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Some Recurrence Relations for Cauchy Numbers of the First Kind

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Abstract

In this paper, we present some recurrence relations for the Cauchy numbers of the first kind, by making use of the Stirling numbers of the first kind.

1 Introduction

The Cauchy numbers of the first kind b_n, $n = 0, 1, 2, \cdots$, are defined by the generating function [3]

$$\sum_{n\geq 0} b_n \frac{t^n}{n!} = \frac{t}{\log(1+t)} . \tag{1}$$

These numbers are also called the Bernoulli numbers of the second kind, and are represented by sequences A006232 and A006233 in Sloane's *Encyclopedia*. The Cauchy numbers play important roles in some fields, and one of the most important applications is the so-called Laplace summation formula [5]. These numbers are related to various special combinatorial numbers, such as the Stirling numbers of both kinds, the Bernoulli numbers (of the first kind) and the harmonic numbers [4, 5, 7]. Furthermore, many generalizations of these numbers are also introduced [2, 6, 8]. The most basic recurrence relation of these numbers is

$$\sum_{j=0}^{n} \frac{(-1)^{n-j} b_j}{j!(n+1-j)} = 0 \qquad (n \geq 1) ,$$

with $b_0 = 1$. It can be derived from (1) by multiplying both sides by $\log(1+t)$, and then equating the coefficients of the powers of x.

The classical Bernoulli numbers (or the Bernoulli numbers of the first kind) B_n, $n = 0, 1, 2, \cdots$, are defined by the generating function [3]

$$\sum_{n\geq 0} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1};$$

see sequences A000367 and A002445. There is much work devoted to the Bernoulli numbers. For example, Agoh and Dilcher [1] derived recently the following recurrence relations:

$$\sum_{j=0}^{n} \binom{n+k}{j} S(n-j+k, k) B_j = \frac{n+k}{k} S(n+k-1, k-1)\,, \tag{2}$$

$$\sum_{j=m}^{n} \binom{n+k}{j-m} S(n-j+k+m, k+m) B_j = \frac{(-1)^m}{k+m} \sum_{j=1}^{m+1} \binom{k+m-1}{j-1}^{-1} N_1(n, m, k, j)\,, \tag{3}$$

where $S(n,k)$ are the Stirling numbers of the second kind and $N_1(n, m, k, j)$ is defined by

$$(n+k)S(m+1, j)S(n+k-1, k+m-j) - mS(m, j)S(n+k, k+m-j).$$

From these relations, we can see that the classical Bernoulli numbers are closely related to the Stirling numbers of the second kind, and we can also find that the classical Bernoulli numbers and the Sirling numbers of the second kind are both associated with the delta series $e^t - 1$.

On the other hand, it is well known that the Stirling numbers of the first kind $s(n,k)$ are defined by the generating function

$$\sum_{n\geq k} s(n,k) \frac{t^n}{n!} = \frac{\log^k(1+t)}{k!}\,. \tag{4}$$

They have the basic recurrence relation

$$s(n,k) = s(n-1, k-1) - (n-1)s(n-1, k) \qquad (n, k \geq 1) \tag{5}$$

as well as the following special values:

$$s(0,0) = 1,\ s(n,0) = s(0,n) = 0,\ s(n,n) = 1, \quad (n \geq 1)\,. \tag{6}$$

From the generating functions, we can find that the Cauchy numbers of the first kind and the Stirling numbers of the first kind are both associated with the delta series $\log(1+t)$, which is the compositional inverse of $e^t - 1$. Motivated by this fact, in this paper, we will give some recurrence relations on the Cauchy numbers of the first kind, in which the Stirling numbers of the first kind are involved. The results of this paper can be viewed as the analogs of those on the classical Bernoulli numbers and the Stirling numbers of the second kind.

2 The main results

Firstly, we give a relation similar to the identity (2).

Theorem 1. *For any integers $k \ge 1$ and $n \ge 0$ we have*

$$\sum_{j=0}^{n} \binom{n+k}{j} s(n-j+k,k) b_j = \frac{n+k}{k} s(n+k-1,k-1)\,.$$

Proof. We can write

$$\left(\frac{\log(1+t)}{t}\right)^k = \frac{k!}{t^k} \frac{\log^k(1+t)}{k!} = \sum_{i\ge 0} \frac{s(i+k,k)}{\binom{i+k}{k}} \frac{t^i}{i!}\,. \tag{7}$$

Putting (7) in the following identity

$$\left(\frac{\log(1+t)}{t}\right)^k \frac{t}{\log(1+t)} = \left(\frac{\log(1+t)}{t}\right)^{k-1},$$

we have

$$\begin{aligned}\sum_{i\ge 0} \frac{s(i+k,k)}{\binom{i+k}{k}} \frac{t^i}{i!} \sum_{j\ge 0} b_j \frac{t^j}{j!} &= \sum_{n\ge 0} \left(\sum_{j=0}^{n} \binom{n}{j} \frac{s(n-j+k,k)}{\binom{n-j+k}{k}} b_j \right) \frac{t^n}{n!} \\ &= \sum_{n\ge 0} \frac{s(n+k-1,k-1)}{\binom{n+k-1}{k-1}} \frac{t^n}{n!}\,. \end{aligned} \tag{8}$$

Thus, comparing the coefficients of $t^n/n!$ in (8) gives the final result. ☐

In order to give a relation similar to (3), we need the following lemma.

Lemma 2. *For any $m \ge 0$ we have*

$$\frac{d^m}{dt^m} \frac{1}{\log(1+t)} = \left(-\frac{1}{1+t}\right)^m \sum_{j=0}^{m} j! \frac{(-1)^{m-j} s(m,j)}{\log^{j+1}(1+t)}\,, \tag{9}$$

$$\frac{d^m}{dt^m} \frac{t}{\log(1+t)} = \left(-\frac{1}{1+t}\right)^m \sum_{j=0}^{m} (-1)^{m-j} j! \frac{t s(m,j) + m(1+t) s(m-1,j)}{\log^{j+1}(1+t)}\,. \tag{10}$$

Proof. We prove (9) by induction. According to (6), it is true for $m = 0$. Suppose it is true for m, then

$$\begin{aligned} &(-1)^{m+1} \frac{d^{m+1}}{dt^{m+1}} \frac{1}{\log(1+t)} \\ &= (-1)^{m+1} \frac{d}{dt} \left(\frac{-1}{1+t}\right)^m \sum_{j=0}^{m} \frac{(-1)^{m-j} s(m,j) j!}{\log^{j+1}(1+t)} \\ &= \frac{m}{(1+t)^{m+1}} \sum_{j=0}^{m} \frac{(-1)^{m-j} s(m,j) j!}{\log^{j+1}(1+t)} + \frac{1}{(1+t)^{m+1}} \sum_{j=0}^{m} \frac{(-1)^{m-j} s(m,j)(j+1)!}{\log^{j+2}(1+t)} \\ &= \frac{1}{(1+t)^{m+1}} \sum_{j=0}^{m+1} \frac{(-1)^{m+1-j} s(m+1,j) j!}{\log^{j+1}(1+t)}\,, \end{aligned}$$

where the last equation is obtained by relation (5). This gives (9) finally. Now, using the Leibniz's rule in the form

$$\frac{d^m}{dt^m}\frac{t}{\log(1+t)}=t\frac{d^m}{dt^m}\frac{1}{\log(1+t)}+m\frac{d^{m-1}}{dt^{m-1}}\frac{1}{\log(1+t)}\,,$$

and using identity (9), we can immediately obtain (10). □

From (1) we have

$$\frac{d^{m-1}}{dt^{m-1}}\frac{1}{\log(1+t)}=\frac{(-1)^{m-1}(m-1)!}{t^m}+\sum_{n\ge 0}\frac{b_{n+m}}{n+m}\frac{t^n}{n!}\,.$$

Then, based on Lemma 2 and the above identity, we obtain the following recurrence relation on the Cauchy numbers of the first kind.

Theorem 3. *For any integers $1\le m\le n$ and $k\ge 0$ we have*

$$\sum_{j=m}^{n}\binom{n+k}{j-m}s(n-j+k+m,k+m)\frac{b_j}{j}=\frac{(-1)^m}{m}\frac{s(n+k+m,k+m)}{\binom{n+k+m}{m}}$$
$$+\sum_{i=0}^{n}\sum_{j=0}^{m-1}(-1)^j\binom{n+k}{i}\binom{k+m}{j}^{-1}\frac{(-m+1)_i}{k+m-j}s(m-1,j)s(n+k-i,k+m-j-1)\,,$$

where $(n)_i=n(n-1)\cdots(n-i+1)$.

Proof. Replacing k by $k+m$ in (7) and multiplying the resulting identity with (9), we have

$$\begin{aligned}\left(\frac{\log(1+t)}{t}\right)^{k+m}&\frac{d^{m-1}}{dt^{m-1}}\frac{1}{\log(1+t)}\\ &=\sum_{i\ge 0}\frac{s(i+k+m,k+m)}{\binom{i+k+m}{k+m}}\frac{(-1)^{m-1}(m-1)!}{i!}t^{i-m}\\ &\quad+\sum_{n\ge 0}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{s(n-j+k+m,k+m)}{\binom{n-j+k+m}{k+m}}\frac{b_{j+m}}{j+m}\right)\frac{t^n}{n!}\,.\end{aligned}\tag{11}$$

Denote the left-hand side of (11) by $A(t)$. By (9) we obtain

$$A(t)=\frac{(-1)^{m-1}}{t^{k+m}}(t+1)^{-m+1}\sum_{j=0}^{m-1}j!(-1)^{m-1-j}s(m-1,j)(\log(1+t))^{k+m-j-1}\,,$$

and by (4) we obtain

$$
\begin{aligned}
A(t) &= \frac{(-1)^{m-1}}{t^{k+m}} \sum_{i\ge 0} \binom{-m+1}{i} t^i \sum_{j=0}^{m-1} j!(-1)^{m-1-j} s(m-1,j) \\
&\quad (k+m-j-1)! \sum_{l\ge 0} s(l,k+m-j-1)\frac{t^l}{l!} \\
&= \sum_{n\ge 0}\sum_{i=0}^{n+m}\sum_{j=0}^{m-1} \binom{-m+1}{i} (-1)^j j! \frac{(k+m-j-1)!}{(n+k+m-i)!} s(m-1,j) \\
&\quad s(n+k+m-i,k+m-j-1) t^n\,. \qquad (12)
\end{aligned}
$$

Comparing the coefficients of t^n in (11) and (12), and replacing n by $n-m$, we have the desired result. $\square$

Let us consider the extreme cases. When $m=1$, we return to Theorem 1. When $m=n$, we have the following corollary.

Corollary 4. *For any integers $n\ge 1$ and $k\ge 0$ we have*

$$
\begin{aligned}
b_n &= (-1)^n \frac{s(2n+k,n+k)}{\binom{2n+k}{n}} \\
&\quad + \sum_{i=0}^{n}\sum_{j=0}^{n-1} (-1)^j \binom{n+k}{i} \binom{n+k}{j}^{-1} \frac{n(-n+1)_i}{n+k-j} s(n-1,j) s(n+k-i,n+k-j-1)\,.
\end{aligned}
$$

In particular when $k=0$, we have

$$
b_n = (-1)^n \frac{s(2n,n)}{\binom{2n}{n}} + \sum_{i=0}^{n}\sum_{j=0}^{n-1} (-1)^j \binom{n}{i} \binom{n}{j}^{-1} \frac{n(-n+1)_i}{n-j} s(n-1,j) s(n-i,n-j-1)\,.
$$

The last identity indicates that many of the results in this section can also be seen as convolution identities for the Stirling numbers of the first kind.

Finally, from (1) we also have

$$
\frac{d^m}{dt^m} \frac{t}{\log(1+t)} = \sum_{n\ge 0} b_{n+m} \frac{t^n}{n!}\,. \qquad (13)
$$

Similarly to the proof of Theorem 3, by identities (10) and (13), we can easily obtain the following theorem.

Theorem 5. *For any integers $0\le m\le n$ and $k\ge 1$ we have*

$$
\begin{aligned}
&\sum_{j=m}^{n} \binom{n+k}{j-m} s(n-j+k+m,k+m) b_j \\
&= \sum_{j=0}^{m}\sum_{l=m-j-1}^{n} (-1)^j \binom{n+k}{n-l} \binom{k+m}{j}^{-1} \frac{(-m)_{n-l}}{k+m-j} N_2(l,m,k,j)\,,
\end{aligned}
$$

where the number $N_2(l, m, k, j)$ *is*

$$(l+k)(s(m,j)+ms(m-1,j))s(l-1+k,k+m-j-1) \\ +ms(m-1,j)s(l+k,k+m-j-1)\,.$$

Setting $m = 0$ in this theorem, we can obtain Theorem 1 once again.

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