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Exponential Riordan Arrays and Permutation Enumeration

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Abstract

We show that the generating function of the symmetric group with respect to five particular statistics gives rise to an exponential Riordan array, whose inverse is the coefficient array of the associated orthogonal polynomials. This also provides us with an LDU factorization of the Hankel matrix of the associated moments.

1 Introduction

In this note, we shall re-interpret some of the results of Zeng [30] in terms of exponential Riordan arrays. For this, we let $\mathfrak{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$. Given a permutation $\sigma \in \mathfrak{S}_n$, each value $x = \sigma(i)$, $1 \leq i \leq n$, can be classified according to one of the five following cases:

1. a peak ("pic de cycle"), if $\sigma^{-1}(x) < x > \sigma(x)$;
2. a valley ("creux de cycle"), if $\sigma^{-1} > x < \sigma(x)$;
3. a double rise ("double montée de cycle"), if $\sigma^{-1}(x) < x < \sigma(x)$;
4. a double descent ("double descente de cycle"), if $\sigma^{-1}(x) > x > \sigma(x)$;
5. a fixed point ("point fixe"), if $\sigma(i) = i$.

Using the notation of Zeng [30], we denote the number of peaks, valleys, double rises, double descents, and fixed points of σ respectively by pic σ, cc σ, dm σ, dd σ, and fix σ. We shall also denote by cyc σ the number of cycles of σ. We set

$$\mu_n = \sum_{\sigma \in \mathbf{S}_n} u_1^{\mathrm{cc}\,\sigma} u_2^{\mathrm{pic}\,\sigma} u_3^{\mathrm{dm}\,\sigma} u_4^{\mathrm{dd}\,\sigma} \alpha^{\mathrm{fix}\,\sigma} \beta^{\mathrm{cyc}\,\sigma}.$$

We then have the following theorem.

Theorem 1. *We let α_1 and α_2 be such that $\alpha_1\alpha_2 = u_1u_2$ and $\alpha_1 + \alpha_2 = u_3 + u_4$. Then the exponential Riordan array*

$$\mathbf{L} = \left[e^{\alpha\beta x} \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 e^{\alpha_2 x} - \alpha_2 e^{\alpha_1 x}} \right)^{\beta}, \frac{e^{\alpha_1 x} - e^{\alpha_2 x}}{\alpha_1 e^{\alpha_2 x} - \alpha_2 e^{\alpha_1 x}} \right]$$

is the inverse of the coefficient array for the family of orthogonal polynomials for which μ_n are the moments. The elements of the first column of $\mathbf{L}$ are given by μ_n. The Hankel matrix $\mathbf{H} = (\mu_{i+j})_{i,j \geq 0}$ has LDU factorization

$$\mathbf{H} = \mathbf{L}\mathbf{D}\mathbf{L}^t.$$

While partly expository in nature, this note assumes a certain familiarity with integer sequences, generating functions, orthogonal polynomials [5, 11, 26], Riordan arrays [21, 25], production matrices [10, 18], and the integer Hankel transform [1, 7, 16]. Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [23, 24]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.

The plan of the paper is as follows:

1. This Introduction
2. Integer sequences, exponential Riordan arrays and orthogonal polynomials
3. Proof of the theorem
4. A matrix product

2 Integer sequences, exponential Riordan arrays and orthogonal polynomials

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f_o(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence, while $f_e(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is called the *exponential generating function* or e.g.f. of the sequence. a_n is thus the coefficient of x^n in $f_o(x)$. We denote this by $a_n = [x^n] f_o(x)$. Similarly, $a_n = n![x^n] f_e(x)$. For instance, $F_n = [x^n] \frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $n! = n![x^n] \frac{1}{1-x}$, which says that $\frac{1}{1-x}$ is the e.g.f. of $n!$ A000142. For a power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ with $f(0)=0$ and $f'(0)\neq 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)=f^{[-1]}(x)$ such that $f(\bar{f}(x))=x$. We sometimes write $\bar{f}=\mathrm{Rev} f$.

The *exponential Riordan group* [2, 10, 8], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x)=g_0+g_1x+g_2x^2+\cdots$ and $f(x)=f_1x+f_2x^2+\cdots$ where $g_0\neq 0$ and $f_1\neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f,g is denoted by $[g,f]$. It is *monic* if $g_0=1$. The group law is given by

$$[g,f]*[h,l]=[g(h\circ f), l\circ f].$$

The identity for this law is $I=[1,x]$ and the inverse of $[g,f]$ is $[g,f]^{-1}=[1/(g\circ\bar{f}),\bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group. If $\mathbf{M}$ is the matrix $[g,f]$, and $\mathbf{u}=(u_n)_{n\geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}(x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g,f]$ are given by $g(x)e^{f(x)}$ since the sequence $1,1,1,\ldots$ has exponential generating function e^x.

Example 2. The *binomial matrix* is the matrix with general term $\binom{n}{k}$. It is realized by Pascal's triangle. As an exponential Riordan array, it is given by $[e^x,x]$. We further have

$$([e^x,x])^m=[e^{mx},x].$$

As an example of the calculation of an inverse, we have the following proposition.

Proposition 3.

$$\mathbf{L}^{-1}=\left[\left(\frac{1+\alpha_2x}{1+\alpha_1x}\right)^{\frac{\alpha\beta}{\alpha_1-\alpha_2}}\left(\frac{(1+\alpha_2x)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}}{(1+\alpha_1x)^{\frac{\alpha_2}{\alpha_1-\alpha_2}}}\right)^{-\beta}, \frac{1}{\alpha_2-\alpha_1}\ln\left(\frac{1+\alpha_2x}{1+\alpha_1x}\right)\right].$$

Proof. This follows since with

$$f(x)=\frac{e^{\alpha_1x}-e^{\alpha_2x}}{\alpha_1e^{\alpha_2x}-\alpha_2e^{\alpha_1x}}$$

we have

$$\bar{f}(x)=\frac{1}{\alpha_2-\alpha_1}\ln\left(\frac{1+\alpha_2x}{1+\alpha_1x}\right).$$

□

We note that we can then write $\mathbf{L}^{-1}$ as

$$\mathbf{L}^{-1}=\left[\frac{(1+\alpha_2x)^{\frac{\beta(\alpha-\alpha_1)}{\alpha_1-\alpha_2}}}{(1+\alpha_1x)^{\frac{\beta(\alpha-\alpha_2)}{\alpha_1-\alpha_2}}}, \frac{1}{\alpha_2-\alpha_1}\ln\left(\frac{1+\alpha_2x}{1+\alpha_1x}\right)\right].$$

An important concept for the sequel is that of production matrix. The concept of a *production matrix* [9, 10] is a general one, but for this work we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting $\mathbf{r}_0$ be the row vector

$$\mathbf{r}_0 = (1, 0, 0, 0, \ldots),$$

we define $\mathbf{r}_i = \mathbf{r}_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P. If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \begin{pmatrix} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{pmatrix}$$

and

$$DA_P = A_P P$$

where $D = (\delta_{i+1,j})_{i,j\geq 0}$ (where δ is the usual Kronecker symbol). In [18, 22] P is called the Stieltjes matrix associated to A_P. In [10], we find the following result concerning matrices that are production matrices for exponential Riordan arrays.

Proposition 4. *Let $A = (a_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \cdots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \cdots \tag{1}$$

be two formal power series such that

$$\begin{aligned} r(f(x)) &= f'(x) & (2)\\ c(f(x)) &= \frac{g'(x)}{g(x)}. & (3) \end{aligned}$$

Then

$$\begin{aligned} (i) \qquad a_{n+1,0} &= \sum_i i! c_i a_{n,i} & (4)\\ (ii) \qquad a_{n+1,k} &= r_0 a_{n,k-1} + \frac{1}{k!}\sum_{i\geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} & (5) \end{aligned}$$

or, defining $c_{-1} = 0$,

$$a_{n+1,k} = \frac{1}{k!}\sum_{i\geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{6}$$

Conversely, starting from the sequences defined by (1), the infinite array $(a_{n,k})_{n,k\geq 0}$ defined by (6) is an exponential Riordan array.

A consequence of this proposition is that $P=(p_{i,j})_{i,j\ge 0}$ where

$$p_{i,j}=\frac{i!}{j!}(c_{i-j}+jr_{r-j+1}) \qquad (c_{-1}=0).$$

Furthermore, the bivariate exponential generating function

$$\phi_P(t,z)=\sum_{n,k} p_{n,k}t^k\frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,z)=e^{tz}(c(z)+tr(z)).$$

Note in particular that we have

$$r(x)=f'(\bar{f}(x))$$

and

$$c(x)=\frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

Example 5. We consider the exponential Riordan array $L=[\frac{1}{1-x},x]$, A094587. This array has elements

$$L=\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 6 & 3 & 1 & 0 & 0 & \cdots \\ 24 & 24 & 12 & 4 & 1 & 0 & \cdots \\ 120 & 120 & 60 & 20 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and general term $[k\le n]\frac{n!}{k!}$ with inverse

$$L^{-1}=\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the array $[1-x,x]$. In particular, we note that the row sums of the inverse, which begin $1,0,-1,-2,-3,\ldots$ (that is, $1-n$), have e.g.f. $(1-x)\exp(x)$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1-x)$ (which is the sequence starting $1,-1,0,0,0,\ldots$). In order to calculate the production matrix $\mathbf{P}$ of $L=[\frac{1}{1-x},x]$ we note that $f(x)=x$, and hence we have $f'(x)=1$ so $f'(\bar{f}(x))=1$. Also $g(x)=\frac{1}{1-x}$ leads to $g'(x)=\frac{1}{(1-x)^2}$, and so, since $\bar{f}(x)=x$, we get

$$\frac{g'(\bar{f}(x))}{g(\bar{f}(x))}=\frac{1}{1-x}.$$

Thus the generating function for $\mathbf{P}$ is

$$e^{tz}\left(\frac{1}{1-z}+t\right).$$

Thus $\mathbf{P}$ is the matrix $[\frac{1}{1-x}, x]$ with its first row removed.

Example 6. We consider the exponential Riordan array $L=[1, \frac{x}{1-x}]$. The general term of this matrix may be calculated as follows:

$$\begin{aligned}
T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\
&= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\
&= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\
&= \frac{n!}{k!}\binom{n-1}{n-k},
\end{aligned}$$

with

$$L=\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 1 & 0 & 0 & 0 & \cdots \\
0 & 6 & 6 & 1 & 0 & 0 & \cdots \\
0 & 24 & 36 & 12 & 1 & 0 & \cdots \\
0 & 120 & 240 & 120 & 20 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1,..,n\}$ into any number of lists, where a list means an ordered subset'. Its general term is equal to $(n-1)!L_{n-1}(1,-1)$. The inverse of $\left[1, \frac{x}{1-x}\right]$ is the exponential Riordan array $L^{-1}=\left[1, \frac{x}{1+x}\right]$, A111596. The row sums of this sequence have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884. To calculate the production matrix of $L=\left[1, \frac{x}{1-x}\right]$ we note that $g'(x)=0$, while $\bar{f}(x)=\frac{x}{1+x}$ with $f'(x)=\frac{1}{(1+x)^2}$. Thus

$$f'(\bar{f}(x))=(1+x)^2,$$

and so the generating function of the production matrix is given by

$$e^{tz}t(1+z)^2.$$

Thus the production matrix of the inverse begins

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 4 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 6 & 6 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 12 & 8 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 20 & 10 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Example 7. The exponential Riordan array $A = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, or

$$A = \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

has general term

$$T_{n,k} = \frac{n!}{k!}\binom{n}{k}.$$

Its inverse is $A^{-1} = \left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ with general term $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. This is A021009, the triangle of coefficients of the Laguerre polynomials $L_n(x)$. The production matrix of A is given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 5 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9 & 7 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16 & 9 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Example 8. The exponential Riordan array $L = \left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, or

$$L = \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 8 & 6 & 1 & 0 & 0 & \cdots \\ 1 & 24 & 29 & 10 & 1 & 0 & \cdots \\ 1 & 89 & 145 & 75 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

is the coefficient array for the polynomials

$$ {}_2F_0(-n, x; -1) $$

which are an unsigned version of the Charlier polynomials (of order 0) [11, 20, 26]. This is A094816. We have

$$ L = [e^x, x] * \left[1, \ln\left(\frac{1}{1-x}\right)\right], $$

or the product of the binomial array $\mathbf{B}$ and the array of (unsigned) Stirling numbers of the first kind. The production matrix of the inverse of this matrix is given by

$$ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3 & -4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4 & -5 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5 & -6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} $$

which indicates the orthogonal nature of these polynomials. We can prove this form as follows. We have

$$ \left[e^x, \ln\left(\frac{1}{1-x}\right)\right]^{-1} = \left[e^{-(1-e^{-x})}, 1-e^{-x}\right]. $$

Hence $g(x) = e^{-(1-e^{-x})}$ and $f(x) = 1 - e^{-x}$. We are thus led to the equations

$$ \begin{aligned} r(1-e^{-x}) &= e^{-x}, \\ c(1-e^{-x}) &= -e^{-x}, \end{aligned} $$

with solutions $r(x) = 1 - x$, $c(x) = x - 1$. Thus the bi-variate generating function for the production matrix of the inverse array is

$$ e^{tz}(z - 1 + t(1-z)), $$

which is what is required.

According to Proposition **4**, for a Riordan array to have a tri-diagonal production array P, it is necessary and sufficient that P be of the form

$$ \begin{pmatrix} c_0 & r_0 & 0 & 0 & 0 & 0 & \cdots \\ c_1 & c_0 + r_1 & r_0 & 0 & 0 & 0 & \cdots \\ 0 & 2(c_1 + r_2) & c_0 + 2r_1 & r_0 & 0 & 0 & \cdots \\ 0 & 0 & 3(c_1 + 2r_2) & r_0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4(c_1 + 3r_2) & c_0 + 4r_1 & r_0 & \cdots \\ 0 & 0 & 0 & 0 & 5(c_1 + 4r_2) & c_0 + 5r_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. $$

We recognize in this the form of Meixner's solution [19, 12] to the question of which families of Sheffer polynomials [15] are orthogonal. Thus P corresponds to the family of orthogonal polynomials $(S_n(x))_{n\geq 0}$ that satisfy

$$S_{n+1}(x) = (x - (c_0 + nr_1))S_n(x) - n(c_1 + nr_2)S_{n-1}(x).$$

Of importance to this study are the following results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [14].

Theorem 9. *[14]* (Cf. [27, Théorème 9, p. I-4] or [28, Theorem 50.1]). *Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$. Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n(x), \quad \textit{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 10. *[14]* (Cf. [27, Prop. 1, (7), p. V-5] or [28, Theorem 51.1]). *Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional L. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n(x), \quad \textit{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = L(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

The Hankel transform of μ_n, which is the sequence with general term $h_n = |\mu_{i+j}|_{0\leq i,j\leq n}$, is then given by

$$h_n = \mu_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n.$$

3 Proof of Theorem 1

Proof. We first note that since

$$g(x) = e^{\alpha\beta x}\left(\frac{\alpha_1 - \alpha_2}{\alpha_1 e^{\alpha_2 x} - \alpha_2 e^{\alpha_1 x}}\right)^{\beta},$$

by [30, Theorem 1], the first column of the Riordan array is indeed $\{\mu_n\}_{n\geq 0}$. We now calculate the production matrix $\mathbf{P_L}$ of $\mathbf{L}$. We have

$$r(x) = f'(\bar{f}(x)) = (1+\alpha_2 x)(1+\alpha_1 x)$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \beta(\alpha + \alpha_1\alpha_2 x).$$

Thus the bivariate generating function for the production matrix $\mathbf{P_L}$ of $\mathbf{L}$ is given by

$$e^{tz}(\beta(\alpha+\alpha_1\alpha_2 x) + t(1+\alpha_2 x)(1+\alpha_1 x)).$$

Now this is equal to

$$e^{tz}(\alpha\beta + \beta u_1 u_2 x + t(1 + (u_3+u_4)x + u_1 u_2 x^2)).$$

But this implies that $\mathbf{P_L}$ is precisely the Jacobi tri-diagonal matrix corresponding to the continued J-fraction

$$\cfrac{1}{1-\alpha\beta x - \cfrac{\beta u_1 u_2 x^2}{1-(\alpha\beta+u_3+u_4)x - \cfrac{2(\beta+1)u_1u_2x^2}{1-(\alpha\beta+2(u_3+u_4))x - \cfrac{3(\beta+2)u_1u_2x^2}{1-\cdots}}}}$$

which by [30] is equal to the generating function

$$\sum_{k=0}^{\infty} \mu_k x^k.$$

The matrix $\mathbf{P_L}$ begins:

$$\left(\begin{array}{ccccccc}
\alpha\beta & 1 & 0 & 0 & 0 & 0 & \cdots \\
\beta u_1 u_2 & \alpha\beta+u_3+u_4 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2(\beta+1)u_1u_2 & \alpha\beta+2(u_3+u_4) & 0 & 0 & 0 & \cdots \\
0 & 0 & 3(\beta+2)u_1u_2 & \alpha\beta+3(u_3+u_4) & 0 & 0 & \cdots \\
0 & 0 & 0 & 4(\beta+3)u_1u_2 & \alpha\beta+4(u_3+u_4) & 0 & \cdots \\
0 & 0 & 0 & 0 & 5(\beta+4)u_1u_2 & \alpha\beta+5(u_3+u_4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

This implies that $\mathbf{L}^{-1}$ is indeed the coefficient array of the set of orthogonal polynomials which correspond to the tri-diagonal array $\mathbf{P_L}$. The results of [17, 18] now imply that if $\mathbf{H} = (\mu_{i+j})_{i,j\geq 0}$ then

$$\mathbf{H} = \mathbf{L}\mathbf{D}\mathbf{L}^t.$$

$\square$

As pointed out by an anonymous referee, this result could also have been arrived at using the theory of orthogonal Sheffer polynomials, as the μ_n are seen to be the moment sequence of the orthogonal polynomials defined by

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

with $P_{-1}(x) = 0$ and $P_1(x) = 1$, where $b_n = \alpha\beta + n(u_3 + u_4)$ and $\lambda_n = n(\beta + n)u_1u_2$. Thus $(P_n(x))_{n\geq 0}$ is a sequence of orthogonal Sheffer polynomials [12]. Note that the interrelationship between Riordan arrays and Sheffer polynomials is comprehensively studied in [15].

The elements of the diagonal matrix $\mathbf{D}$ are the successive products of the elements of the sub-diagonal of $\mathbf{P_L}$:

$$\beta u_1u_2, \quad 2\beta(\beta+1)u_1^2u_2^2, \quad 6\beta(\beta+1)(\beta+2)u_1^3u_2^3, \ldots$$

Corollary 11. *The Hankel transform of μ_n is given by*

$$h_n = (u_1u_2)^{\binom{n+1}{2}} \prod_{k=0}^{n} k!(\beta+k)^{n-k}.$$

We notice in particular that this is independent of α, u_3 and u_4.
We have the following factorization.

$$\begin{aligned}
\mathbf{L} &= \left[e^{\alpha\beta x}\left(\frac{\alpha_1-\alpha_2}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right)^{\beta}, \frac{e^{\alpha_1 x}-e^{\alpha_2 x}}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right] \\
&= [e^{\alpha\beta x}, x]\left[\left(\frac{\alpha_1-\alpha_2}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right)^{\beta}, \frac{e^{\alpha_1 x}-e^{\alpha_2 x}}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right] \\
&= [e^x, x]^{\alpha\beta}\mathbf{L}_0,
\end{aligned}$$

where the matrix $\mathbf{L_0}$ has production matrix generated by

$$e^{tz}(\beta u_1u_2x + t(1 + (u_3+u_4)x + u_1u_2x^2)).$$

This matrix begins

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\beta u_1u_2 & u_3+u_4 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2(\beta+1)u_1u_2 & 2(u_3+u_4) & 0 & 0 & 0 & \cdots \\
0 & 0 & 3(\beta+2)u_1u_2 & 3(u_3+u_4) & 0 & 0 & \cdots \\
0 & 0 & 0 & 4(\beta+3)u_1u_2 & 4(u_3+u_4) & 0 & \cdots \\
0 & 0 & 0 & 0 & 5(\beta+4)u_1u_2 & 5(u_3+u_4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Thus $\mathbf{L}_0^{-1}$ is the coefficient array of the orthogonal polynomials whose moments have generating function given by

$$\cfrac{1}{1 - \cfrac{\beta u_1u_2x^2}{1-(u_3+u_4)x - \cfrac{2(\beta+1)u_1u_2x^2}{1-\cdots}}}.$$

In fact, we have

$$\mathbf{L}_0^{-1} = \left[\left(\frac{(1+\alpha_2 x)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}}{(1+\alpha_1 x)^{\frac{\alpha_2}{\alpha_1-\alpha_2}}}\right)^{-\beta}, \frac{1}{\alpha_2-\alpha_1}\ln\left(\frac{1+\alpha_2 x}{1+\alpha_1 x}\right)\right].$$

Example 12. The exponential Riordan array $\left[\frac{e^x}{2e^x-e^{2x}}, \frac{e^{2x}-e^x}{2e^x-e^{2x}}\right]$ begins:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 13 & 31 & 12 & 1 & 0 & 0 & \cdots \\ 75 & 233 & 133 & 22 & 1 & 0 & \cdots \\ 541 & 2071 & 1560 & 385 & 35 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Its first column is the sequence A000670, known as the ordered Bell numbers. The production matrix of this array is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 7 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 18 & 10 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 32 & 13 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 50 & 16 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus the ordered Bell numbers are the moments μ_n of the family of orthogonal polynomials whose coefficient array is given by

$$\left[\frac{e^x}{2e^x-e^{2x}}, \frac{e^{2x}-e^x}{2e^x-e^{2x}}\right]^{-1} = \left[\frac{1}{1+x}, \ln\left(\frac{1+2x}{1+x}\right)\right],$$

and whose generating function is given by

$$\cfrac{1}{1-x-\cfrac{2x^2}{1-4x-\cfrac{8x^2}{1-7x-\cfrac{18x^2}{1-\cdots}}}}.$$

4 A matrix product

We recall that the matrix $[\frac{1}{1-rx}, \frac{x}{1-rx}]$ has production matrix

$$\left(\begin{array}{ccccccc} r & 1 & 0 & 0 & 0 & 0 & \cdots \\ r^2 & 3r & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4r^2 & 5r & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9r^2 & 7r & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16r^2 & 9r & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25r^2 & 11r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We use the notation $\mathbf{Lag}[r]$ for this matrix $\left[\frac{1}{1-rx}, \frac{x}{1-rx}\right]$ [3]. We now form the product $\mathbf{L} \cdot \mathbf{Lag}[r]$ to get

$$\begin{aligned} \mathbf{L} \cdot \mathbf{Lag}[r] &= \left[e^{\alpha\beta x}\left(\frac{\alpha_1-\alpha_2}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right)^{\beta}, \frac{e^{\alpha_1 x}-e^{\alpha_2 x}}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right] \cdot \left[\frac{1}{1-rx}, \frac{x}{1-rx}\right] \\ &= \left[e^{\alpha\beta x}\left(\frac{\alpha_1-\alpha_2}{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}\right)^{\beta} \frac{\alpha_1 e^{\alpha_2 x}-\alpha_2 e^{\alpha_1 x}}{(\alpha_1+r)e^{\alpha_2 x}-(\alpha_2+r)e^{\alpha_1 x}}, \frac{e^{\alpha_1 x}-e^{\alpha_2 x}}{(\alpha_1+r)e^{\alpha_2 x}-(\alpha_2+r)e^{\alpha_1 x}}\right]. \end{aligned}$$

For $\beta = 1$, this product is equal to

$$\left[e^{\alpha x}\frac{\alpha_1-\alpha_2}{(\alpha_1+r)e^{\alpha_2 x}-(\alpha_2+r)e^{\alpha_1 x}}, \frac{e^{\alpha_1 x}-e^{\alpha_2 x}}{(\alpha_1+r)e^{\alpha_2 x}-(\alpha_2+r)e^{\alpha_1 x}}\right].$$

This matrix has a tri-diagonal production array which starts

$$\left(\begin{array}{ccccccc} \alpha+r & 1 & 0 & 0 & 0 & 0 & \cdots \\ (\alpha_1+r)(\alpha_2+r) & \alpha+\alpha_1+\alpha_2+3r & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4(\alpha_1+r)(\alpha_2+r) & \alpha+2(\alpha_1+\alpha_2)+5r & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9(\alpha_1+r)(\alpha_2+r) & \alpha+3(\alpha_1+\alpha_2)+7r & 1 & 0 & \cdots \\ 0 & 0 & 0 & 16(\alpha_1+r)(\alpha_2+r) & \alpha+4(\alpha_1+\alpha_2)+9r & 1 & \cdots \\ 0 & 0 & 0 & 0 & 25(\alpha_1+r)(\alpha_2+r) & \alpha+5(\alpha_1+\alpha_2)+11r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus for $\beta = 1$, this product matrix is again the inverse of the coefficient array of a family of orthogonal polynomials. Taking inverses, we arrive at the following product of orthogonal polynomial coefficient arrays:

$$\begin{aligned} &\left[\frac{1}{1+rx}, \frac{x}{1+rx}\right] \cdot \left[\left(\frac{1+\alpha_2 x}{1+\alpha_1 x}\right)^{\frac{\alpha}{\alpha_1-\alpha_2}} \frac{(1+\alpha_1 x)^{\frac{\alpha_2}{\alpha_1-\alpha_2}}}{(1+\alpha_2 x)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}}, \frac{1}{\alpha_2-\alpha_1}\ln\left(\frac{1+\alpha_2 x}{1+\alpha_1 x}\right)\right] \\ &= \left[\frac{1}{1+rx}, \frac{x}{1+rx}\right] \cdot \left[\frac{(1+\alpha_2 x)^{\frac{\alpha-\alpha_1}{\alpha_1-\alpha_2}}}{(1+\alpha_1 x)^{\frac{\alpha-\alpha_2}{\alpha_1-\alpha_2}}}, \frac{1}{\alpha_2-\alpha_1}\ln\left(\frac{1+\alpha_2 x}{1+\alpha_1 x}\right)\right] \\ &= \left[\frac{(1+(\alpha_1+r)x)^{\frac{\alpha_2-\alpha}{\alpha_1-\alpha_2}}}{(1+(\alpha_2+r)x)^{\frac{\alpha_1-\alpha}{\alpha_1-\alpha_2}}}, \frac{1}{\alpha_1-\alpha_2}\ln\left(\frac{1+(\alpha_1+r)x}{1+(\alpha_2+r)x}\right)\right]. \end{aligned}$$

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