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# Running Modulus Recursions

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**Abstract**

Fix integers $b \geq 2$ and $k \geq 1$. Define the sequence $\{z_n\}$ recursively by taking $z_0$ to be any integer, and for $n \geq 1$, take $z_n$ to be the least nonnegative residue of $bz_{n-1}$ modulo $(n+k)$. Since the modulus increases by 1 when stepping from one term to the next, such a definition will be called a *running modulus recursion* or *rumor* for short. While the terms of such sequences appear to bounce around irregularly, empirical evidence suggests the terms will eventually be zero. We prove this is so when one additional assumption is made, and we conjecture that this additional condition is always met.

## 1 Introduction

Fix integers $b \geq 2$ and $k \geq 1$. Define the sequence $\{z_n\}$ recursively by taking $z_0$ to be any integer, and for $n \geq 1$, $z_n = bz_{n-1} \bmod (n+k)$, the least nonnegative residue of $bz_{n-1}$ modulo $(n+k)$. Since the modulus increases by 1 when stepping from one term to the next, a sequence generated in this fashion will be called *running modulus recursion* or *rumor* for short.

Since only the eventual behavior of rumors is of interest, we may as well assume $0 \leq z_0 \leq k$: using the same $b$ parameter, the sequence with $z'_0 = z_0 \bmod (k+1)$, and, for $n \geq 1$, $z'_n = bz'_{n-1} \bmod (n+k)$, differs from the original sequence only in the first term.

The terms of such sequences appear to bounce around irregularly, at least for a while, but empirical evidence suggests that eventually a 0 term will occur in the sequence, and of course, if 0 occurs in the sequence, all following terms will also be 0.

For example, with $z_0 = 1$, and for $n \geq 1$, $z_n = 3z_{n-1} \bmod (n+11)$, the sequence begins

$$1, 3, 9, 13, 9, 11, 16, 12, 17, 11, 12, 14, 19, 9, 2, 6, 18, 26, 20, 0, \ldots.$$

For this example, $z_{19}$ is the first 0 term.

While there are many tantalizing patterns, the location of the first 0 does not seem to obey any simple general rule. For example, for $k \geq 1$, let $s_{b,k}$ denote the index of the first 0 in the rumor defined by $z_0 = 1$, and $z_n = bz_{n-1} \bmod (n+k)$. The sequence $\{s_{2,k}\}$ begins

$$1, 2, 5, 10, 3, 18, 7, 24, 23, 22, 13, 4, 19, 18, 9, 6, 15, 374, 13, 12, 11, 370, 369, \ldots.$$

This erratic behavior continues, as evidenced by the few additional values given in the tables below.

| $z_0$ | $b$ | $k$ | first 0 term, $s_{2,k}$ |
|---|---|---|---|
| 1 | 2 | 68 | 324 |
| 1 | 2 | 69 | 12161 |
| 1 | 2 | 70 | 322 |
| 1 | 2 | 71 | 25 |

| $z_0$ | $b$ | $k$ | first 0 term, $s_{3,k}$ |
|---|---|---|---|
| 1 | 3 | 68 | 784 |
| 1 | 3 | 69 | 783 |
| 1 | 3 | 70 | 782 |
| 1 | 3 | 71 | 10 |

## 2 Eventual Behavior: Condition and Conjecture

We show that the initial value $z_0$ of a rumor may be naturally written in terms of a series expansion related to the base-$b$ expansion of a related real number $\zeta_0$. Under the additional condition that $\zeta_0$ is rational we show that the rumor is eventually zero. First we establish the natural expansion of $z_0$ relative to the base-$b$ and basic properties of its coefficients.

**Lemma 1.** *Let $b \geq 2$ and $k \geq 1$. Suppose $z_0$ is any integer and, for $n \geq 1$, let $z_n = bz_{n-1} \bmod (n+k)$. For $n \geq 1$, let $d_n = \left\lfloor \frac{bz_{n-1}}{n+k} \right\rfloor$ so that $z_n = bz_{n-1} - d_n(n+k)$. Then*

*(a)* $\displaystyle\sum_{n \geq 1} \frac{d_n(n+k)}{b^n} = z_0$,

*(b) For $n \geq 2$, $0 \leq d_n \leq b-1$,*

*(c) $d_n < b-1$ infinitely often,*

*(d) $z_n$ is eventually 0 iff $d_n$ is eventually 0.*

*Proof.* (a) Since

$$\frac{z_{n-1}}{b^{n-1}} - \frac{z_n}{b^n} = \frac{d_n(n+k)}{b^n},$$

we have the telescoping sum

$$\sum_{1 \leq n \leq j} \frac{d_n(n+k)}{b^n} = \frac{z_0}{b^0} - \frac{z_j}{b^j}.$$

The last term on the right goes to 0 as $j \to \infty$ since $0 \le z_j < (j+k)$ and $b \ge 2$.

(b) For $n \ge 2$, $0 \le z_{n-1} < n-1+k$, and so

$$0 \le d_n = \left\lfloor \frac{bz_{n-1}}{n+k} \right\rfloor \le \frac{bz_{n-1}}{n+k} \le \frac{b(n+k)-b}{n+k} = b - \frac{b}{n+k} < b.$$

(c) Suppose $d_n = b-1$ for all $n \ge n_0$ for some $n_0$. Then, since $z_{n-1} < n+k-1$,

$$z_n = bz_{n-1} - (b-1)(n+k) < bz_{n-1} - (b-1)z_{n-1} = z_{n-1}.$$

So $\{z_n\}$ would be an eventually strictly decreasing sequence of nonnegative integers.

(d) Suppose $d_n$ is eventually 0. Then for large enough $n$,

$$z_n = bz_{n-1} - 0(n+k) = bz_{n-1},$$

which means $z_n$ grows like a constant times $b^n$. Since $z_n < n+k$, that implies $z_n$ is eventually 0. The converse is obvious. $\square$

We now turn to the main result of this note.

**Theorem 2.** *Let $b \ge 2$ and $k \ge 1$ be integers. Let $\{z_n\}$ be the integer sequence with initial term $z_0$, and for $n \ge 1$, $z_n = bz_{n-1} \bmod (n+k)$, the least nonnegative residue of $bz_{n-1}$ modulo $(n+k)$. Let $d_n = \left\lfloor \frac{bz_{n-1}}{n+k} \right\rfloor$. If $\zeta_0 = \sum_{n\ge1} \frac{d_n}{b^n}$ is rational, then $z_n$ is eventually zero.*

*Proof.* Suppose $\{z_n\}$ and $\{d_n\}$ are constructed as described in the statement of the theorem.

Notice that for any $m \ge 0$, we can generate the tail $\{z_n\}_{n\ge m}$ of the sequence $\{z_n\}$ as a rumor. To do that we set $z'_0 = z_m$, and, for $n \ge 1$, let $z'_n = bz'_{n-1} \bmod (n+k+m)$. Refer to this operation as *restarting* the sequence $\{z_n\}$ *(at $m$)*.

Now suppose $w = \sum_{n\ge1} \frac{d_n}{b^n}$ is rational. Since, by Lemma 1(b), $0 \le d_n \le b-1$ for $n \ge 2$, we can, after restarting $\{z_n\}$ at 2 if necessary, view $\sum_{n\ge1} \frac{d_n}{b^n}$ as the base-$b$ expansion of $w$. Because $w$ is rational, that expansion will be eventually periodic. Again, restarting $\{z_n\}$ at some index if necessary, we may as well assume that expansion of $w$ is purely periodic. Let $p$ be the minimal period length of that expansion.

From Lemma 1(a) we see

$$
\begin{aligned}
z_0 &= \sum_{m\geq 1} \frac{d_m(m+k)}{b^m} = \sum_{j\geq 0}\sum_{1\leq n\leq p} \frac{d_{pj+n}(pj+n+k)}{b^{pj+n}} \\
&= \sum_{1\leq n\leq p}\sum_{j\geq 0} \frac{d_{pj+n}(pj+n+k)}{b^{pj+n}} \\
&= \sum_{1\leq n\leq p} \left( \frac{d_n p}{b^n} \sum_{j\geq 0} \frac{j}{(b^p)^j} + \frac{nd_n}{b^n}\sum_{j\geq 0}\frac{1}{(b^p)^j} + \frac{kd_n}{b^n}\sum_{j\geq 0}\frac{1}{(b^p)^j} \right) \\
&= \sum_{1\leq n\leq p} \left( \frac{d_n p}{b^n} \frac{\frac{1}{b^p}}{(1-\frac{1}{b^p})^2} + \frac{nd_n}{b^n}\frac{1}{1-\frac{1}{b^p}} + \frac{kd_n}{b^n}\frac{1}{1-\frac{1}{b^p}} \right) \\
&= \sum_{1\leq n\leq p} \left( \frac{d_n p}{b^n} \frac{b^p}{(b^p-1)^2} + \frac{nd_n}{b^n}\frac{b^p}{b^p-1} + \frac{kd_n}{b^n}\frac{b^p}{b^p-1} \right) \\
&= \frac{pb^p}{(b^p-1)^2}\sum_{1\leq n\leq p}\frac{d_n}{b^n} + \frac{b^p}{b^p-1}\sum_{1\leq n\leq p}\frac{nd_n}{b^n} + \frac{kb^p}{b^p-1}\sum_{1\leq n\leq p}\frac{d_n}{b^n}.
\end{aligned}
$$

Multiplying both sides of

$$
z_0 = \frac{pb^p}{(b^p-1)^2}\sum_{1\leq n\leq p}\frac{d_n}{b^n} + \frac{b^p}{b^p-1}\sum_{1\leq n\leq p}\frac{nd_n}{b^n} + \frac{kb^p}{b^p-1}\sum_{1\leq n\leq p}\frac{d_n}{b^n}
$$

by $b^p-1$ gives

$$
z_0(b^p-1) = \frac{p\sum_{1\leq n\leq p} d_n b^{p-n}}{b^p-1} + \sum_{1\leq n\leq p} nd_n b^{p-n} + k\sum_{1\leq n\leq p} d_n b^{p-n}
$$

Since the terms $z_0(b^p-1)$, $\sum_{1\leq n\leq p} nd_n b^{p-n}$, and $k\sum_{1\leq n\leq p} d_n b^{p-n}$ are integers, it must be that $\frac{p\sum_{1\leq n\leq p} d_n b^{p-n}}{b^p-1}$ is also an integer, and so $b^p-1$ must divide $p\sum_{1\leq n\leq p} d_n b^{p-n}$. From Lemma 1(c), $d_n < b-1$ for at least one $n$ with $1\leq n\leq p$, and so $\sum_{1\leq n\leq p} d_n b^{p-n} < \sum_{1\leq n\leq p}(b-1)b^{p-n} = b^p-1$. It follows that either (1) $\sum_{1\leq n\leq p} d_n b^{p-n} = 0$, and hence $p=1$, and $d_1=0$ or (2) $\sum_{1\leq n\leq p} d_n b^{p-n} > 0$, and hence $\gcd(p, b^p-1) > 1$ and $p>1$. To complete the proof, we will show case (1) must be correct, and so, by Lemma 1(d), the (original) sequence $\{z_n\}$ is eventually 0. To that end, we assume $\gcd(p, b^p-1) = s > 1$ (and consequently $p>1$) and derive a contradiction.

Let $r$ be the order of $b$ modulo $s$, so that $r\leq\varphi(s)$ where $\varphi$ is Euler's phi function. Since $b^p \equiv 1 \pmod{s}$, it follows that $r$ divides $p$.

Now $s>1$ and $s|p$, and so it follows that $r<p$ since $r\leq\varphi(s)<s\leq p$. Since $r$ is a divisor of $p$, but strictly less than $p$, we have $p=mr$ for some integer $m>1$. Moreover, there is an integer $a$, with $1\leq a<b^r-1$, such that $sa=b^r-1$. By telescoping we see

$$
s\left[a + ab^r + ab^{2r} + \cdots + ab^{(m-1)r}\right] = b^p - 1.
$$

It follows that

$$u = \frac{b^p - 1}{s} = a + ab^r + ab^{2r} + \cdots + ab^{(m-1)r}$$

has periodic base-$b$ digits with a period length $r$.

Now $u = \frac{b^p-1}{s}$ and $\frac{p}{s}$ are relatively prime, while

$$u = \frac{b^p - 1}{s} \quad \text{divides} \quad \frac{p}{s} \sum_{1 \le n \le p} d_n b^{p-n}.$$

So for some $v > 0$, we have that

$$vu = \sum_{1 \le n \le p} d_n b^{p-n} < b^p - 1 = su.$$

That implies $0 < v < s$ and consequently $0 < va < sa = b^r - 1$.

Hence, the integer $va$ has base-$b$ expansion involving only powers of $b$ less than $r$. Consequently

$$\sum_{1 \le n \le p} d_n b^{p-n} = vu = v\left[a + ab^r + ab^{2r} + \cdots + ab^{(m-1)r}\right]$$

has periodic base-$b$ digits with period length $r$. It follows that the base-$b$ expansion $\sum_{n \ge 1} \frac{d_n}{b^n}$ is periodic with a period $r < p$, which violates the choice of $p$ as the minimal period length and the proof is complete. □

Without regard to the rationality condition, our data shows that, in every tested instance, a rumor as in Theorem 2 is eventually zero. Thus, we have been led to postulate the following conjecture.

**Conjecture 3.** *With the notation as in Theorem 2,* $\displaystyle\sum_{n \ge 1} \frac{d_n}{b^n}$ *is rational, and hence every such rumor is eventually* 0.

# 3 Rumors and the Josephus Problem

Rumors are related to the Josephus problem variation as described by Graham, Knuth, and Patashnik [1] and Jensen [2]. Arrange the integers 0 to $n-1$ in a circle. Beginning at 0, delete every second remaining number until only one number remains. That last remaining value is denoted by $J_n$. Taking $J_0 = 0$, it is easy to see that the numbers $J_n$ obey the running modulus recursion

$$\begin{aligned} &J_0 = 0 \\ &J_n = (J_{n-1} + 2) \bmod n \ \text{ for } n \ge 1. \end{aligned}$$

Generalizing that rumor leads to consideration of rumors of the form

$$\begin{aligned} x_0 &= \text{ some nonnegative integer} \\ x_n &= (bx_{n-1} + c) \bmod n \ \text{ for } n \geq 1 \end{aligned}$$

where $b, c$ are positive integers.

Such generalized Josephus rumors exhibit a number of unexpected behaviors, most of which are supported presently by empirical evidence only. The most remarkable observation is that, for certain pairs $b, c$, all such sequences appear to synchronize in the sense described in the following conjecture.

**Conjecture 4.** *For the sequence $\{x_n\}$ given above, if $c = j(b-1)+1$ for some $j \geq 2$, then eventually $x_n = n - j + 1$.*

So when $b$ and $c$ are related as in the conjecture, it appears the sequence eventually becomes a sequence of consecutive integers. In this case we say the rumor sequence eventually *walks.* In particular then, for any initial term, $x_0$, as long as $c = j(b-1)+1$, any two such sequences will eventually synchronize in the sense that eventually corresponding terms will differ by a fixed constant, and for a fixed $j \geq 2$ for any two $b \geq 2$ and any $x_0$, the sequences are eventually identical.

On the other hand, if $b$ and $c$ are not related as described, it appears the sequence never walks, and in fact the terms show no easily discernible pattern at all.

When a sequence does walk, the point at which the walk begins is surprisingly variable, which may partly explain the difficulty establishing the conjecture. For example, the rumor

$$\begin{aligned} x_0 &= 0 \\ x_n &= (2x_{n-1} + 291) \bmod n \ \text{ for } n \geq 1 \end{aligned}$$

begins walking at term number 570.

On the other hand, the rumor

$$\begin{aligned} x_0 &= 0 \\ x_n &= (2x_{n-1} + 292) \bmod n \ \text{ for } n \geq 1 \end{aligned}$$

begins walking at term number 69921608358, while

$$\begin{aligned} x_0 &= 0 \\ x_n &= (2x_{n-1} + 293) \bmod n \ \text{ for } n \geq 1 \end{aligned}$$

begins walking at term number 13942.

The rumor given by $x_0 = 0$ and $x_n = (2x_{n-1} + c) \bmod n$ for $n \geq 1$ will eventually walk for all $c \geq 3$ according to Conjecture 4. For $3 \leq c \leq 23$, the walks begin at

$$1, 2, 8, 32, 6, 8, 38, 38, 48, 16, 384, 174, 24, 34, 16, 120, 26, 26, 680, 680, 5600, \ldots$$

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## References

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[2] B. Jensen, The Running Modulus, Master's Independent Study, University of North Dakota, 2000.

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