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Variations on Euclid's Formula for Perfect Numbers

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Abstract

We study several families of solutions to equations of the form $\sigma(n) = A\,n + B(n)$, where B is a function that may depend on properties of n.

1 Introduction

We recall that *perfect numbers* (sequence A000396 of Sloane's *Encyclopedia* [8]) are defined as solutions to the equation $\sigma(x) = 2\,x$, where $\sigma(x)$ denotes the sum of all positive divisors of x, including 1 and x itself. Euclid showed around 300 BCE [2, Proposition IX.36] that all numbers of the form $x = 2^{q-1}M_q$, where $M_q = 2^q - 1$ is prime (A000668), are perfect numbers.

While it is still not known whether there exist any odd perfect numbers, Euler [3] proved a converse of Euclid's proposition, showing that there are no other *even* perfect numbers (cf. A000043, A006516). (As a side note, this can also be stated by saying that the even perfect numbers are exactly the triangular numbers (A000217$(n) = n(n+1)/2$) whose indices are Mersenne primes A000668.)

One possible generalization of perfect numbers is the multiply or k–fold perfect numbers (A007691, A007539) such that $\sigma(x) = k\,x$ [1, 7, 9, 10]. Here we consider some modified equations, where a second term is added on the right hand side. The starting point for these investigations was the following observation, which to the best knowledge of the authors has not been discussed earlier in the literature:

1.1 The equation $\sigma(x) = 2\,(x + m)$

Here and throughout this paper, letters $k, \ell, m, ..., x$ will always denote integers.

Theorem 1.1. *If $p = 2^k - 1 - 2\,m$ is an odd prime, then $x = 2^{k-1}\,p$ is a solution to $\sigma(x) = 2\,(x + m)$.*

Proof. This result is an immediate consequence of the formula

$$\sigma(2^{k-1}\,p) = (2^k - 1)\,(p + 1) \ , \tag{1}$$

and the relation $2^k - (p + 1) = 2\,m$. □

Actually, this relation could be seen as the definition of m (which may take any positive, zero or negative value), as function of the prime p and $k \in \mathbb{N}$, which could both be chosen completely arbitrarily. However, in the spirit of this paper, we would rather consider the value of m to be given, and vary k to see when we get a prime p and thus a solution to the given equation.

Example 1.2. As function of m, we get the following minimal solution to $\sigma(x) = 2(x+m)$: (For positive m these k–values are listed in A096502.)

m	-5	-4	-3	-2	-1	0	1	2	3	4	5
k	1	2	1	1	1	2	3	3	39	4	4
p	11	11	7	5	3	3	5	3	549755813881	7	5
x	11	22	7	5	3	6	20	12	$2^{38}p \approx 1.5e24$	56	40

In spite of the simplicity of this theorem and its proof, we think that this way of writing the formula provides a nice generalization of the well-known formula for the even perfect numbers, which are obtained as the special case $m = 0$. In some sense, it shows how the "deviation" $(2\,m)$ of p from a Mersenne prime $M_k = 2^k - 1$, modifies the corresponding value of $\sigma(x)$. In the sequel, we attempt a systematic presentation of a certain number of similar cases.

Theorem 1.3. *If m is a perfect number (A000396), then the equation $\sigma(x) = 2\,(x + m)$ has infinitely many solutions, including all numbers of the form $x = m\,p$, where p is a prime not dividing m.*

Proof. The simple calculation is developed for Theorem 1.13 below. □

Remark 1.4. Numbers which are solutions to $\sigma(x) = 2\,(x+m)$, with m being a proper divisor of x, are called admirable numbers (A111592). In the particular case where they are of the form $x = m\,p$ with $p < m$, the "subtracted divisor" m is larger than $\sqrt{x}$, so they are in A109321.

Although Theorem 1.3 gives solutions to the same equation as Theorem 1.1 (if m is the same), the spirit is quite different: While m could take any value in Theorem 1.1, we have here an infinite sequence of solutions for each of the 47 known perfect numbers, and this sequence is different (even disjoint) from the set of solutions to the same equation which follows from Theorem 1.1:

Example 1.5. If we choose the perfect number $m = 6$, the respective theorems provide the following solutions to the equation $\sigma(x) = 2\,x + 12$:

- Theorem 1.1: $x = 24, 304, 127744, 33501184, \ldots$ (= A096821), by scanning k–values to see whether $2^k - 13$ is prime, i.e., whether $k \in$ A096818.
- Theorem 1.3: $x = 30, 42, 66, 78, 102, 114, \ldots$, i.e., $6\,p$ for all primes $p > 3$.

The solution $x = 54 = 2 \cdot 3^3$, in neither of these lists, is covered by the following Proposition 1.6. For $m = 28$, we get the following solutions to $\sigma(x) = 2\,x + 56$:

- From Theorem 1.1, we have $x = 224, 4544, 25472, 495104, \ldots$, by searching for $k > \log_2(57)$ such that $p = 2^k - 57$ is prime.
- From Theorem 1.3, we have $x = 84, 140, 308, 364, 476, 532, 644, 812, \ldots$, i.e., $x = 28\,p$ for all odd primes $p \neq 7$.

Again, there are solutions $x = 1372, 9272, 14552, \ldots$ which are not in these two lists, but discussed below.
For $m = 496$, we get the following solutions to $\sigma(x) = 2\,x + 992$:

- From Theorem 1.1, we get $x = 15872, 126083072, 8524857344, \ldots$, by searching $k > \log_2(993)$ such that $p = 2^k - 993$ is prime.
- From Theorem 1.3, we have $x = 1488, 2480, 3472, 5456, 6448, 8432, 9424, 11408, 14384, 18352, \ldots$, i.e., $x = 496\,p$ for all odd primes $p \neq 31$.

Solutions to this equation not obtained from Theorem 1.1 or 1.3 include $x = 2892 = 2^2 \cdot 3 \cdot 241$ and $x = 6104 = 2^3 \cdot 7 \cdot 109$, they result from Proposition 1.9, see Example 1.10.

For each perfect number m, there is a particular solution not covered by the preceding theorems:

Proposition 1.6. *If $m = 2^{k-1}\,p$ is the perfect number associated to the prime $p = 2^k - 1$, then $x = m\,p^2 = 2^{k-1}p^3$ is a solution to $\sigma(x) = 2\,(x+m)$.*

Proof. For $x = 2^{k-1}p^3$ we have $\sigma(x) - 2(x+m) = (2^k-1)(p^4-1)/(p-1) - 2^k(p^3+p) = p(p^2+1)(p+1) - (p+1)(p^3+p) = 0$. □

Example 1.7. For $m = 6$, 28 and 496, we get, respectively, the solutions $x = 54 = 2 \cdot 3^3$, $1372 = 2^2 \cdot 7^3$ and $476656 = 2^4 \cdot 31^3$ to $\sigma(x) = 2(x + m)$.

Remark 1.8. With this result, we cover all solutions to $\sigma(x) = 2(x + 6)$ (at least up to 10^{10}), while the equations $\sigma(x) = 2\,(x + 28)$ and $\sigma(x) = 2\,(x + 496)$ have several other solutions with more prime divisors, which we did not consider so far. Some of them are consequence of the following result:

Proposition 1.9. *If $m = 2^k\,(M_k + \frac{Q+1}{2})$, where $M_k = 2^k - 1$ and Q are two distinct odd primes, then a solution to $\sigma(x) = 2\,(x + m)$ is given by $x = 2^k\,M_k\,Q$.*

Proof. As before, we have $\sigma(2^k\,M_k\,Q) - 2 \cdot 2^k\,M_k\,Q = 2^{k+1}(M_k + (Q + 1)/2)$. □

Example 1.10. We have $m = 496 = 2^2\,(3 + (241 + 1)/2) = 2^3\,(7 + (109 + 1)/2)$, which leads to the two solutions $x = 2^2 \cdot 3 \cdot 241 = 2892$ and $x = 2^3 \cdot 7 \cdot 109 = 6104$.

The above proposition is a particular case of

Theorem 1.11. *If $m = 2^k(2^{k+1} - 1) - 2\,\ell_1\,\ell_2$ such that $p_i = 2^{k+1} - 1 + 2\,\ell_i$ $(i = 1, 2)$ are two distinct odd primes, then $x = 2^k\,p_1\,p_2$ is a solution of the equation $\sigma(x) = 2\,(x + m)$.*

Example 1.12. For $m = 496$, we have (among others) the following solutions:

k	ℓ_1	p_1	p_2	x
2	-2	3	241	2892
3	-4	7	109	6104
5	20	103	139	458144
5	5	73	367	857312
5	4	71	443	1006496
5	2	67	823	1764512
6	6	139	1399	12445504
6	2	131	3943	33058112
7	82	419	647	34699904
7	56	367	829	38943104

(For given m, the solutions are found by taking $k = 1, 2, 3, ...$ and checking factorizations $\ell_1\,\ell_2$ of $(2^k(2^{k+1} - 1) - m)/2$, to see whether the resulting p_i are prime.) In the first two lines we recognize the previous examples. Indeed, for $\ell_1 = -2^{k-1}$, $p_1 = 2^k - 1$, $p_2 = Q$, we get Proposition 1.9.

Proof. For $x = 2^k\,p_1\,p_2$, $\sigma(x) - 2\,x = (2^{k+1} - 1)(p_1 + 1)(p_2 + 1) - 2^{k+1}\,p_1\,p_2 = 2^{k+1}\,(p_1 + p_2 + 1) - (p_1 + 1)(p_2 + 1) = 2^{k+1}\,(2^{k+1} - 1) - 4\,\ell_1\,\ell_2 = 2\,m$. □

1.2 The equation $\sigma(x) = k\,(x + m)$

We conclude this introduction by the following generalizations of the preceding theorems to k–fold perfect numbers.

Theorem 1.13. *If m is a k–perfect number, then the equation $\sigma(x) = k(x+m)$ has infinitely many solutions, including all numbers of the form $x = p\,m$, where p is a prime not dividing m.*

Proof. The proof is the same as before, except for replacing 2 by k. To write out the straightforward calculation here,

$$\sigma(x) = \sigma(p\,m) = (p+1)\,\sigma(m) = (p+1)\,k\,m = k\,(p\,m+m) = k\,(x+m).$$

□

The particular cases of this Theorem 1.13, corresponding to $k=3$ and $k=4$, will be developed in Corollary 4.1 and 5.1.

Theorem 1.14. *If m is a k–perfect number and p is a prime such that $p^t \mid m$ but $p^{t+1} \nmid m$, (i.e., $t \geq 0$ is the p–adic valuation of m), then $x = p^{t+1}\,m$ is a solution of the equation $\sigma(x) = k\,(x+m)$.*

Proof. For $m = p^t\,m'$, $\gcd(m',p) = 1$, and $x = p^{t+1}\,m = p^{2t+1}\,m'$, we have $\sigma(x) = \sigma(p^{2t+1})\,\sigma(m') = \sigma(p^{2t+1})\,\sigma(m)/\sigma(p^t) = k\,m\,(p^{2t+2}-1)/(p^{t+1}-1) = k\,m\,(p^{t+1}+1) = k\,(x+m)$. □

Example 1.15. For $t=0$, $k=2$, we get Theorem 1.3, for $t=1$, $k=2$ we get Proposition 1.6, and for $t=0$ and any k, we get the preceding Theorem 1.13.
Other examples with $t>0$ include the following cases:

- $k=3$, $T_1 = 120 = 2^3\,3\,5$ with solutions $x = 1080 = 2^3\,3^{1+2}\,5$ and $x = 1920 = 2^{3+4}\,3\,5$.
- $k=3$, $T_2 = 672 = 2^5 \cdot 3 \cdot 7$, with solution $x = 3^2\,T_2 = 6048$.
- $k=4$, $Q_1 = 30240 = 672 = 2^5 \cdot 3 \cdot 7$, with solution $x = 5^2\,Q_1 = 756000$.

For the case where $q = k+1$ is a prime, we have the following additional results:

Theorem 1.16. *If Q is a $(q-1)$–perfect number for some prime q, and $q^k - q - 1$ is a prime such that $\gcd(q\,(q^k-q-1), Q) = 1$, then $x = q^{k-1}\,(q^k-q-1)\,Q$ is a solution of the equation $\sigma(x) = q\,(x+Q)$.*

Example 1.17.
- For $q=2$, we have $Q=1$ and we get the same as Theorem 1.1 with $m=1$.
- Examples for $q=3$ and $q=5$ will be given in Example 4.12 and 5.23.

Proof. This is the following theorem with $m=1$. □

Theorem 1.18. *If Q is a $(q-1)$–perfect number for some prime q and $m \in \mathbb{Z}$, then the equation*

$$\sigma(x) = q\,(x + m\,Q)$$

admits the solution $x = q^{k-1}\,p\,Q$ whenever $p = q^k - m\,q - 1$ is prime and such that $\gcd(q^{k-1}\,p, Q) = 1$ and $\gcd(q^{k-1}, p) = 1$ (i.e., $q \neq p$ or $k = 1$).

Proof. With the given definitions and assumptions, we have
$\sigma(x) = (q^k - 1)/(q - 1) \cdot (p + 1) \cdot (q - 1)\, Q = (q^k\, p + q^k - p - 1)\, Q = q\,(x + m\, Q)$. □

Example 1.19. For $m = 1$, we get the preceding Theorem 1.16. For $q = 2$, we have $Q = 1$ and Theorem 1.1; and thus, for $m = 0$, Euclid's proposition.

Theorem 1.20. *If m is a k–perfect number and $2\,t$ is the sum of two distinct primes p_1 and p_2 such that $\gcd(p_1\, p_2, m) = 1$, then $p_1\, p_2\, m$ are solutions of the equation $\sigma(x) = k\,(x + (2\,t + 1)\, m)$.*

Example 1.21. For $k = 2$, $m = 6$, $t = 50$ one has $2\,t = 100 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53$, so all the five numbers $11 \cdot 89 \cdot 6, 17 \cdot 83 \cdot 6$, $29 \cdot 71 \cdot 6$, $41 \cdot 59 \cdot 6$ and $47 \cdot 53 \cdot 6$ are solutions of the equation $\sigma(x) = 2\,(x + 606)$.

Corollary 1.22. *All semi-primes of the form $p\,(2t - p)$, $p < t$, are solutions for the equation $\sigma(x) = x + 2\,t + 1$.*

Proof of the Theorem. With given hypotheses, for $x = p_1\, p_2\, m$ we have $\sigma(x) = (p_1 + 1)\,(p_2 + 1)\, k\, m = k\,(p_1\, p_2\, m + (p_1 + p_2 + 1)\, m) = k\,(x + (2\,t + 1)\, m)$. □

2 Equations of the form $\sigma(x) = 2\,x - f(x)$

In this section, we consider solutions to equations of the form $\sigma(x) = 2\,x - f(x)$, where the right hand side may involve different functions of x.

In some sense this was already considered in the introduction, since the equation $\sigma(x) = 2\,(x + m)$ simply corresponds to a constant function equal to an even number, $f(x) = -2\,m$.

2.1 Solutions to $\sigma(x) - 2\,x = k \in 1 + 2\,\mathbb{Z}$

It is seen that for any even constant on the right hand side, there are usually many solutions. For an odd constant, however, things are quite different. A short remark about this can be found in §196 of De Koninck's book [6], who conjectures that there is at most a finite number of solutions for odd k. (Of course this does not apply to the special case $k = -1$, since any power of 2 is a solution to $\sigma(x) = 2\,x - 1$, while it is not known whether there are any other such *almost perfect numbers*.) If k is odd, then $\sigma(x)$ must be odd, which is the case if and only if x is a square or twice a square. (Indeed, the factor $\sigma(p^e) = (p^{e+1} - 1)/(p - 1) = 1 + p + \cdots + p^e$ is odd iff $p = 2$ or there is an odd number $e + 1$ of terms.)

Sequence A140863 = (3, 7, 17, 19, 31, 39, 41, 51, 59, 65, 71, 89, 115, ...) lists odd numbers k for which $\sigma(x) = 2\,x + k$ is known to have a solution:

k	3	7	17	19	31	39	41	51	59	65	71	89
x	18	196	100	36	15376	162	1352	72	968	200	392	144

Sequence A082731 lists the smallest solution to $\sigma(x) = 2\,x + k$ for "any" $k \in \mathbb{N}$.

The list of odd k such that $\sigma(x) = 2\,x - k$ has a solution starts as follows:

k	1	5	7	11	19	25	37	41	47	61	71	85	109
x	1	9	50	494^2	25	98	484	49	225	2888	676	242	121

Sequence A082730 lists the smallest solution to $\sigma(x) = 2\,x - k$ for $k \geq 0$.

Concerning subsequences of A140863 (odd k such that $\sigma(x) - 2\,x = k$ has a solution), we have the following results.

Proposition 2.1. *The sequence A000668 of Mersenne primes is a subsequence of A140863, since if $M_p = 2^p - 1$ is prime, then $x = 2^{p-1}\,(2^p - 1)^2$ is a solution to $\sigma(x) - 2\,x = M_p$.*

Example 2.2.

- $p = 3$; $x = 2^2\,7^2 = 196$ is a solution to $\sigma(x) = 2\,x + 7$.
- $p = 5$; $x = 2^4\,31^2 = 15376$ is a solution for the equation $\sigma(x) = 2\,x + 31$.
- $p = 7$; $x = 2^6\,127^2 = 1032256$ is a solution of $\sigma(x) = 2\,x + 127$.

Proof. For prime $M_p = 2^p - 1$ and $x = 2^{p-1}\,M_p^2$, we have $\sigma(x) - 2\,x = M_p\,(M_p^3 - 1)/(M_p - 1) - 2^p\,M_p^2 = M_p\,(M_p^2 + M_p + 1 - (M_p + 1)\,M_p) = M_p$. $\square$

This Proposition is actually a special case of the following

Theorem 2.3. *If $M = 2^p - 1$ is a Mersenne prime and t is a non-negative integer then $x = (2^t M)^2$ is a solution for the equation $\sigma(x) = 2\,x + k$, where $k = (M+1)\,(2^{2\,t+1} - 1) - M^2$. Such values of k are therefore a subsequence of A140863.*

Proof. $\sigma(x) - 2\,x = (2^{2t+1} - 1)\,(M^2 + M + 1) - 2^{2t+1}M^2 = (M+1)\,(2^{2t+1} - 1) - M^2 = k$. $\square$

Example 2.4. For $t = (p-1)/2$, this yields the preceding Proposition 2.1, and the previous examples apply. For $t = 0$ we get: If $M = 2^p - 1$ is prime, then $x = M^2$ is a solution for the equation

$$\sigma(x) = 2x - 4^p + 3 \cdot 2^p - 1\ .$$

Some particular cases are:

- $p = 2$; $x = 9$ is a solution for the equation $\sigma(x) = 2\,x - 5$.
- $p = 3$; $x = 49$ is a solution for the equation $\sigma(x) = 2\,x - 41$.
- $p = 5$; $x = 961$ is a solution for the equation $\sigma(x) = 2\,x - 929$.

We also have the following similar result, related to A140863 and Mersenne primes:

Proposition 2.5. *If $M = 2^p - 1$ is prime, then $x = (M+1)\,M^2 = 2^p(2^p - 1)^2$ is a solution for the equation $\sigma(x) = 2\,x + k$, where $k = 4^p + 2^p - 1$.*

Proof. As before, a straightforward calculation gives $\sigma(x) - 2\,x = \sigma(2^p(2^p - 1)^2) - 2^{p+1}(2^p - 1)^2 = (2^{p+1} - 1)\,((2^p - 1)^3 - 1)/(2^p - 2) - 2^{p+1}(2^p - 1)^2 = 4^p + 2^p - 1 = k$. $\square$

Example 2.6.

- For $p = 2$, we get $x = 36$ as a solution to $\sigma(x) = 2\,x + 19$.
- The prime $p = 3$ yields $x = 392$ as solution of $\sigma(x) = 2\,x + 71$.

- For $p = 17$ we have $x = 2251765454077952$ as a solution of the equation $\sigma(x) = 2\,x + 17180000255$.

An extended numerical search motivates the following conjectures:

Conjecture 2.7. *If M is a Mersenne prime, then the equation $\sigma(x) - 2\,x = M$ has only one solution, $x = M^2\,(M+1)/2$.*

Conjecture 2.8. *If M is a Mersenne prime, then the equation $\sigma(x) - 2\,x = M^2 + 3\,M + 1$ has only one solution, $x = (M+1)\,M^2$.*

We have verified that there is no counterexample less than $L = 2{\cdot}10^{17}$ to these conjectures, for all Mersenne primes $\leq M(89) = 2^{89} - 1$. This seems significant at least for the smaller Mersenne primes, for which the known solution is significantly smaller than this search limit. Obviously, there was not much hope to find a counter-example below this limit for the larger Mersenne primes, since from $M(19)$ on, the known solution is already of the order of magnitude of L or above.

We conclude this subsection with two results concerning solutions for a negative constant term.

Proposition 2.9. *The equation $\sigma(x) = 2\,x - k$ has the solution $x = p^2$ whenever $k = p(p-1) - 1$ for some prime p.*

This is a special case ($t = 2$) of the following

Theorem 2.10. *When k is of the form $(p^{t+1} - 2\,p^t + 1)/(p-1)$ where p is prime, then $x = p^t$ is a solution for the equation $\sigma(x) = 2\,x - k$.*

Proof. Again by a straightforward calculation, $\sigma(p^t) - 2\,p^t = (p^{t+1} - 1)/(p-1) - 2\,(p^{t+1} - p^t)/(p-1) = -(p^{t+1} - 2\,p^t + 1)/(p-1)$. □

2.2 Equations involving the number-of-divisors function

We denote by $d(n)$ the number of positive divisors of n.

Theorem 2.11. *If m is a natural number and $4^m + 2^m - 2t - 1$ is an odd prime, then $x = 2^{m-1}\,(4^m + 2^m - 2\,t - 1)$ is a solution of the equation*

$$\sigma(x) = 2\,x - 2^{d(x)} + 2\,t.$$

Proof. This follows by straightforward calculation, using again equation (1) and $d(x) = 2\,m$ here: $\sigma(x) = (2^m - 1)\,(4^m + 2^m - 2\,t) = 2^m\,(4^m + 2^m - 2\,t - 1 + 1) - (4^m + 2^m - 2\,t) = 2^m\,(4^m + 2^m - 2\,t - 1) - (4^m - 2\,t) = 2\,x - 2^{2\,m} + 2\,t = 2\,x - 2^{d(x)} + 2\,t$. □

Corollary 2.12. *For $t = 0$, we get: If $4^m + 2^m - 1$ is prime (A098855), then $2^{m-1}\,(4^m + 2^m - 1)$ (A110082) is a solution of the equation $\sigma(x) = 2\,x - 2^{d(x)}$.*

Corollary 2.13. *For $t = 1$, we find solutions to $\sigma(x) = 2\,x - 2^{d(x)} + 2$ of the form $x = 2^{m-1}\,(4^m + 2^m - 3)$, with $m =$* 1, 2, 4, 6, 8, 24, 44, 76, 144, 492, 756, 1564, ...

Corollary 2.14. *For $t = 2$, solutions to $\sigma(x) = 2\,x - 2^{d(x)} + 4$ are given by $x = 2^{m-1}\,(4^m + 2^m - 5)$ with $m = 3, 5, 9, 11, 25, 33, 133, 189, 635, 1235, 1685, 1849, \ldots$.*

3 Equations involving Euler's totient function ϕ

In this section we deal with equations which are not really "deformations" of Euclid's formula in the strict sense, since the right hand side does not start with the term $2\,x$. However, calculations are very similar to what precedes, so it seemed justified to present the following results in this place.

We use the standard notation $\phi(n)$ for Euler's totient function, giving the order of the group $(\mathbb{Z}/n\mathbb{Z})^*$, i.e., the number of positive integers not exceeding n and coprime to n. Equations similar to those discussed here have been studied by Guy [5] and Zhang [11].

Theorem 3.1. *If m is a natural number and $2^{m+2}-2\,t\,(m+1)-2\,k-1$ is an odd prime, then $x = 2^m\,(2^{m+2}-2\,t\,(m+1)-2\,k-1)$ is a solution of the equation $\sigma(x) = 4\,\phi(x)+t\,d(x)+2\,k$.*

Proof. In this case, $d(x) = 2\,(m+1)$ and $\phi(x) = 2^{m-1}\,(2^{m+2} - 2\,t\,(m+1) - 2\,k - 2)$, so

$$\begin{aligned} 4\,\phi(x) + t\,d(x) + 2\,k &= 2^{m+1}\,(2^{m+2} - 2\,t\,(m+1) - 2\,k - 2) + 2\,t\,(m+1) + 2\,k \\ &= (2^{m+1} - 1)\,(2^{m+2} - 2\,t\,(m+1) - 2\,k - 2) \\ &\quad + (2^{m+2} - 2\,t\,(m+1) - 2\,k - 2) + 2\,t\,(m+1) + 2\,k \\ &= (2^{m+1} - 1)\,(2^{m+2} - 2\,t\,(m+1) - 2\,k) = \sigma(x). \end{aligned}$$

□

Corollary 3.2. *For $t = 0$, we get: If m is a natural number and $2^{m+2} - 2\,k - 1$ is an odd prime, then $x = 2^m \cdot (2^{m+2} - 2\,k - 1)$ is a solution of the equation $\sigma(x) = 4\,\phi(x) + 2\,k$.*

Corollary 3.3. *For $k = 0$, we have: If m is a natural number and $2^{m+2} - 2\,t\,(m+1) - 1$ is an odd prime, then $x = 2^m \cdot (2^{m+2} - 2\,t\,(m+1) - 1)$ is a solution of the equation $\sigma(x) = 4\,\phi(x) + t\,d(x)$.*

Corollary 3.4. *Take $t = k = 0$ and $m = p - 2$. If $2^p - 1$ is a Mersenne prime, then $2^{p-2}\,(2^p - 1)$ is a solution of the equation $\sigma(x) = 4\,\phi(x)$.*

Parts of these results have already been contributed as comments to sequence A068390 by the first author.

Theorem 3.5. *If $3 \cdot 2^m - 2\,t\,(m+1) - 1$ is an odd prime, then $x = 2^m\,(3 \cdot 2^m - 2\,t\,(m+1) - 1)$ is a solution of the equation $\sigma(x) = x + 2\,\phi(x) + t\,d(x)$.*

Proof. Here, $d(x) = 2\,(m+1)$ and $\phi(x) = 2^{m-1}\,(3 \cdot 2^m - 2\,t\,(m+1) - 2)$, so

$$\begin{aligned} &x + 2\,\phi(x) + t\,d(x) \\ &\quad = 2^m\,(3 \cdot 2^m - 2\,t\,(m+1) - 1) \cdot 2 - 2 \cdot 2^{m-1} + 2\,t\,(m+1) \\ &\quad = (2^{m+1} - 1 + 1)\,(3 \cdot 2^m - 2\,t\,(m+1) - 1) - 2^m + 2\,t\,(m+1) \\ &\quad = (2^{m+1} - 1)\,(3 \cdot 2^m - 2t(m+1)) - 2^{m+1} + 1 \\ &\qquad\qquad + (3 \cdot 2^m - 2\,t\,(m+1) - 1) - 2^m + 2\,t\,(m+1) \\ &\quad = \sigma(2^m)\;\sigma(3 \cdot 2^m - 2\,t\,(m+1) - 1) \\ &\quad = \sigma(2^m\,(3 \cdot 2^m - 2\,t\,(m+1) - 1)) = \sigma(x)\;. \end{aligned}$$

□

Corollary 3.6. *For $t = 0$, we get: If $3 \cdot 2^m - 1$ is an odd prime, then $x = 2^m\,(3 \cdot 2^m - 1)$ is a solution of the equation $\sigma(x) = x + 2\,\phi(x)$.*

Remark 3.7. Numbers of this form are listed in sequence A097215.

Theorem 3.8. *If $p = 7 \cdot 2^m - k\,(m+3) - 2\,t - 1$ is a prime greater than 3, then $x = 3 \cdot 2^{m+2}\,p$ is a solution of the equation $\phi(x) + \sigma(x) = 3\,x + k\,d(x) + 8\,t$.*

Proof. For $x = 3 \cdot 2^{m+2}\,p$, $d(x) = 2 \cdot (m+3) \cdot 2 = 4\,(m+3)$, $\phi(x) = 2^{m+2}\,(p-1)$ and $\sigma(x) = 4\,(2^{m+3} - 1)\,(p+1)$, so $\phi(x) + \sigma(x) = p\,(2^{m+2} + 8 \cdot 2^{m+2} - 4) - 2^{m+2} + 8 \cdot 2^{m+2} - 4 = 9 \cdot 2^{m+2}\,p - 4\,p + 4\,(7 \cdot 2^m) - 4 = 3\,(3 \cdot 2^{m+2}\,p) - 4\,(p - 7 \cdot 2^m + 1) = 3\,x - 4\,(-k\,(m+3) - 2\,t) = 3\,x + k\,d(x) + 8\,t$. □

Corollary 3.9. *Using the theorem with $k = 0$, we get: If $p = 7 \cdot 2^m - 2\,t - 1$ is a prime greater than 3, then $x = 3 \cdot 2^{m+2}\,p$ is a solution of the equation $\phi(x) + \sigma(x) = 3\,x + 8\,t$.*

Corollary 3.10. *Consider $t = 0$: If $p = 7 \cdot 2^m - k\,(m+3) - 1$ is a prime greater than 3, then $x = 3 \cdot 2^{m+2}\,p$ is a solution of the equation $\sigma(x) + \phi(x) = 3\,x + k\,d(x)$.*

Corollary 3.11. *For $t = k = 0$, we get: If $p = 7 \cdot 2^m - 1$ is prime, then $x = 3 \cdot 2^{m+2}\,p$ is a solution of the equation $\sigma(x) + \phi(x) = 3\,x$.*

Parts of these results have been submitted as comments to sequence A011251.

4 Solutions to $\sigma(x) = 3\,x - f(x)$

In this section we consider equations which can be seen as "deformations" of the definition of triply-perfect numbers, $\sigma(T) = 3\,T$.

4.1 The special case $\sigma(x) = 3\,x - C$

First we discuss solutions to equations $\sigma(x) = 3\,x - C$ for some particular constants C. Let us first recall that Theorem 1.13 with $k = 3$ yields

Corollary 4.1. *If T is a triperfect number (A005820), then the equation $\sigma(x) = 3\,(x + T)$ has infinitely many solutions, which include all numbers of the form $x = p\,T$, where p is a prime not dividing T.*

Example 4.2. For $T_2 = 672 = 2^5 \cdot 3 \cdot 7$, $p = 5$ gives the solution $x = 5\,T_2 = 3360$.
For $T_1 = 120 = 2^3 \cdot 3 \cdot 5$, all primes $p > 5$ yield a solution $x = 120\,p$.

Proposition 4.3. *If P is a perfect number, then $x = P$ is a solution to the equation $\sigma(x) = 3\,x - P$, and if 3 does not divide P, then this equation has infinitely many solutions, including all numbers of the form $x = 3^m\,P$, $m \in \mathbb{N}$.*

This is a particular case ($q = 3$) of the more general

Theorem 4.4. *If q is a prime and Q is a $(q-1)$–perfect number, then $x = Q$ is a solution to the equation $\sigma(x) = q\,x - Q$, and if q does not divide Q, then this equation has infinitely many solutions, including all numbers of the form $x = q^m\,Q$, $m \in \mathbb{N}$.*

Proof. The first statement is obvious, and assuming that $q \not| \, Q$, we have
$\sigma(x) = \sigma(q^m)\,\sigma(Q) = (q^{m+1}-1)/(q-1)\cdot(q-1)\,Q = (q^{m+1}-1)\,Q = q\,x - Q.$ □

Example 4.5.

(i) Since $P = 2^2\,(2^3-1) = 28$ is a perfect number, $x = 28,\ 28\cdot 3,\ 28\cdot 3^2,\ \ldots$ are some solutions of the equation $\sigma(x) = 3\,x - 28$.

(ii) For $P = 2^4\,(2^5-1) = 496$, we get that $496,\ 496\cdot 3,\ 496\cdot 3^2,\ \ldots$ are some solutions of the equation $\sigma(x) = 3\,x - 496$.

(iii) With $P = 2^6\,(2^7-1) = 8128$, we have that $8128,\ 8128\cdot 3,\ 8128\cdot 3^2,\ \ldots$ are some solutions of the equation $\sigma(x) = 3\,x - 8128$.
The number 6508 is another solution, which is not of the mentioned form.

The following theorem gives solutions to a more general equation than the one considered in the previous proposition. The special case $t = 0$ yields the same equation, but not all of the earlier solutions.

Theorem 4.6. *If t is an integer, $p = 3^m + 2\,t$ is prime and P is a perfect number such that $\gcd(3\,p, P) = 1$, then $x = 3^{m-1}\,p\,P$ is a solution of the equation $\sigma(x) = 3\,x - (2\,t+1)\,P$.*

Proof. This is a special case ($q = 3$) of the following Theorem 4.9. □

Example 4.7. For $t = -1$, all numbers m such that $m < 2000$ and $3^m - 2$ is prime, are: 2, 4, 5, 6, 9, 22, 37, 41, 90, 102, 105, 317, 520, 541, 561, 648, 780, 786, 957 and 1353 (A014224). From these we obtain 20 solutions $3\,(3^2-2)\,P$, $3^3\,(3^4-2)\,P$, ... , $3^{956}\,(3^{957}-2)\,P$ and $3^{1352}\,(3^{1353}-2)\,P$ of the equation $\sigma(x) = 3\,x + P$, for any even perfect number P.

Example 4.8. For $t = 1$, the numbers $m > 0$ such that $3^m + 2$ is prime are 1, 2, 3, 4, 8, 10, 14, 15, 24, 26, 36, 63, 98, 110, 123, 126, 139, 235, 243, 315, 363, 386, 391, 494, 1131, 1220, 1503, 1858... (A051783). From these we obtain the solutions $(3^1+2)\,P$, $3^1\,(3^2+2)\,P$, ..., $3^{1502}\,(3^{1503}+2)\,P$, $3^{1857}\,(3^{1858}+2)\,P$,... of the equation $\sigma(x) = 3\,(x-P)$, for any even perfect number P.

The previous theorem is a special case of the following more general result:

Theorem 4.9. *If Q is a $(q-1)$–perfect number for some prime q, then for all integers t, the equation*

$$\sigma(x) = q\,x - (2\,t+1)\,Q \tag{2}$$

has the solution $x = q^{k-1}\,p\,Q$ whenever $k \in \mathbb{N}$ is such that $p = q^k + 2\,t$ is prime, $\gcd(q^{k-1}, p) = 1$ (i.e., $q \neq p$ or $k = 1$) and $\gcd(q^{k-1}\,p, Q) = 1$.

Proof. $\sigma(x) = \sigma(q^{k-1})\,\sigma(p)\,\sigma(Q) = (q^k-1)/(q-1)\cdot(p+1)\,(q-1)\,Q = q^k\,(p+1)\,Q - (p+1)\,Q = q^k\,p\,Q + (q^k-(p+1))\,Q = q\,x - (2\,t+1)\,Q$. □

Remark 4.10. For $t = -\frac{1}{2}(m\,q+1)$, we get back theorem 1.18.

Finally, recall that from Theorem 1.16 we have for $q = 3$:

Proposition 4.11. *If P is an even perfect number not divisible by 3 (i.e., $P > 6$), then*

$$\sigma(x) = 3\,(x + P)$$

admits the solution $x = 3^{k-1}\,p\,P$ whenever $p = 3^k - 4$ is prime ($k \in$ A058959).

Example 4.12. For $q = 3$, $k \in$ A058959 = (2, 3, 5, 21, 31, 37, 41, 53, 73, 101, 175, 203, 225, 455, 557, 651, ...) corresponds to a prime $3^k - 4 \in$ A156555 = (5, 23, 239, 10460353199, 617673396283943,...) and yields solutions $x = 3^{k-1}\,(3^k - 4)\,P$ to $\sigma(x) = 3\,(x + P)$ for all even perfect numbers $P > 6$ (since $3^k - 4$ is always odd and can never be equal to $2^p - 1$):
For $P = 28$ we have $x = 420$, 5796, 542052, ...; for $P = 496$ we get $x = 7440$, 102672, 9602064, ..., and $P = 8128$ leads to $x = 121920$, 1682496, 157349952, ...

Proposition 4.13. *If $p = (m+8-2^{k+4})/(2^k-8)$ is a prime greater than 5, then $x = 2^k \cdot 15\,p$ is a solution for the equation $\sigma(x) = 3\,(x + m)$.*

Proof. For any prime $p > 5$, $\sigma(2^k\,15\,p) - 3 \cdot 2^k \cdot 15\,p = 3\left[8\,(2^{k+1} - 1) + p\,(2^k - 8)\right]$, which equals $3\,m$ by hypothesis. □

Example 4.14. If $m = 672$, we need a prime $p = (680 - 2^{k+4})/(2^k - 8)$.
For $k = 4$, we obtain $p = 53$ and $x = 2^4 \cdot 3 \cdot 5 \cdot 53 = 12720$.
For $k = 5$, we get $p = 7$ and the solution $x = 2^5 \cdot 3 \cdot 5 \cdot 7 = 3360$ already known as $x = 5\,m$ from Corollary 4.1 of Theorem 1.13.

4.2 The case $\sigma(x) = 3\,x - 3^{d(x)} - k$

Theorem 4.15. *If $2 \cdot 3^m - 1$ is prime ($m \in$ A003307), then $x = 3^{m-1}\,(2 \cdot 3^m - 1)$ is a solution of the equation*

$$\sigma(x) = 3\,x - 3^{d(x)}\ . \tag{3}$$

Proof. $d(x) = 2m$ and $\sigma(x) = \sigma(3^{m-1}(2 \cdot 3^m - 1)) = \sigma(3^{m-1}) \cdot \sigma(2 \cdot 3^m - 1) = (3^m - 1)/2 \cdot (2 \cdot 3^m) = 3 \cdot (3^{m-1} \cdot (2 \cdot 3^m - 1 - 3^m)) = 3\,x - 3^{2m} = 3\,x - 3^{d(x)}$. □

Example 4.16. Sequence A003307 starts with $m = 1$, 2, 3, 7, 8, 12, 20, 23, 27, 35, 56, 62, 68, ... These numbers correspond to solutions $x = 5$, 51, 477, 3187917, 28695627, 188286180507, 8105110304875691067, ... of equation (3).

Let us now consider the following equation for some particular $k \in \mathbb{N}$,

$$\sigma(x) = 3\,x - 3^{d(x)} - k\ . \tag{4}$$

Let us start with two families of very simple solutions:

Proposition 4.17. *If $k = 4\,\ell$ and $p = 2\,\ell + 5$ is prime, then p is a solution of (4).*

Proof. We have $\sigma(p) = p + 1 = 2\,\ell + 6 = 3\,(2\,\ell + 5) - 3^2 - 4\,\ell = 3\,p - 3^{d(p)} - k$, so p is a solution as claimed. □

Example 4.18. Below we list a table with values $k = 4\,\ell$ for which $p = 2\,\ell+5$ is a solution to equation (4), according to this proposition:

k	0	4	12	16	24	28	36	48	52	64	72	76	84	96	108	112
p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61

- Indeed, for $\ell = 0$ we get equation (3), and from Proposition 4.17 the solution $x = 5$, which is the first one given in the preceding Example 4.16.
- For $\ell = 1$, $2\,\ell + 5 = 7$ is prime, therefore $p = 7$ is a solution of the equation $\sigma(x) = 3\,x - 3^{d(x)} - 4$.
- For $\ell = 3$, $p = 2\,\ell + 5 = 11$ is prime; accordingly, $p = 11$ is a solution to $\sigma(x) = 3\,x - 3^{d(x)} - 12$.

Proposition 4.19. *If* $k = 3\,(2\,\ell + 1)$ *and* $p = 2\,\ell + 29$ *is prime, then* $x = 2\,p$ *is a solution of* (4)*.*

Proof. Again, $\sigma(x) = \sigma(2\,p) = 3\,(p+1) = 3\,(2p) - 3^4 - 3\,(p-28) = 3\,x - 3^{d(x)} - 3\,(2\,\ell+1)$. □

Example 4.20. For $\ell = -5$, $p = 19$ is prime, and one can check that $2 \cdot 19$ is indeed a solution to $\sigma(x) = 3\,x - 3^{d(x)} + 27$.

Example 4.21. For $\ell = 1$, $p = 31$ is prime, and $2{\cdot}31$ is a solution to $\sigma(x) = 3\,x - 3^{d(x)} - 9$.

The following is a considerable generalization of the above Theorem 4.15:

Theorem 4.22. *Let p and q be two distinct primes, and t a nonnegative integer. Then $x = p\,q^t$ is a solution for the equation* (4) *iff* $p = (k\,(q-1) + q^{t+1} - 1 + (q-1)3^{2\,t+2})/(2\,q^{t+1} - 3\,q^t + 1)$.

Proof. For $x = p\,q^t$, we have $3x - 3^{d(x)} - \sigma(x) = k \iff 3\,p\,q^t - 3^{2\,t+2} - (p+1)(q^{t+1} - 1)/(q-1) = k \iff 3pq^t(q-1) - 3^{2t+2}(q-1) - (p+1)(q^{t+1} - 1) = k(q-1) \iff p(2q^{t+1} - 3q^t + 1) = (k(q-1) + q^(t+1) - 1 + (q-1)3^(2t+2)) \iff p = (k(q-1) + q^{t+1} - 1 + (q-1)3^{2t+2})/(2q^{t+1} - 3q^t + 1)$. So $x = pq^t$ is a solution of the equation (4) iff $p = (k(q-1) + q^{t+1} - 1 + (q-1)3^{2t+2})/(2q^{t+1} - 3q^t + 1)$. □

5 Results related to quadruply-perfect numbers

Quadruply-perfect numbers Q are such that $\sigma(Q) = 4\,Q$. They are listed in sequence A027687. There exist only 36 such numbers, which are all known since the early 20th century [4].

We consider first one "deformed" version of this defining equation, and then some other results related to quadruply-perfect numbers, or, to relations of the form $\sigma(x) = 4\,x - f(x)$.

5.1 Solutions to $\sigma(x) = 4\,x - C$

As before, we first consider a constant additional term, and recall that Theorem 1.13 (with $k = 4$) yields

Corollary 5.1. *If Q is a quadruply-perfect number (A027687) then the equation*

$$\sigma(x) = 4\,(x + Q)$$

has infinitely many solutions, including all numbers of the form $x = p\,Q$, where p is a prime which does not divide Q.

A numerical search for solutions to the preceding equation reveals additional solutions, as $x = 2^3 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 13$, which are not yet described by previous theorems. In such a case we can often, somehow systematically, construct a relation which describes a family of solutions to which the given number x belongs. To this end, we consider (for example) one of the prime factors of x as a parameter, and express it in terms of the constant term occurring in the equation. Then, whenever this expression yields a prime, coprime to the other factors of x, we have again a solution. In the case at hand, we can look for solutions of the form $x = 2^3 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot p$, where $p > 7$ is prime. For such x, we find that $\sigma(x) = 4\,(x + m)$ is equivalent to $2^2 \cdot 5 \cdot 3^3\,(5 \cdot 19 - 3\,p) = m$. Otherwise stated,

Proposition 5.2. *If $p = (95 - m/540)/3$ is a prime greater than 7, i.e., $m = 51300 - 1620\,p$, then $x = 2^3 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot p$ is a solution for the equation $\sigma(x) = 4\,(x + m)$.*

Example 5.3. Considering primes $p > 7$, this proposition yields a solution for the following values of m:

m	p	x
33480	11	582120
30240	13	687960
23760	17	899640
20520	19	1005480
14040	23	1217160
4320	29	1534680
1080	31	1640520
−8640	37	1958040
−15120	41	2169720
−18360	43	2275560

m	p	x
−24840	47	2487240
−34560	53	2804760
−44280	59	3122280
−47520	61	3228120
−57240	67	3545640
−63720	71	3757320
−66960	73	3863160
−76680	79	4180680
−83160	83	4392360
−92880	89	4709880

etc.

5.2 Solutions to $\sigma(x) = 4\,x - k\,d(x)$

Let us now consider the following equation for some particular $k \in \mathbb{N}$,

$$\sigma(x) = 4\,x - k\,d(x)\ . \tag{5}$$

Proposition 5.4. *A prime p is solution to equation (5) (with $d(x) = 2$) if, and only if, $k = (3\,p - 1)/2$. Otherwise said, this equation has a prime as solution, if and only if $k \equiv 1 \pmod 3$ and $p = (2\,k + 1)/3$ is prime.*

Proof. We have $\sigma(p) = 4\,p - (3\,p - 1) = 4\,p - 2\,\frac{3\,p-1}{2} = 4\,p - d(p)\,k$. □

Example 5.5. For the following values of k (non exhaustive list) we get a prime p which is solution to (5) according to the preceding Proposition:

k	4	7	10	16	19	25	28	34	43	46	55	61	64	70	79	88	91
p	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61

The above proposition is actually a special case of the following one, similar to Theorem 4.22 (and with analogous proof):

Theorem 5.6. *Let p and q be two distinct primes and t a nonnegative integer. Then $x = p\,q^t$ is a solution for the equation (5) iff $p = (2\,k\,(q-1)(t+1) + q^{t+1} - 1)/(3\,q^{t+1} - 4\,q^t + 1)$.*

Example 5.7. For $t = 0$, we get the earlier Proposition 5.4. For $t = 1$, we get the following

Corollary 5.8. *Let p and q be two distinct primes, then $x = p\,q$ is a solution for the equation (5) iff $p = (4\,k + q + 1)/(3\,q - 1)$.*

Example 5.9. Take $q = 3$, then $3p$ is a solution iff $p = (k+1)/2$; thus for $k = 61$ we obtain $p = 31$ and the solution $x = 3 \cdot 31 = 93$ of the equation

$$\sigma(x) = 4\,x - 61\,d(x). \tag{6}$$

Proposition 5.10. *Let p and q be two distinct primes, and t a nonnegative integer. Then $x = 2^t\,p\,q$ is a solution for the equation (5) iff $p = (4\,k\,(t+1) + (q+1)(2^{t+1} - 1))/((2^{t+1} + 1)\,q - 2^{t+1} + 1)$.*

Corollary 5.11. *We get that $x = 2^t\,p\,q$ is a solution for the equation (6) iff $p = (244\,(t+1) + (q+1)(2^{t+1} - 1))/((2^{t+1} + 1)\,q - 2^{t+1} + 1)$.*

Example 5.12. For $t = 0$, we get $p = (244 + q + 1)/(3\,q - 1)$, so q should be less than 123 and we have $x = 3.31 = 93$, which is the only semiprime solution of the equation (6).

Example 5.13. For $t = 1$, we get $p = (3q + 491)/(5q - 3)$, so q should be less than 247 and we obtain as the only solution $x = 2 \cdot 5 \cdot 23 = 230$.

Example 5.14. The only other $t < 22$ with a corresponding solution is $t = 5$, with $(p, q) = (3, 13)$. So $x = 2^5\,3 \cdot 13 = 1248$ is the largest solution of this form we know.

Proposition 5.15. *Let p and q be two distinct primes greater than 3, then $x = 2^t\,3^2\,p\,q$ is a solution of the equation (5) iff $p = (12\,k\,(t+1) + 13(2^{t+1} - 1)(q+1))/(q(5 \cdot 2^{t+1} + 13) - 13(2^{t+1} - 1))$.*

So $x = 2^t\,3^2\,p\,q$ is a solution of the equation(6) iff $p = (13\,(2^{1+t} - 1)(1+q) + 732\,(1+t))/((13 + 5 \cdot 2^{1+t})q - 13\,(2^{t+1} - 1))$.

Example 5.16. The only number t we know such that there is a corresponding solution is $t = 5$, with $x = 2^5\,3^2\,5 \cdot 13 = 15840$.

There are no other solutions of the equation (6) below 10^{10} than those described by one of the preceding propositions, namely $x \in \{\,41,\ 93,\ 230,\ 1248,\ 15840\,\}$.

5.3 Solutions to $\sigma(x) = 5\,x - C$

Proposition 5.17. *If Q is a quadruply-perfect number, then $x = Q$ is a solution to the equation $\sigma(x) = 5\,x - Q$, and if 5 does not divide Q, then the equation has infinitely many solutions, including $x = 5^m\,Q$, $m = 0, 1, 2, 3, \ldots$.*

Proof. This is Theorem 4.4 for $q = 5$. □

Example 5.18. The quadruply-perfect numbers which are not divisible by 5 are: $Q_6 = 142990848$, $Q_{11} = 403031236608$, $Q_{12} = 704575228896$, $Q_{13} = 181742883469056$, and $Q_{16}, \ldots, Q_{36}$ except for Q_{21}, Q_{26} and Q_{28} (cf. OEIS [8], b-file for sequence A027687). Each of them gives rise to an infinite sequence of solutions to $\sigma(x) = 5\,x - Q$.

For $t = 0$, the following theorem solves the same equation as the previous proposition; in this case, however, it gives only one of the solutions found above.

Theorem 5.19. *If Q is a quadruply-perfect number and $t \in \mathbb{Z}$, then the equation*

$$\sigma(x) = 5\,x - (2t + 1)\,Q$$

has $x = 5^{k-1}\,p\,Q$ as solution, whenever $p = 5^m + 2\,t$ is prime, with $m \in \mathbb{N}$ such that $\gcd(5^{m-1}, p) = 1$ (i.e., $p \neq 5$ or $m = 1$) and $\gcd(5\,p, Q) = 1$.

Proof. This is Theorem 4.9 with $q = 5$. □

Example 5.20. For $t = -1$, all numbers $m < 2000$ such that $p = 5^m - 2$ is prime, are: 1, 2, 14, 26, 50, 126, 144, 260, 624 and 1424 (A109080). From these we obtain 10 solutions $(5^1 - 2)\,Q$, $5\,(5^2 - 2)\,Q$, ..., $5^{623}\,(5^{624} - 2)\,Q$ and $5^{1423}\,(5^{1424} - 2)\,Q$ for the equation $\sigma(x) = 5\,x + Q$.

Example 5.21. For $t = 2$, all numbers m such that $m < 2000$ and $5^m + 4$ is prime, are: 2, 6, 10, 102, 494, 794 and 1326 (A124621). So we obtain 6 solutions $5\,(5^2 + 4)\,Q$, $5^5\,(5^6 + 4)\,Q$, ..., $5^{793}\,(5^{794} + 4)\,Q$ and $5^{1325}\,(5^{1326} + 4)\,Q$ for the equation $\sigma(x) = 5\,(x - Q)$.

Finally, recall that from Theorem 1.16 we have for $q = 5$:

Proposition 5.22. *If Q is a quadruply perfect number not divisible by 5, then*

$$\sigma(x) = 5\,(x + Q)$$

admits the solution $x = 5^{k-1}\,p\,Q$ whenever $p = 5^k - 6$ is prime ($k \in$ A165701) and does not divide Q.

Example 5.23. For $k \in$ A165701 $= (2, 4, 5, 6, 10, 53, 76, 82, 88, 242, 247{,}473, 586, 966, \ldots)$ we get primes $p = 5^k - 6 \in (19, 619, 3119, 15619, 9765619, \ldots)$ which lead to solutions $x = 5^{k-1}\,p\,Q$ to $\sigma(x) = 5\,(x + Q)$ for all quadruply perfect numbers not divisible by 5, listed in Example 5.18, except for the case $k = 2$ where $p = 19$ divides the first two (Q_6, Q_{11}), the seventh (Q_{18}), and the last six $(Q_{31}, \ldots, Q_{36})$ of these quadruply perfect numbers.
The smallest example is therefore $x = 5 \cdot 19 \cdot 142990848$, solution to $\sigma(x) = 5\,(x + 142990848)$.

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