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Sums of Products of Bernoulli Numbers, Including Poly-Bernoulli Numbers

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Abstract

We investigate sums of products of Bernoulli numbers including poly-Bernoulli numbers. A relation among these sums and explicit expressions of sums of two and three products are given. As a corollary, we obtain fractional parts of sums of two and three products for negative indices.

1 Introduction and main results

Bernoulli numbers B_n $(n = 0, 1, 2, \ldots)$ are defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

The following identity on sums of two products of Bernoulli numbers is known as Euler's formula:

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n \geq 1). \tag{1}$$

When n is an even integer, the identity (1) can be written as

$$\sum_{i=1}^{n-1} \binom{2n}{2i} B_{2i} B_{2n-2i} = -(2n+1)B_{2n} \quad (n \geq 2), \tag{2}$$

because $B_n = 0$ for any odd integer $n \geq 3$. Many generalizations of (1) and (2) have been considered. As a generalization of (2), Dilcher [7] gave closed formulas of sums of N products of Bernoulli numbers for any positive integer N. Chen [6] gave generalizations of (1) for sums of N products of Bernoulli polynomials, generalized Bernoulli numbers and Euler polynomials by using special values of certain zeta functions at non-positive integers. Other types of sums of products have been also studied; see, for example, [1, 2, 8, 12, 13, 14].

The reason why these formulas are valid is that the generating function of Bernoulli numbers satisfies simple differential equations. For example, Euler's formula (1) is derived by comparing the coefficients of the following identity:

$$F(t)^2 = -tF'(t) + (1-t)F(t), \tag{3}$$

where $F(t) = t/(e^t - 1)$.

For any integer k, Kaneko [10] introduced poly-Bernoulli numbers of index k (denoted by $B_n^{(k)}$) by the following generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}}{n!} t^n, \tag{4}$$

where $\mathrm{Li}_k(x)$ is the k-th polylogarithm defined by $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} x^n/n^k$. The list of poly-Bernoulli numbers $B_n^{(k)}$ with $-5 \leq k \leq 5$ and $0 \leq n \leq 7$ are given by Arakawa and Kaneko [3]. The numbers $B_n^{(k)}$ are rational numbers, in particular, are positive integers for $k \leq 0$ (e.g., [10, Section 1]). When $k = 1$, the left-hand side of (4) is equal to $te^t/(e^t - 1)$ because of $\mathrm{Li}_1(x) = -\log(1-x)$. Since

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} t^n, \tag{5}$$

we have $B_n^{(1)} = (-1)^n B_n$ for $n \geq 0$ (actually $B_n^{(1)} = B_n$ except for $n = 1$). Poly-Bernoulli numbers of positive index are related to multiple zeta functions. To be more precise, special values of certain multiple zeta functions at non-positive integers are described in terms of poly-Bernoulli numbers (cf. [4]). For combinatorial interpretations of poly-Bernoulli numbers of negative index, see Brewbaker [5] and Launois [11].

In this paper we investigate the following type of sums of products of Bernoulli numbers including poly-Bernoulli numbers:

$$S_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\ldots,i_m \geq 0}} \binom{n}{i_1,\ldots,i_m} B_{i_1} \cdots B_{i_{m-1}} B_{i_m}^{(k)} \quad (m \geq 1, n \geq 0), \tag{6}$$

where $\binom{n}{i_1,\ldots,i_m}$ are multinomial coefficients defined by

$$\binom{n}{i_1,\ldots,i_m} = \frac{n!}{i_1! \cdots i_m!}.$$

Table 1: $S_2^{(k)}(n)$

$k\backslash n$	0	1	2	3	4	5	6
-4	1	$\frac{31}{2}$	$\frac{781}{6}$	855	$\frac{147479}{30}$	26025	$\frac{5474701}{42}$
-3	1	$\frac{15}{2}$	$\frac{229}{6}$	165	$\frac{19559}{30}$	2435	$\frac{367669}{42}$
-2	1	$\frac{7}{2}$	$\frac{61}{6}$	27	$\frac{2039}{30}$	165	$\frac{16381}{42}$
-1	1	$\frac{3}{2}$	$\frac{13}{6}$	3	$\frac{119}{30}$	5	$\frac{253}{42}$
0	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$
1	1	0	$-\frac{1}{6}$	0	$\frac{1}{10}$	0	$-\frac{5}{42}$
2	1	$-\frac{1}{4}$	$-\frac{1}{9}$	$\frac{1}{8}$	$\frac{17}{450}$	$-\frac{1}{8}$	$-\frac{23}{1470}$
3	1	$-\frac{3}{8}$	$-\frac{1}{108}$	$\frac{13}{96}$	$-\frac{733}{13500}$	$-\frac{131}{1440}$	$\frac{65953}{617400}$
4	1	$-\frac{7}{16}$	$\frac{43}{648}$	$\frac{115}{1152}$	$-\frac{70271}{810000}$	$-\frac{233}{9600}$	$\frac{26855027}{259308000}$

Clearly, it holds that $S_1^{(k)}(n) = B_n^{(k)}$. We list $S_2^{(k)}(n)$ and $S_3^{(k)}(n)$ with $-4 \le k \le 4$ and $0 \le n \le 6$ in Tables 1 and 2. We note that the numbers $S_m^{(k)}(n)$ appear as the coefficients of the following generating function:

$$\left(\frac{t}{e^t - 1}\right)^{m-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} S_m^{(k)}(n) \frac{t^n}{n!}. \tag{7}$$

The left-hand side of (7) satisfies a certain differential equation like (3) (see Proposition 5), thus this type of sums (6) is one of natural extensions of the classical sums of products of Bernoulli numbers.

Now we state our main results of this paper.

Theorem 1. *For $k \in \mathbb{Z}$ and $m \ge 1$, we have*

$$\begin{aligned} &\sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) \\ &= \begin{cases} n(n-1)\cdots(n-m+1) \displaystyle\sum_{l=1}^{m} \begin{bmatrix} m \\ l \end{bmatrix} B_{n-m+l}^{(k)}, & \text{if } n \ge m; \\ 0, & \text{if } 0 \le n \le m-1, \end{cases} \end{aligned} \tag{8}$$

where $\begin{bmatrix} m \\ l \end{bmatrix}$ are (unsigned) Stirling numbers of the first kind.

The definition of Stirling numbers of the first kind $\begin{bmatrix} m \\ l \end{bmatrix}$ will be given in Section 2. Although Theorem 1 only gives relations among sums of products $S_m^{(k)}(n)$, explicit formulas of $S_m^{(k)}(n)$ can be obtained for $m = 2$ and 3.

Table 2: $S_3^{(k)}(n)$

$k\backslash n$	0	1	2	3	4	5	6
-4	1	15	$\frac{689}{6}$	$\frac{1335}{2}$	$\frac{33361}{10}$	$\frac{30315}{2}$	$\frac{2708995}{42}$
-3	1	7	$\frac{185}{6}$	$\frac{223}{2}$	$\frac{3601}{10}$	$\frac{6473}{6}$	$\frac{128515}{42}$
-2	1	3	$\frac{41}{6}$	$\frac{27}{2}$	$\frac{241}{10}$	$\frac{79}{2}$	$\frac{2515}{42}$
-1	1	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{10}$	$-\frac{1}{6}$	$-\frac{5}{42}$
0	1	0	$-\frac{1}{6}$	0	$\frac{1}{10}$	0	$-\frac{5}{42}$
1	1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$-\frac{1}{10}$	$-\frac{1}{4}$	$\frac{5}{21}$
2	1	$-\frac{3}{4}$	$\frac{11}{36}$	$\frac{1}{6}$	$-\frac{107}{300}$	$\frac{11}{360}$	$\frac{209}{392}$
3	1	$-\frac{7}{8}$	$\frac{115}{216}$	$-\frac{11}{288}$	$-\frac{6619}{18000}$	$\frac{899}{2700}$	$\frac{134563}{493920}$
4	1	$-\frac{15}{16}$	$\frac{869}{1296}$	$-\frac{755}{3456}$	$-\frac{273653}{1080000}$	$\frac{279877}{648000}$	$-\frac{10347133}{207446400}$

Theorem 2. *For $k \ge 1$ and $n \ge 0$, it holds that*

$$S_2^{(0)}(n) = B_n^{(1)}, \tag{9}$$

$$S_2^{(k)}(n) = B_n^{(1)} - n\sum_{j=1}^{k} B_n^{(j)}, \tag{10}$$

$$S_2^{(-k)}(n) = B_n^{(1)} + n\sum_{j=0}^{k-1} B_n^{(-j)}. \tag{11}$$

When $k = 1$ in (10), we have

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}B_iB_{n-i} = -(n-1)B_n. \tag{12}$$

The identity (12) is equivalent to (1) because $B_n = 0$ for any odd integer $n \ge 3$. Therefore Theorem 2 can be regarded as a generalization of Euler's formula (1).

Theorem 3. *For $k \ge 1$ and $n \ge 1$, it holds that*

$$S_3^{(0)}(n) = -(n-1)B_n, \tag{13}$$

$$\begin{aligned} S_3^{(k)}(n) =& (-1)^n(1-2^{-k})B_{n-1} - (n-1)B_n \\ &+ n(n-1)\sum_{j=1}^{k}(1-2^{j-k-1})\left(B_n^{(j)} + B_{n-1}^{(j)}\right), \end{aligned} \tag{14}$$

$$\begin{aligned} S_3^{(-k)}(n) =& n(2^k-1)(-1)^{n-1}B_{n-1} - (n-1)B_n \\ &+ n(n-1)\sum_{j=0}^{k-2}(2^{k-1-j}-1)\left(B_n^{(-j)} + B_{n-1}^{(-j)}\right). \end{aligned} \tag{15}$$

As a corollary, for a negative index $-k$, we obtain the following formulas on fractional parts of $S_2^{(-k)}(n)$ and $S_3^{(-k)}(n)$.

Corollary 4. *For $k \geq 1$ and $n \geq 0$, we have*

$$S_2^{(-k)}(n) \equiv B_n \pmod 1, \tag{16}$$

$$S_3^{(-k)}(n) \equiv \begin{cases} n(2^k-1)B_{n-1} \pmod 1, & \text{if } n \text{ is odd;} \\ -(n-1)B_n \pmod 1, & \text{if } n \text{ is even.} \end{cases} \tag{17}$$

Here $\alpha \equiv \beta \pmod 1$ means $\alpha - \beta \in \mathbb{Z}$ for rational numbers α and β.

The classical von Staudt-Clausen theorem (e.g., [9, Section 7.9]) states that

$$B_n \equiv -\sum_{\substack{p:\text{prime} \\ (p-1)|n}} \frac{1}{p} \pmod 1$$

for any even integer $n \geq 2$. Therefore, by using Corollary 4, we can determine the fractional parts of $S_2^{(-k)}(n)$ and $S_3^{(-k)}(n)$ if $k \geq 1$ and $n \geq 0$ are given.

2 Proof of Theorem 1

We first recall (unsigned) Stirling numbers of the first kind. Let m be a positive integer. For $0 \leq l \leq m$, Stirling numbers of the first kind $\left[{m \atop l}\right]$ are defined as

$$x(x+1)\cdots(x+m-1) = \sum_{l=0}^{m} \left[{m \atop l}\right] x^l. \tag{18}$$

It follows immediately that $\left[{m \atop 0}\right] = 0$ and $\left[{m \atop m}\right] = 1$ for all $m \geq 1$. For $l \geq m+1$ and $l \leq -1$, we define $\left[{m \atop l}\right] = 0$. Then the recurrence relation

$$\left[{m+1 \atop l}\right] = \left[{m \atop l-1}\right] + m\left[{m \atop l}\right] \tag{19}$$

holds for all $m \geq 1$ and $l \in \mathbb{Z}$.

We set the generating function of poly-Bernoulli numbers of index k as $F_k(t)$, i.e.,

$$F_k(t) := \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}.$$

For $k = 1$, 0 and -1, they have simple expressions; $F_1(t) = te^t/(e^t-1)$, $F_0(t) = e^t$ and $F_{-1}(t) = e^{2t}$.

Now let us prove Theorem 1. The n-th coefficient of $t^m \frac{d^l}{dt^l} F_k(t)$ is equal to

$$\begin{cases} \dfrac{n(n-1)\cdots(n-m+1)B_{n-m+l}^{(k)}}{n!}, & \text{if } n \geq m; \\ 0, & \text{if } 0 \leq n \leq m-1. \end{cases}$$

Therefore it suffices to show the following proposition to get Theorem 1.

Proposition 5. *For $k \in \mathbb{Z}$ and $m \geq 1$ we have*

$$\begin{aligned}
&\left(\begin{bmatrix} m \\ m \end{bmatrix} \frac{d^m}{dt^m} + \begin{bmatrix} m \\ m-1 \end{bmatrix} \frac{d^{m-1}}{dt^{m-1}} + \cdots + \begin{bmatrix} m \\ 1 \end{bmatrix} \frac{d}{dt} \right) F_k(t) \\
&= \frac{1}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l}(t).
\end{aligned} \tag{20}$$

Proof. We prove the proposition by induction on m. Since $\frac{d}{dt}F_k(t) = F_{k-1}(t)/t$, we can easily prove that

$$\frac{d}{dt}F_k(t) = \frac{1}{e^t-1}(F_{k-1}(t) - F_k(t)) \quad (k \in \mathbb{Z}). \tag{21}$$

Hence the case $m = 1$ holds.

We assume that (20) holds for a certain m. By (19), we have

$$\begin{aligned}
&\left(\begin{bmatrix} m+1 \\ m+1 \end{bmatrix} \frac{d^{m+1}}{dt^{m+1}} + \begin{bmatrix} m+1 \\ m \end{bmatrix} \frac{d^m}{dt^m} + \cdots + \begin{bmatrix} m+1 \\ 1 \end{bmatrix} \frac{d}{dt} \right) F_k(t) \\
&= \frac{d}{dt} \left(\begin{bmatrix} m \\ m \end{bmatrix} \frac{d^m}{dt^m} + \cdots + \begin{bmatrix} m \\ 1 \end{bmatrix} \frac{d}{dt} \right) F_k(t) + m \left(\begin{bmatrix} m \\ m \end{bmatrix} \frac{d^m}{dt^m} + \cdots + \begin{bmatrix} m \\ 1 \end{bmatrix} \frac{d}{dt} \right) F_k(t).
\end{aligned} \tag{22}$$

By the inductive assumption and (21), the right-hand side of (22) is equal to

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{1}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l}(t) \right) + \frac{m}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l}(t) \\
=&\frac{-me^t}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l}(t) \\
&+ \frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} (F_{k-l-1}(t) - F_{k-l}(t)) \\
&+ \frac{m}{(e^t-1)^m} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l}(t) \\
=&\frac{-m-1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l}(t) \\
&+ \frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} F_{k-l-1}(t) \\
=&\frac{1}{(e^t-1)^{m+1}} \sum_{l=0}^{m+1} (-1)^{(m+1)-l} \left((m+1) \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} + \begin{bmatrix} m+1 \\ l \end{bmatrix} \right) F_{k-l}(t).
\end{aligned}$$

As a consequence, by using the relation (19) again, we obtain

$$\begin{aligned}&\left(\begin{bmatrix}m+1\\m+1\end{bmatrix}\frac{d^{m+1}}{dt^{m+1}}+\begin{bmatrix}m+1\\m\end{bmatrix}\frac{d^{m}}{dt^{m}}+\cdots+\begin{bmatrix}m+1\\1\end{bmatrix}\frac{d}{dt}\right)F_k(t)\\
=&\frac{1}{(e^t-1)^{m+1}}\sum_{l=0}^{m+1}(-1)^{(m+1)-l}\begin{bmatrix}(m+1)+1\\l+1\end{bmatrix}F_{k-l}(t).\end{aligned}$$

Therefore (20) also holds for $m+1$ and this completes the proof. □

3 Explicit formulas of $S_2^{(k)}(n)$ and $S_3^{(k)}(n)$

In this section, we prove Theorem 2, Theorem 3 and Corollary 4.

Proof of Theorem 2. First we prove (9). We recall $F_0(t)=e^t$. By setting $m=2$ and $k=0$ in (7), we obtain that the generating function of $S_2^{(0)}(n)$ is equal to $te^t/(e^t-1)$. Then (9) follows from (5).

Next we prove the positive index case (10). Since the negative index case (11) can be proved similarly, we omit its proof. By (21), we have

$$\sum_{j=1}^{k}\frac{d}{dt}F_j(t)=\frac{1}{e^t-1}(F_0(t)-F_k(t)).$$

Since $F_0(t)=e^t$, it holds that

$$\frac{t}{e^t-1}F_k(t)=\frac{te^t}{e^t-1}-t\sum_{j=1}^{k}\frac{d}{dt}F_j(t).$$

By comparing the coefficients of both sides, we obtain (10). □

Proof of Theorem 3. By setting $m=3$ and $k=0$ in (7), we obtain that the generating function of $S_3^{(0)}(n)$ is $t^2e^t/(e^t-1)^2$. This is exactly the same as the generating function of $S_2^{(1)}(n)$, therefore (13) follows from the relation (12).

We prove the positive index case (14). We also omit the proof of the negative index case (15) because it can be proved similarly. Setting $m=2$ in Proposition 5, we have

$$\left(\frac{d^2}{dt^2}+\frac{d}{dt}\right)F_k(t)=\frac{1}{(e^t-1)^2}\left((2F_k(t)-F_{k-1}(t))-(2F_{k-1}(t)-F_{k-2}(t))\right). \tag{23}$$

By this equation, we get

$$\frac{2F_l(t)}{(e^t-1)^2}-\frac{F_{l-1}(t)}{(e^t-1)^2}=\frac{2F_0(t)-F_{-1}(t)}{(e^t-1)^2}+\sum_{j=1}^{l}\left(\frac{d^2}{dt^2}+\frac{d}{dt}\right)F_j(t). \tag{24}$$

In fact, this can be proved by replacing k with j in (23) and summing over j from 1 to l. Furthermore we multiply both sides of (24) by 2^{l-1} and sum over l from 1 to k. Then we obtain

$$\begin{aligned}
2^k \frac{F_k(t)}{(e^t-1)^2} =& \frac{F_0(t)}{(e^t-1)^2} + \left(\sum_{l=1}^{k} 2^{l-1}\right) \frac{2F_0(t)-F_{-1}(t)}{(e^t-1)^2} \\
&+ \sum_{l=1}^{k} 2^{l-1} \sum_{j=1}^{l} \left(\frac{d^2}{dt^2}+\frac{d}{dt}\right) F_j(t) \\
=& \frac{(2^{k+1}-1)F_0(t)-(2^k-1)F_{-1}(t)}{(e^t-1)^2} \\
&+ \sum_{j=1}^{k} \left(\sum_{l=j}^{k} 2^{l-1}\right) \left(\frac{d^2}{dt^2}+\frac{d}{dt}\right) F_j(t).
\end{aligned}$$

Hence we have

$$\begin{aligned}
2^k \left(\frac{t}{e^t-1}\right)^2 F_k(t) =& (2^{k+1}-1)\frac{t}{e^t-1}\frac{te^t}{e^t-1} - (2^k-1)\frac{te^t}{e^t-1}\frac{te^t}{e^t-1} \\
&+ t^2 \sum_{j=1}^{k} (2^k-2^{j-1}) \left(\frac{d^2}{dt^2}+\frac{d}{dt}\right) F_j(t).
\end{aligned} \tag{25}$$

By comparing the coefficients of both sides, we obtain for $n \geq 1$

$$\begin{aligned}
2^k S_3^{(k)}(n) =& (2^{k+1}-1)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}B_iB_{n-i} - (2^k-1)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n}B_iB_{n-i} \\
&+ n(n-1)\sum_{j=1}^{k}(2^k-2^{j-1})\left(B_n^{(j)}+B_{n-1}^{(j)}\right).
\end{aligned}$$

By (1), (12) and the fact $(-1)^n(n-1)B_n = (n-1)B_n$ for all $n \geq 1$, it holds that

$$\begin{aligned}
&(2^{k+1}-1)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}B_iB_{n-i} - (2^k-1)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n}B_iB_{n-i} \\
&= -(2^{k+1}-1)(n-1)B_n - (-1)^n(2^k-1)(-nB_{n-1}-(n-1)B_n) \\
&= -n(2^k-1)(-1)^{n-1}B_{n-1} - (n-1)2^kB_n.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
2^k S_3^{(k)}(n) =& -n\left(2^k-1\right)(-1)^{n-1}B_{n-1} - (n-1)2^kB_n \\
&+ n(n-1)\sum_{j=1}^{k}(2^k-2^{j-1})(B_n^{(j)}+B_{n-1}^{(j)}).
\end{aligned} \tag{26}$$

Dividing both sides of (26) by 2^k, we get (14). □

Proof of Corollary 4. The congruence (16) immediately follows from (11) and the fact $B_n^{(-k)}$ are integers for $k \geq 0$.

The congruence (17) holds for $n = 0$ because $S_3^{(-k)}(0) = 1$ for any $k \geq 1$. We assume that $n \geq 1$. By (15), the fractional part of $S_3^{(-k)}(n)$ is

$$n(2^k - 1)(-1)^{n-1}B_{n-1} - (n-1)B_n. \tag{27}$$

If n is odd, then $(-1)^{n-1}B_{n-1} = B_{n-1}$ and $(n-1)B_n = 0$. Thus we have $S_3^{(-k)}(n) \equiv n(2^k - 1)B_{n-1} \pmod 1$. If $n \geq 4$ is even, then $B_{n-1} = 0$. Thus we have $S_3^{(-k)}(n) \equiv -(n-1)B_n$ (mod1) for even $n \geq 4$. This congruence also holds for $n = 2$ because the first term of (27) becomes $2^k - 1 \in \mathbb{Z}$, and this completes the proof of (17). □

Remark 6. For $m \geq 4$ we may give explicit formulas of $S_m^{(k)}(n)$ by the method similar to the proof of Theorem 2 and Theorem 3. However, these formulas seem to be complicated to describe.

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