Journal of Integer Sequences, Vol. 13 (2010),
Article 10.4.4

A Note on Nested Sums

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Abstract

We consider several nested sums, and show how binomial coefficients, Stirling numbers of the second kind and Gaussian binomial coefficients can be written as nested sums. We use this to find the rate of growth for diagonals of Stirling numbers of the second kind, as well as another proof of a known identity for Gaussian binomial coefficients.

1 Introduction

In this note we are interested in looking at nested sums and finding simple closed form expressions for them. Our motivation for looking at these sums was the following identity.

$$\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2}1=\binom{n+p}{n} \tag{1}$$

One way to see this is to note that on the left hand side we count the number of occurrences of $n \geq k_p \geq k_{p-1} \geq \cdots \geq k_1 \geq 0$. There is a one-to-one correspondence between these and

[1]Supported by an NSF postdoctoral fellowship.

nonnegative $(p+1)$-tuples which sum to n, i.e., $(n-k_p, k_p-k_{p-1}, \ldots, k_1-0)$. The number of such tuples is the same as the number of ways to put n identical balls into $p+1$ distinct bins. This can be counted by arranging the n balls and p dividers between bins in a row which can be done in $\binom{n+p}{n}$, the right hand side.

More generally one can ask for closed formed expressions for sums of the form

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} f(k_1, \ldots, k_p).$$

We will show in Section 2 that when $f(k_1, \ldots, k_p) = \binom{k_i}{j}$ the sum has a simple closed expression, and in Section 3 we consider functions of the form $f(k_1, \ldots, k_p) = g(k_1) \cdots g(k_p)$ and show how to get Stirling numbers of the second kind, the central factorial numbers and Gaussian binomial coefficients as nested sums.

2 Sums involving binomial coefficients

When the function being summed is a binomial coefficient then the sum has a simple closed form.

Lemma 1. *For $1 \le i \le p$ and $j \ge 0$ we have*

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} \binom{k_i}{j} = \binom{j+i-1}{i-1}\binom{n+p}{p+j}.$$

For the proof of Lemma 1 we will make use of two basic identities for binomial coefficients (see [1]). The first is a way to rewrite the product of binomial coefficients,

$$\binom{q}{b}\binom{q+d}{d} = \binom{b+d}{d}\binom{q+d}{b+d}, \tag{2}$$

and the second is a way to sum on the upper index

$$\sum_{k=0}^{m} \binom{k+n}{d} = \binom{m+n+1}{d+1}, \tag{3}$$

where more generally we can start our sum at any $k=a$ for $a \le d-n$.

Proof of Lemma 1. We have

$$\begin{aligned}
\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2}\binom{k_i}{j} &= \sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_i=0}^{k_{i+1}}\binom{k_i}{j}\sum_{k_{i-1}=0}^{k_i}\cdots\sum_{k_1=0}^{k_2}1 \\
&= \sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_i=0}^{k_{i+1}}\binom{k_i}{j}\binom{k_i+i-1}{i-1} \\
&= \sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_i=0}^{k_{i+1}}\binom{j+i-1}{i-1}\binom{k_i+i-1}{i-1+j} \\
&= \binom{j+i-1}{i-1}\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_i=0}^{k_{i+1}}\binom{k_i+i-1}{i-1+j} \\
&= \binom{j+i-1}{i-1}\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_{i+1}=0}^{k_{i+2}}\binom{k_{i+1}+i}{i+j} \\
&= \cdots \\
&= \binom{j+i-1}{i-1}\binom{n+p}{p+j},
\end{aligned}$$

where we used (1) in going from the first to the second line, (2) in going from the second to the third line, and repeated application of (3) from the fifth line down. $\square$

Theorem 2. *For $j \geq 0$ we have*

$$\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2}\left(\binom{k_1}{j}+\binom{k_2}{j}+\cdots+\binom{k_p}{j}\right) = \frac{p}{j+1}\binom{n}{j}\binom{n+p}{n}, \tag{4}$$

Proof. We have

$$\begin{aligned}
&\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2}\left(\binom{k_1}{j}+\binom{k_2}{j}+\cdots+\binom{k_p}{j}\right) = \sum_{q=1}^{p}\left(\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2}\binom{k_q}{j}\right) \\
&= \sum_{q=1}^{p}\binom{j+q-1}{q-1}\binom{n+p}{p+j} = \binom{n+p}{p+j}\sum_{q=1}^{p}\binom{j+q-1}{j} = \binom{n+p}{p+j}\binom{j+p}{j+1} \\
&= \frac{p}{j+1}\binom{n}{j}\binom{n+p}{n}. \quad \square
\end{aligned}$$

Putting $j=1$ into (4) we have

$$\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2}(k_1+k_2+\cdots+k_p) = \frac{pn}{2}\binom{n+p}{n}.$$

When $p = 2$ this generates the sequence A027480 and when $p = 3$ this generates the sequence A033487 in Sloane's *Encyclopedia* [3].

Using that $k_i^2 = 2\binom{k_i}{2} + \binom{k_i}{1}$ and (4) we have

$$\sum_{k_p=0}^{n} \cdots \sum_{k_1=0}^{k_2} (k_1^2 + \cdots + k_p^2) = \left(2\frac{p}{3}\binom{n}{2} + \frac{p}{2}\binom{n}{1}\right)\binom{n+p}{n} = \frac{p(2n^2+n)}{6}\binom{n+p}{n}.$$

Since any polynomial can be written as a sum of binomial coefficients we can use Lemma 1 to give a simple expression for functions of the form

$$f(k_1, k_2, \ldots, k_p) = f_1(k_1) + f_2(k_2) + \cdots + f_p(k_p),$$

where each f_i is a polynomial.

3 Sums of products

Looking at the proof of Lemma 1 we see that the two main ingredients were the identities which allowed us to write a sum of binomial coefficients as a single binomial coefficient and rewrite a product of two binomial coefficients so that we could pull one term out. When our function is no longer a single binomial coefficient then we might no longer be able to rewrite the product so simply, though the general technique of summing, rewriting and repeating still works. In this section we consider functions of the form $f(k_1, \ldots, k_p) = g(k_1) \cdots g(k_p)$. We begin with the function $g(k) = k$.

Theorem 3. *Let $a(p, k)$ be defined by $a(1, 1) = 1$, $a(1, i) = 0$ for $i \neq 1$ and*

$$a(p+1, k) = k\big(a(p, k-2) - a(p, k-1)\big) \qquad \text{for } p \geq 1.$$

Then

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} k_1 k_2 \cdots k_p = \sum_{k} a(p, k)\binom{n+k}{k+1}.$$

The first few rows for a table of $a(p, k)$ are given below

$a(p,k)$	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=11
p=1	1										
p=2		−2	3								
p=3			6	−20	15						
p=4				−24	130	−210	105				
p=5					120	−924	2380	−2520	945		
p=6						−720	7308	−26432	44100	−34650	10395

Using this table we can now see, for example, that

$$\sum_{k_4=0}^{n} \sum_{k_3=0}^{k_4} \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} k_1 k_2 k_3 k_4 = -24\binom{n+4}{5} + 130\binom{n+5}{6} - 210\binom{n+6}{7} + 105\binom{n+7}{8}.$$

Looking at the table we see that the row sums are 1, this is easily verified by Theorem 3 by putting $n = 1$ into both sides. Similarly it is easy to see that the terms on the diagonals are the signed factorials and the rightmost term in the pth row is the product of the first p odd numbers. The values in the rows are closely related to A111999 in Sloane's *Encyclopedia* [3] which are defined with a similar recurrence (though in that table the rows are written in reverse order and shifted to the left as well as differing in the signs).

Proof of Theorem 3. We proceed by induction on p. For $p = 1$ we have by a simple application of (3) that

$$\sum_{k_1=0}^{n} k_1 = \binom{n+1}{2} = \sum_k a(1,k)\binom{n+k}{k+1}.$$

So now we assume it is true for p and consider the case $p+1$. By our induction hypothesis we have

$$\sum_{k_{p+1}=0}^{n}\sum_{k_p=0}^{k_{p+1}}\cdots\sum_{k_1=0}^{k_2} k_1k_2\cdots k_pk_{p+1} = \sum_{k_{p+1}=0}^{n} k_{p+1}\sum_k a(p,k)\binom{k_{p+1}+k}{k+1}. \tag{5}$$

Since

$$\begin{aligned} k_{p+1}\binom{k_{p+1}+k}{k+1} &= (k_{p+1}+k+1)\binom{k_{p+1}+k}{k+1} - (k+1)\binom{k_{p+1}+k}{k+1} \\ &= (k+2)\binom{k_{p+1}+k+1}{k+2} - (k+1)\binom{k_{p+1}+k}{k+1}, \end{aligned}$$

we can use this to rewrite (5) to get

$$\begin{aligned} &\sum_{k_{p+1}=0}^{n}\sum_k a(p,k)\left((k+2)\binom{k_{p+1}+k+1}{k+2} - (k+1)\binom{k_{p+1}+k}{k+1}\right) \\ &\qquad = \sum_k a(p,k)\left((k+2)\binom{n+k+2}{k+3} - (k+1)\binom{n+k+1}{k+2}\right) \\ &\qquad\qquad = \sum_\ell \underbrace{\big(\ell a(p,\ell-2) - \ell a(p,\ell-1)\big)}_{=a(p+1,\ell)}\binom{n+\ell}{\ell+1}. \qquad \square \end{aligned}$$

For fixed values of p this summand produces diagonals for Stirling numbers of the second kind. For instance $p = 1, 2, 3, 4, 5$ corresponds (respectively) to the sequences A000217, A001296, A001297, A001298 and A112494 in Sloane's *Encyclopedia* [3] which are the second, third, fourth, fifth and sixth diagonals in the table of Stirling numbers of the second kind. In general we have for $p \geq 1$

$$\sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2} k_1k_2\cdots k_p = \left\{{n+p \atop n}\right\}. \tag{6}$$

To see this we use the recurrence relationship for the Stirling numbers of the second kind (see [1]) to get

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} = k\left\{ {n-1 \atop k} \right\} + (k-1)\left\{ {n-2 \atop k-1} \right\} + \left\{ {n-2 \atop k-2} \right\}$$
$$= \cdots = \sum_{j=0}^{k} j\left\{ {n-k-1+j \atop j} \right\}.$$

Now for $p = 1$ we have

$$\sum_{k_1=0}^{n} k_1 = \binom{n+1}{2} = \left\{ {n+1 \atop n} \right\},$$

and by induction we have

$$\sum_{k_p=0}^{n} \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} k_1 k_2 \cdots k_p = \sum_{k_p=0}^{n} k_p \left\{ {k_p + p - 1 \atop k_p} \right\}$$
$$= \sum_{k_p=0}^{n} k_p \left\{ {n+p-n-1+k_p \atop k_p} \right\} = \left\{ {n+p \atop n} \right\}.$$

We can also prove this directly. Since $\left\{ {n+p \atop n} \right\}$ counts the number of ways to partition $1, 2, \ldots, n+p$ into n sets, for each such partition there will be exactly p elements which are *not* the first element in their partition. Fix the elements which are not first as

$$1 \le a_1 < a_2 < \cdots < a_p \le n + p.$$

The remaining n elements are initially in sets by themselves and we distribute the a_i among these sets. Each a_i can go into $a_i - i$ sets, i.e., the number of elements which are less then a_i but not an a_j. Therefore the number of partitions with the a_i not being the first element is

$$\prod_{i=1}^{p} (a_i - i).$$

So now we sum over all possibilities to get

$$\left\{ {n+p \atop n} \right\} = \sum_{1 \le a_1 < a_2 < \cdots < a_p \le n+p} \left(\prod_{i=1}^{p} (a_i - i) \right).$$

Finally if we let $k_i = a_i - i$ then this becomes

$$\left\{ {n+p \atop n} \right\} = \sum_{0 \le k_1 \le k_2 \le \cdots \le k_p \le n} k_1 k_2 \cdots k_p.$$

We can now use Theorem 3 to express diagonals for Stirling numbers of the second kind as sums of binomial coefficients. By looking at the leading coefficient $a(p, 2p-1)$ this can be used to show that for p fixed

$$\left\{ {n+p \atop n} \right\} = \prod_{i=1}^{p}(2i-1)\binom{n+2p}{2p} + O(n^{2p-1})$$
$$= \frac{\prod_{i=1}^{p}(2i-1)}{(2p)!}n^{2p} + O(n^{2p-1}) = \frac{1}{p!2^p}n^{2p} + O(n^{2p-1}).$$

3.1 Other recurrences

The proof using the recurrence to relate the Stirling numbers of the second kind to nested sums generalizes to give the following result.

Theorem 4. *Let $G(n,k)$ satisfy $G(n,n) = 1$, $G(n,-k) = 0$ for all $k \geq 1$ and the recurrence*

$$G(n,k) = G(n-1,k-1) + g(k)G(n-1,k). \tag{7}$$

Then for $p \geq 1$

$$G(n+p,n) = \sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2} g(k_1)g(k_2)\cdots g(k_p).$$

The case $g(k) = 1$ gives the binomial coefficients (1), and we have seen that $g(k) = k$ gives the Stirling numbers of the second kind. It is easy to show that $g(k) = k^2$ gives the central factorial numbers (A036969 in Sloane's *Encyclopedia* [3]).

Another important recurrence of the type given in (7) is for the Gaussian binomial coefficients $\left[{n \atop k}\right]_q$ (see [2]). These are polynomials in q which satisfy $\left[{n \atop n}\right]_q = 1$ and the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

This is the case $g(k) = q^k$. In particular, we have

$$\begin{bmatrix} n+p \\ n \end{bmatrix}_q = \sum_{k_p=0}^{n}\sum_{k_{p-1}=0}^{k_p}\cdots\sum_{k_1=0}^{k_2} q^{k_1+k_2+\cdots+k_p}.$$

From this we see that the coefficient of q^m in $\left[{n+p \atop n}\right]_q$ is the number of solutions of

$$k_1 + k_2 + \cdots + k_p = m \quad \text{with} \quad 0 \leq k_1 \leq k_2 \leq \cdots \leq k_p \leq n.$$

If we let $k_i' = k_i + i$ then we have

$$m = k_1+k_2+\cdots+k_p = k_1'+k_2'+\cdots+k_p' - \frac{p(p+1)}{2} \quad \text{with} \quad 1 \leq k_1' < k_2' < \cdots < k_p' \leq n+p.$$

note that the k_i' form a p element subset of $\{1, 2, \ldots, n+p\}$ and that we now sum over all possible p element subsets. From this we can recover the following known identity [2] for Gaussian binomial coefficients

$$\begin{bmatrix} n+p \\ n \end{bmatrix}_q = \sum_{\substack{S \subset \{1,2,\ldots,n+p\} \\ |S|=p}} q^{\left(\sum_{i \in S} i\right) - \frac{p(p+1)}{2}}.$$

References

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[2] V. Kac and P. Cheung, *Quantum Calculus*, Springer-Verlag, 2002.

[3] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://www.research.att.com/~njas/sequences/.

2010 *Mathematics Subject Classification*: Primary 05A10; Secondary 11B73, 11B65.
Keywords: nested sums; Stirling numbers; q-nomial coefficients.

(Concerned with sequences A000217, A001296, A001297, A001298, A027480, A033487, A036969, A111999, and A112494.)

Received October 8 2009; revised version received April 5 2010. Published in *Journal of Integer Sequences*, April 5 2010.
