N_p is prime then $S_{p-1} \equiv -34 \pmod{N_p}$.

Similar congruences involving Fibonacci numbers and more general Lucas sequences instead of only Mersenne numbers appear in [1] and [3].

It is easy to see that the reduction of a $2p$-bit number modulo N_p is also very fast. However, it is not known whether the numbers $\{N_p\}_{p\geq 3}$ have a very important property enjoyed by the numbers $\{M_p\}_{p\geq 3}$. Specifically, it is not known if $S_{p-1} \equiv -34 \pmod{N_p}$ implies that N_p is prime.

The numbers $\{N_p\}_{p\geq 3}$ were studied by Bateman, Selfridge, and Wagstaff, Jr. [2] who proposed the following conjecture.

Conjecture 3. *If two of the following statements about an odd positive integer p are true, then the third one is also true.*

- *$p = 2^k \pm 1$ or $p = 4^k \pm 3$;*
- *M_p is prime;*
- *N_p is prime.*

Currently, there are forty known primes/probable primes N_p, sometimes called Wagstaff primes/PRP. See, for example, [8]. Probable primes N_p can be discovered via any of the known pseudoprime tests. Examples of such tests are the strong pseudoprime test (or the Miller-Rabin test [6]), and the Grantham test [4]. They can also be discovered with the use of Melham's probable prime test, given in Theorem 2 above. This test has the computational advantadge of involving the computation of only $p-1$ modular squares, the subtraction of 2 in each step being neglected.

Melham's probable prime test for N_p can be derived by the application of a Frobenius test to $1+\sqrt{2}$ in the finite field $\mathbb{K} := \mathbb{Z}[\sqrt{2}]/N_p$. The application of the Frobenius test is equivalent to the determination of the quadratic character of $1+\sqrt{2}$ in $\mathbb{K}$.

Similarly, we will see that, by the application of a Frobenius test to $2+\sqrt{2}$, one can obtain the following weaker variant of Melham's test: If N_p is prime, then the sequence given by

$$\begin{aligned} R_0 &= 4, \\ R_{k+1} &= R_k^2 - 2^{2^k+1}, \qquad k \geq 0, \end{aligned}$$

satisfies $R_{p-1}^2 \equiv 64 \pmod{N_p}$ (see Lemma 5 below).

Curiously, we noticed experimentally that whenever N_p is prime, then $R_{p-1} \equiv 8 \pmod{N_p}$ holds. The object of this paper is to show that this is indeed the case. Our proof hinges on the determination of the biquadratic character of $2+\sqrt{2}$ in $\mathbb{K}$, a problem that we consider to be interesting in its own right.

2 The Main Result

Theorem 4. *If $p > 3$ is prime, and N_p is prime, then $R_{p-1} \equiv 8 \pmod{N_p}$.*

Let $\alpha := 2+\sqrt{2}$ and $\beta := 2-\sqrt{2}$. It is easy to see, by induction on n, that the formula

$$R_n = \alpha^{2^n} + \beta^{2^n} \qquad \text{holds for all } n \geq 0. \tag{1}$$

Since $p > 3$, it follows easily that $N_p \equiv 3 \pmod 8$. In particular,

$$\left(\frac{-1}{N_p}\right) = \left(\frac{2}{N_p}\right) = -1, \tag{2}$$

where, as usual, for integers a, and $q \geq 3$ odd, we write $\left(\frac{a}{q}\right)$ for the Jacobi symbol of a with respect to q.

We start by giving a short proof of a somewhat weaker congruence using nothing else but the properties of the Frobenius automorphism.

Lemma 5. *Let $p > 3$ be prime. If N_p is prime, then $R_{p-1}^2 \equiv 64 \pmod{N_p}$.*

Proof. Assume that $q := N_p$ is prime. Again let $\mathbb{K} := \mathbb{F}_q[\sqrt{2}]$. By equation (2), it follows that $\mathbb{K}$ is a finite field with q^2 elements. Since $\alpha \notin \mathbb{F}_q$, we have that $\alpha^q = \beta$ in $\mathbb{K}$. Then $\alpha^{3q} = \beta^3$. Since $3q = 2^p + 1$, it follows that

$$\alpha^{2^p} = \beta^3 \alpha^{-1}.$$

Conjugating the above relation, we get

$$R_p = \alpha^{2^p} + \beta^{2^p} = \beta^3\alpha^{-1} + \alpha^3\beta^{-1} = \frac{\alpha^4+\beta^4}{\alpha\beta} = 68,$$

where we have used the relations

$$\alpha^4 = 68 + 48\sqrt{2}, \qquad \beta^4 = 68 - 48\sqrt{2}, \qquad \alpha\beta = 2.$$

However, again by formula (2), we have

$$2^{(q-1)/2} = -1 \text{ in } \mathbb{F}_q.$$

Since $(q-1)/2 = (2^{p-1}-1)/3$, we conclude that

$$2^{2^{p-1}-1} = -1. \tag{3}$$

Thus, $2^{2^{p-1}+1} = -4$. The desired relation now follows because

$$R_{p-1}^2 = R_p + 2^{2^p+1} = 68 - 4 = 64,$$

which is what we wanted. □

Let us now go to the proof of Theorem 4. We shall assume that $p > 3$, since for $p = 3$ the congruence can be verified directly. We keep the previous notations. Let i be a fixed square-root of -1 in $\mathbb{K}$. Put

$$\gamma := 1 + i + \sqrt{2}.$$

Let

$$\sigma := 1 + i - \sqrt{2} \qquad \text{and} \qquad \tau := 1 - i + \sqrt{2}.$$

Note that none of the elements γ, σ, τ is in $\mathbb{F}_q$. Indeed, assume say, that $\tau \in \mathbb{F}_q$. Then by writing $-i + \sqrt{2} = a$ with some $a \in \mathbb{F}_q$, rearranging the above relation and squaring it, we get

$$-1 = (-i)^2 = (a - \sqrt{2})^2 = a^2 - 2a\sqrt{2} + 2,$$

so that $a\sqrt{2} \in \mathbb{F}_q$, which is possible only if $a = 0$. However, with $a = 0$ the above relation becomes $-1 = 2$, which is false because $q = N_p > 3$.

Observe now that

$$\sigma\tau = 1 - (i - \sqrt{2})^2 = 2i\sqrt{2} = 2\sqrt{-2} \in \mathbb{F}_q,$$

where the last relation follows from the fact that -2 is a quadratic residue modulo q. Thus, $(\sigma\tau)^{q-1} = 1$. Since $(q^2-1)/4 = (q-1)((q+1)/4)$ is a multiple of $q-1$, we see that $(\sigma\tau)^{(q^2-1)/4} = 1$, which can be rewritten as

$$(\gamma\tau)^{(q^2-1)/4} = (\gamma\sigma)^{(q^2-1)/4}\left(\tau^2\right)^{(q^2-1)/4}. \tag{4}$$

Now,

$$\gamma\sigma = (1+i)^2 - 2 = 2(i-1), \qquad \text{and} \qquad \gamma\tau = (1+\sqrt{2})^2 - i^2 = 2\alpha. \tag{5}$$

Observe that $2(i-1) = -2\sqrt{2}\omega$, where $\omega = (1-i)/\sqrt{2}$ is a root of unity of order 8. Since $p \geq 5$, it follows that $q \equiv 3^{-1} \equiv 11 \pmod{32}$, which implies easily that $(q^2-1)/4 \equiv -2 \pmod{8}$. Thus, the left side of formula (4) is

$$(\gamma\sigma)^{(q^2-1)/4} = (-2\sqrt{2})^{(q^2-1)/4}\omega^{(q^2-1)/4} = (-1)^{(q^2-1)/4}2^{3(q^2-1)/8}\omega^{-2} = -i. \tag{6}$$

Next, observe that

$$(\tau^2)^{(q^2-1)/4} = \left(\tau^{q+1}\right)^{(q-1)/2}.$$

By Frobenius, we have that $\tau^{q+1} = \tau^q\tau = \sigma\tau = 2i\sqrt{2}$. Thus,

$$(\tau^2)^{(q^2-1)/4} = (2i\sqrt{2})^{(q-1)/2} = i^{(q-1)/2}2^{(q-1)/2}(\sqrt{2})^{(q-1)/2} = -i(\sqrt{2})^{(q-1)/2}, \tag{7}$$

where we have used the fact that $(q-1)/2 \equiv 1 \pmod{4}$, which follows easily from the fact that $q \equiv 11 \pmod{32}$. Inserting (6) and (7) into (4), and using also (5), we obtain

$$(2\alpha)^{(q^2-1)/4} = (-i)(-i)(\sqrt{2})^{(q-1)/2} = -(\sqrt{2})^{(q-1)/2}.$$

Using now $2^{(q^2-1)/4} = (2^{q-1})^{(q+1)/4} = 1$, and $\alpha^{q-1} = \alpha^q\alpha^{-1} = \beta/\alpha$, we deduce that

$$\left(\frac{\beta}{\alpha}\right)^{(q+1)/4} = \alpha^{(q^2-1)/4} = (2\alpha)^{(q^2-1)/4} = -(\sqrt{2})^{(q-1)/2}.$$

Now, $(q+1)/4 = (2^p+4)/12 = (2^{p-2}+1)/3$. Thus,

$$\left(\frac{\beta}{\alpha}\right)^{2^{p-2}} = -(\sqrt{2})^{3(q-1)/2}\left(\frac{\alpha}{\beta}\right).$$

Applying the Frobenius automorphism, and summing the resulting relations, we arrive at

$$\left(\frac{\beta}{\alpha}\right)^{2^{p-2}} + \left(\frac{\alpha}{\beta}\right)^{2^{p-2}} = -(\sqrt{2})^{3(q-1)/2}\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right).$$

In the line immediately above, the left side is $R_{p-1}/(\alpha\beta)^{2^{p-2}} = R_{p-1}/2^{2^{p-2}}$. The right side is

$$-(\sqrt{2})^{3(q-1)/2}\left(\frac{\alpha^2-\beta^2}{\alpha\beta}\right) = -(\sqrt{2})^{3(q-1)/2}4\sqrt{2} = -2^{(3q+7)/4}.$$

Since $(3q+7)/4 = 2^{p-2}+2$, we obtain

$$\frac{R_{p-1}}{2^{2^{p-2}}} = -2^{2^{p-2}+2},$$

which finally leads to $R_{p-1} = -2^{2^{p-1}+2}$. Using (3), we obtain the desired result.

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