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The Restricted Toda Chain, Exponential Riordan Arrays, and Hankel Transforms

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

We re-interpret results on the classification of Toda chain solutions given by Sheffer class orthogonal polynomials in terms of exponential Riordan arrays. We also examine associated Hankel transforms.

1 Introduction

The restricted Toda chain equation [15, 23] is simply described by

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad n = 1, 2, \ldots \quad \dot{b}_n = u_{n+1} - u_n, \quad n = 0, 1, \ldots \tag{1}$$

with $u_0 = 0$, where the dot indicates differentiation with respect to t. In this note, we shall show how solutions to this equation can be formulated in the context of exponential Riordan arrays. The Riordan arrays we shall consider may be considered as parameterised (or "time"-dependent) Riordan arrays. We have already considered parameterized Riordan arrays [1], exploring the links between these Riordan arrays and orthogonal polynomials.

The restricted Toda chain equation is closely related to orthogonal polynomials, since the functions u_n and b_n can be considered as the coefficients in the usual three-term recurrence [4, 10, 22] satisfied by orthogonal polynomials:

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad n = 1, 2, \ldots \tag{2}$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - b_0$.

Nakamura and Zhedanov [15, 16] give solutions of the restricted Toda chain equation in terms of Sheffer polynomials. Given the close links between Sheffer class polynomials and

Riordan arrays [6, 11], we find it instructive to re-interpret these results in terms of Riordan arrays. Thus in the cases discussed below, we find that the coefficient arrays of the orthogonal polynomials linked to solutions of the restricted Toda chain are exponential Riordan arrays. We believe that this result concerning an application of exponential Riordan arrays to the Toda chain equation is new. In particular, the parameters of the solution are obtained by calculating the terms of the corresponding production matrices, using the procedures of Deutsch, Ferrari, and Rinaldi [8, 9].

The reader is referred to [4, 10, 22] for basic information on orthogonal polynomials (see also [18]), to [2, 3, 7, 19] for details of Riordan arrays, to [8, 9] for information on production (Stieltjes) matrices (see also [17]), and to [5, 14, 24] for examples of Hankel transforms. Many interesting examples of Riordan arrays (both ordinary and exponential), and Hankel transforms, can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS) [20, 21]. Integer equences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.

2 Integer sequences, Hankel transforms, exponential Riordan arrays, orthogonal polynomials

In this section, we recall known results on integer sequences, Hankel transforms, exponential Riordan arrays and orthogonal polynomials that will be useful for the sequel.

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f_o(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence, while $f_e(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is called the *exponential generating function* or e.g.f. of the sequence. a_n is thus the coefficient of x^n in $f_o(x)$. We denote this by $a_n = [x^n] f_o(x)$. Similarly, $a_n = n![x^n] f_e(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $n! = n![x^n]\frac{1}{1-x}$, which says that $\frac{1}{1-x}$ is the e.g.f. of $n!$ A000142. For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ and $f'(0) \neq 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x) = f^{[-1]}(x)$ such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev} f$.

The *Hankel transform* [14] of a given sequence $A = (a_n)_{n\geq 0}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ where $h_n = |a_{i+j}|_{i,j=0}^{n}$, i.e

$$A = (a_n)_{n\in\mathbb{N}_0} \quad \rightarrow \quad h = (h_n)_{n\in\mathbb{N}_0} : \quad h_n = \left| \begin{array}{cccc} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{array} \right| . \tag{3}$$

The Hankel transform of a sequence a_n and that of its binomial transform are equal.

In the case that a_n has g.f. $g(x)$ expressible in the form

$$g(x) = \cfrac{a_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}},$$

(with $\beta_i \neq 0$ for all i) then we have [12, 13, 25]

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-1}^2 \beta_n = a_0^n \prod_{k=1}^{n} \beta_k^{n-k+1}. \tag{4}$$

Note that this is independent of α_n. In general α_n and β_n are not integers. Such a continued fraction is associated to a monic family of orthogonal polynomials which obey the three term recurrence

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The terms appearing in the first column of the inverse of the coefficient array of these polynomials are the moments of family.

The *exponential Riordan group* [2, 7, 9], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1 x + g_2 x^2 + \ldots$ and $f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x) f(x)^i / i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. It is *monic* if $g_0 = 1$. If in addition $f_1 = 1$, then it is *proper*. The group law is given by

$$[g, f] * [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group. If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n \geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}\ (x)$, then the sequence $\mathbf{M}\mathbf{u}^T$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ are given by $g(x) e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

Example 1. The *binomial matrix* is the matrix with general term $\binom{n}{k}$. It is realized by Pascal's triangle. As an exponential Riordan array, it is given by $[e^x, x]$. We further have

$$([e^x, x])^m = [e^{mx}, x].$$

An important concept for the sequel is that of production matrix. The concept of a *production matrix* [8, 9] is a general one, but for this note we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting $\mathbf{r}_0$ be the row vector

$$\mathbf{r}_0 = (1, 0, 0, 0, \ldots),$$

we define $\mathbf{r}_i = \mathbf{r}_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P. If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \begin{pmatrix} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{pmatrix}$$

and

$$DA_P = A_P P$$

where $D = (\delta_{i,j+1})_{i,j \geq 0}$ (where δ is the usual Kronecker symbol). Note that [17] P is also called the Stieltjes matrix associated to A_P. The following result [9] concerning matrices that are production matrices for exponential Riordan arrays is important for the sequel.

Proposition 2. *Let $A = (a_{n,k})_{n,k \geq 0} = [g(x), f(x)]$ be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \ldots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \ldots \tag{5}$$

be two formal power series that that

$$r(f(x)) = f'(x) \tag{6}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{7}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{8}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!} \sum_{i \geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{9}$$

or, defining $c_{-1} = 0$,

$$a_{n+1,k} = \frac{1}{k!} \sum_{i \geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{10}$$

Conversely, starting from the sequences defined by (5), the infinite array $(a_{n,k})_{n,k \geq 0}$ defined by (10) is an exponential Riordan array.

A consequence of this proposition is that $P = (p_{i,j})_{i,j \geq 0}$ where

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{r-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate exponential generating function

$$\phi_P(t,z)=\sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,z)=e^{tz}(c(z)+tr(z)).$$

Note in particular that we have

$$r(x)=f'(\bar{f}(x))$$

and

$$c(x)=\frac{g'(\bar{f}(x))}{g(\bar{f}(x))}.$$

Example 3. We consider the exponential Riordan array $[\frac{1}{1-x},x]$, A094587. This array [3] has elements

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 6 & 6 & 3 & 1 & 0 & 0 & \ldots \\ 24 & 24 & 12 & 4 & 1 & 0 & \ldots \\ 120 & 120 & 60 & 20 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and general term $[k \le n]\frac{n!}{k!}$ with inverse

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & -5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

which is the array $[1-x,x]$. In particular, we note that the row sums of the inverse, which begin $1,0,-1,-2,-3,\ldots$ (that is, $1-n$), have e.g.f. $(1-x)\exp(x)$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1-x)$ (which is the sequence starting $1,-1,0,0,0,\ldots$). In order to calculate the production matrix P of $[\frac{1}{1-x},x]$ we note that $f(x)=x$, and hence we have $f'(x)=1$ so $f'(\bar{f}(x))=1$. Also $g(x)=\frac{1}{1-x}$ leads to $g'(x)=\frac{1}{(1-x)^2}$, and so, since $\bar{f}(x)=x$, we get

$$\frac{g'(\bar{f}(x))}{g(\bar{f}(x))}=\frac{1}{1-x}.$$

Thus the generating function for P is

$$e^{tz}\left(\frac{1}{1-z}+t\right).$$

Thus P is the matrix $[\frac{1}{1-x}, x]$ with its first row removed.

Example 4. We consider the exponential Riordan array $[1, \frac{x}{1-x}]$. The general term of this matrix [3] may be calculated as follows:

$$\begin{aligned}
T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\
&= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\
&= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\
&= \frac{n!}{k!}\binom{n-1}{n-k}.
\end{aligned}$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1,..,n\}$ into any number of lists, where a list means an ordered subset'. Its general term is equal to $(n-1)!L_{n-1}(1,-1)$. The inverse of $\left[1, \frac{x}{1-x}\right]$ is the exponential Riordan array $\left[1, \frac{x}{1+x}\right]$, A111596. The row sums of this sequence have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884. To calculate the production matrix of $\left[1, \frac{x}{1+x}\right]$ we note that $g'(x) = 0$, while $\bar{f}(x) = \frac{x}{1+x}$ with $f'(x) = \frac{1}{(1+x)^2}$. Thus

$$f'(\bar{f}(x)) = (1+x)^2,$$

and so the generating function of the production matrix is given by

$$e^{tz}t(1+z)^2.$$

The production matrix of the inverse begins

$$\left(\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 2 & 1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 4 & 1 & 0 & 0 & \ldots \\
0 & 0 & 6 & 6 & 1 & 0 & \ldots \\
0 & 0 & 0 & 12 & 8 & 1 & \ldots \\
0 & 0 & 0 & 0 & 20 & 10 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right).$$

Example 5. The exponential Riordan array $\mathbf{A} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

has general term

$$T_{n,k} = \frac{n!}{k!}\binom{n}{k}.$$

Its inverse is $\left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ with general term $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. This is A021009, the triangle of coefficients of the Laguerre polynomials $L_n(x)$. The production matrix of $\left[\frac{1}{1-x}, \frac{x}{1-x}\right]$ is given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & 5 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9 & 7 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16 & 9 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25 & 11 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Example 6. The exponential Riordan array $\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 8 & 6 & 1 & 0 & 0 & \dots \\ 1 & 24 & 29 & 10 & 1 & 0 & \dots \\ 1 & 89 & 145 & 75 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

is the coefficient array for the polynomials

$$_2F_0(-n, x; -1)$$

which are an unsigned version of the Charlier polynomials (of order 0) [10, 18, 22]. This is A094816. It is equal to

$$[e^x, x]\left[1, \ln\left(\frac{1}{1-x}\right)\right],$$

or the product of the binomial array $\mathbf{B}$ and the array of (unsigned) Stirling numbers of the first kind. The production matrix of the inverse of this matrix is given by

$$\left(\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & -3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3 & -4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 4 & -5 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 5 & -6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which indicates the orthogonal nature of these polynomials. We can prove this as follows. We have

$$\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]^{-1} = \left[e^{-(1-e^{-x})}, 1-e^{-x}\right].$$

Hence $g(x) = e^{-(1-e^{-x})}$ and $f(x) = 1 - e^{-x}$. We are thus led to the equations

$$\begin{aligned} r(1-e^{-x}) &= e^{-x}, \\ c(1-e^{-x}) &= -e^{-x}, \end{aligned}$$

with solutions $r(x) = 1 - x$, $c(x) = x - 1$. Thus the bivariate generating function for the production matrix of the inverse array is

$$e^{tz}(z - 1 + t(1-z)),$$

which is what is required.

3 Hermite polynomials and the Toda chain

We recall that the Hermite polynomials may be defined as

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

The generating function for $H_n(x)$ is given by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

The following result was proved [1]:

Proposition 7. *The proper exponential Riordan array*

$$\mathbf{L} = \left[e^{2rx-x^2}, x\right]$$

has as first column the Hermite polynomials $H_n(r)$. This array has a tri-diagonal production array.

Proof. The first column of $\mathbf{L}$ has generating function e^{2rx-x^2}, from which the first assertion follows. Standard Riordan array techniques show us that the production array P of $\mathbf{L}$ is indeed tri-diagonal, beginning with

$$\left(\begin{array}{ccccccc} 2r & 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 2r & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 2r & 1 & 0 & 0 & \cdots \\ 0 & 0 & -6 & 2r & 1 & 0 & \cdots \\ 0 & 0 & 0 & -8 & 2r & 1 & \cdots \\ 0 & 0 & 0 & 0 & -10 & 2r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is so since $f(x) = x$ gives us $\bar{f}(x) = x$ and $f'(x) = 1$, and thus

$$r(x) = f'(\bar{f}(x)) = f'(x) = 1.$$

Similarly,

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = 2(r - x),$$

and thus the bivariate generating function $\phi_P(y, w)$ of P is given by

$$\phi_P(y, z) = e^{yz}(2(r - z) + y),$$

as required. $\square$

We note that $\mathbf{L}$ starts with

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2r & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2(2r^2-1) & 4r & 1 & 0 & 0 & 0 & \cdots \\ 4r(2r^2-3) & 6(2r^2-1) & 6r & 1 & 0 & 0 & \cdots \\ 4(4r^3-12r^2+3) & 16r(2r^2-3) & 12(2r^2-1) & 8r & 1 & 0 & \cdots \\ 8r(4r^4-20r^2+15) & 20(4r^4-12r^2+3) & 40r(2r^2-3) & 20(2r^2-1) & 10r & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus

$$\mathbf{L}^{-1} = \left[e^{-2rx+x^2}, x\right]$$

is the coefficient array of a set of orthogonal polynomials which have as moments the Hermite polynomials. These new orthogonal polynomials satisfy the three-term recurrence

$$\mathfrak{H}_{n+1}(x) = (x - 2r)\mathfrak{H}_n(x) + 2n\mathfrak{H}_{n-1}(x),$$

with $\mathfrak{H}_0 = 1$, $\mathfrak{H}_1 = x - 2r$.

We can now modify this result to give us our first Toda chain result.

Proposition 8. *The exponential Riordan array*

$$\left[e^{-2(z-t)x+x^2}, x\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),$$

where (u_n, b_n) is a solution to the restricted Toda chain.

Proof. We easily determine that the inverse matrix

$$\left[e^{2(z-t)x-x^2}, x\right]$$

has the production matrix P given by

$$\begin{pmatrix} 2(z-t) & 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 2(z-t) & 1 & 0 & 0 & 0 & \dots \\ 0 & -4 & 2(z-t) & 1 & 0 & 0 & \dots \\ 0 & 0 & -6 & 2(z-t) & 1 & 0 & \dots \\ 0 & 0 & 0 & -8 & 2(z-t) & 1 & \dots \\ 0 & 0 & 0 & 0 & -10 & 2(z-t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is so since $f(x) = x$ gives us $\bar{f}(x) = x$ and $f'(x) = 1$, and thus

$$r(x) = f'(\bar{f}(x)) = f'(x) = 1.$$

Similarly,

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = -2(x - z + t),$$

and thus the bivariate generating function $\phi_P(y, w)$ of P is given by

$$\phi_P(y, w) = e^{yw}(-2(w - z + t) + y),$$

as required.

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = -2n, \quad b_n(t) = 2(z-t).$$

It is immediate that these satisfy Eq. (1). $\square$

We now note that the moments of this polynomial family (the first column of the inverse matrix) m_n satisfy the following relation:

$$\begin{aligned}
m_n &= n![x^n]e^{2(z-t)x-x^2} \\
&= \frac{1}{e^{-t^2+2tz}}\frac{d^n}{dt^n}e^{-t^2+2tz} \\
&= n!\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-k}{k}(-1)^k\frac{(2(z-t))^{n-2k}}{(n-k)!} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{k}\binom{n-k}{k}k!(2(z-t))^{n-2k} \\
&= H_n(z-t).
\end{aligned}$$

From the form of the production matrix P, we infer that the generating function of m_n may be expressed as

$$\cfrac{1}{1-2(z-t)x+\cfrac{2x^2}{1-2(z-t)x+\cfrac{4x^2}{1-2(z-t)x+\cfrac{6x^2}{1-2(z-t)x+\cdots}}}}.$$

The Hankel transform of m_n, defined as the sequence h_n where

$$h_n = |m_{i+j}|_{0\le i,j\le n}$$

is then given by

$$h_n = (-2)^{\binom{n+1}{2}}\prod_{k=0}^{n}k!.$$

4 Charlier polynomials and the Toda chain

Proposition 9. *The exponential Riordan array*

$$\left[e^{-xe^t}, \ln(1+x)\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x)+b_nP_n(x)+u_nP_{n-1}(x)=xP_n(x),$$

where (u_n, b_n) is a solution to the restricted Toda chain.

Proof. We determine that the inverse matrix

$$\left[e^{e^{t+x}-e^t}, e^x - 1\right]$$

has the production matrix

$$\begin{pmatrix} e^t & 1 & 0 & 0 & 0 & 0 & \dots \\ e^t & e^t+1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2e^t & e^t+2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3e^t & e^t+3 & 1 & 0 & \dots \\ 0 & 0 & 0 & 4e^t & e^t+4 & 1 & \dots \\ 0 & 0 & 0 & 0 & 5e^t & e^t+5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

as follows. We have $\bar{f}(x) = \ln(1+x)$ and $f'(x) = e^x$, and thus

$$r(x) = f'(\bar{f}(x)) = 1 + x.$$

Similarly,

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{e^{xe^t+t}(1+x)}{e^{xe^t}} = e^t(1+x).$$

Thus the bivariate generating function for the production matrix P is given by

$$\phi_P(y,z) = e^{yz}(e^t(1+z) + y(1+z)),$$

as required. This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = ne^t, \quad b_n(t) = n + e^t.$$

It is easy now to verify that, with these values, (u_n, b_n) satisfies the Toda chain equations Eq. (1). $\square$

The moments m_n of this family of orthogonal polynomials may be expressed as:

$$\begin{aligned} m_n &= n![x^n]e^{e^{t+x}-e^t} \\ &= \frac{1}{e^{e^t-1}}\frac{d^n}{dt^n}e^{e^t-1} \\ &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} e^{kt}, \end{aligned}$$

where

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

are the Stirling numbers of the second kind. From the form of the production matrix P, we infer that the generating function of m_n may be expressed as

$$\cfrac{1}{1-e^t x-\cfrac{e^t}{1-(e^t+1)x-\cfrac{2e^t}{1-(e^t+2)x-\cfrac{3e^t}{1-(e^t+3)x-\cdots}}}}.$$

The Hankel transform of m_n is then given by

$$h_n = (e^t)^{\binom{n+1}{2}} \prod_{k=0}^{n} k!$$

In particular we have

$$m_n(0) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} = \text{Bell}(n),$$

where Bell(n) denotes the n-th Bell number, with Hankel transform

$$h_n = \prod_{k=0}^{n} k!.$$

5 Laguerre polynomials and the Toda chain

We have seen that the exponential Riordan array

$$\left[\frac{1}{1-x}, \frac{x}{1-x}\right]$$

is closely related to the Laguerre polynomials. We introduce a "time" parameter t as follows:

$$\left[\frac{1}{1-\frac{x}{1+t}}, \frac{x}{1-\frac{x}{1+t}}\right] = \left[\left(1-\frac{x}{1+t}\right)^{-1}, \frac{x}{1-\frac{x}{1+t}}\right],$$

and further generalize this array by introducing a general power factor α, to get the array

$$\left[\left(1-\frac{x}{1+t}\right)^{\alpha}, \frac{x}{1-\frac{x}{1+t}}\right].$$

We then have the following proposition.

Proposition 10. *The exponential Riordan array*

$$\left[\left(1-\frac{x}{1+t}\right)^{\alpha}, \frac{x}{1-\frac{x}{1+t}}\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),$$

where (u_n, b_n) is a solution to the restricted Toda chain.

Proof. The inverse matrix

$$\left[\left(\frac{1+t+x}{1+t}\right)^{\alpha}, \frac{(1+t)x}{1+t+x}\right]$$

has the production matrix

$$\begin{pmatrix} \frac{\alpha}{1+t} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{-\alpha}{(1+t)^2} & \frac{\alpha-2}{1+t} & 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{2(1-\alpha)}{(1+t)^2} & \frac{\alpha-4}{1+t} & 1 & 0 & 0 & \cdots \\ 0 & 0 & \frac{3(2-\alpha)}{(1+t)^2} & \frac{\alpha-6}{1+t} & 1 & 0 & \cdots \\ 0 & 0 & 0 & \frac{4(3-\alpha)}{(1+t)^2} & \frac{\alpha-8}{1+t} & 1 & \cdots \\ 0 & 0 & 0 & 0 & \frac{5(4-\alpha)}{(1+t)^2} & \frac{\alpha-10}{1+t} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = \frac{n(n-\alpha-1)}{(1+t)^2}, \quad b_n(t) = \frac{\alpha-2n}{1+t}.$$

It is easy now to verify that, with these values, (u_n, b_n) satisfies the Toda chain equations Eq. (1). □

For this family of orthogonal polynomials, the moments m_n may be expressed as:

$$m_n = n![x^n]\left(1+\frac{x}{1+t}\right)^{\alpha} = \frac{1}{(1+t)^{\alpha}}\frac{d^n}{dt^n}(1+t)^{\alpha} = \frac{(\alpha)_n}{(1+t)^n}, \tag{11}$$

where

$$(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1).$$

From the form of the production matrix, we infer that the generating function of m_n may be expressed as

$$\cfrac{1}{1-\frac{\alpha}{1+t}x-\cfrac{\frac{0-\alpha}{(1+t)^2}x^2}{1-\frac{\alpha-2}{1+t}x-\cfrac{\frac{2(1-\alpha)}{(1+t)^2}x^2}{1-\frac{\alpha-4}{1+t}x-\cfrac{\frac{3(2-\alpha)}{(1+t)^2}x^2}{1-\frac{\alpha-6}{1+t}x-\cdots}}}}.$$

The Hankel transform of m_n is then given by

$$h_n = \frac{(-1)^{\binom{n+1}{2}}}{(1+t)^{n(n+1)}}\prod_{k=0}^{n} k!(\alpha-k)^{n-k}.$$

Noticing that

$$\lim_{c \to 1} \frac{c(1+t)}{c-1} \ln\left(\frac{1-\frac{x}{c(1+t)}}{1-\frac{x}{1+t}}\right) = \frac{x}{1-\frac{x}{1+t}},$$

we can 'embed' this previous result in the following:

Proposition 11. *The exponential Riordan array*

$$\left[\left(1-\frac{x}{1+t}\right)^{\alpha}, \frac{c(1+t)}{c-1} \ln\left(\frac{1-\frac{x}{c(1+t)}}{1-\frac{x}{1+t}}\right)\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),$$

where for $c = 1$, (u_n, b_n) is a solution to the restricted Toda chain.

Proof. We can show that the inverse array has a tri-diagonal production matrix with

$$b(n) = \frac{\alpha c - n(c+1)}{c(1+t)},$$

and

$$u(n) = \frac{n(n-1-\alpha)}{c(1+t)^2}.$$

Letting $c \to 1$ now gives us the result. $\square$

Another direction of generalization is given be looking at the related exponential Riordan array

$$\left[e^{\alpha x}, \frac{x}{1+\frac{x}{1+t}}\right],$$

which has inverse

$$\left[e^{\frac{\alpha x}{1-\frac{x}{1+t}}}, \frac{x}{1-\frac{x}{1+t}}\right].$$

For this, we have the following proposition.

Proposition 12. *The exponential Riordan array*

$$\left[e^{\frac{\alpha x}{1-\frac{x}{1+t}}}, \frac{x}{1-\frac{x}{1+t}}\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),$$

where (u_n, b_n) is a solution to the restricted Toda chain.

Proof. We find that

$$u_n(t) = \frac{n(n-1)}{(1+t)^2}$$

and

$$b_n(t) = -\frac{\alpha(1+t)+2n}{1+t}.$$

$\square$

In this case, the moments are simply

$$m_n = (-\alpha)^n.$$

6 Meixner polynomials and the Toda chain

Proposition 13. *The exponential Riordan array*

$$\left[\frac{1}{\sqrt{1-2x\tanh(t)-x^2\text{sech}(t)^2}}, \ln\left(\sqrt{\frac{1+xe^{-t}\text{sech}(t)}{1-xe^{t}\text{sech}(t)}}\right)\right]$$

is the coefficient array of a family of orthogonal polynomials $P_n(x)$ with

$$P_{n+1}(x)+b_nP_n(x)+u_nP_{n-1}(x) = xP_n(x),$$

where (u_n, b_n) is a solution to the restricted Toda chain.

Proof. The inverse matrix

$$\left[\frac{\text{sech}(x+t)}{\text{sech}(t)}, \sinh(x)\frac{\text{sech}(x+t)}{\text{sech}(t)}\right]$$

has the production matrix

$$\begin{pmatrix} -\tanh(t) & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\text{sech}^2(t) & -3\tanh(t) & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4\,\text{sech}^2(t) & -5\tanh(t) & 1 & 0 & 0 & \cdots \\ 0 & 0 & -9\,\text{sech}^2(t) & -7\tanh(t) & 1 & 0 & \cdots \\ 0 & 0 & 0 & -16\,\text{sech}^2(t) & -9\tanh(t) & 1 & \cdots \\ 0 & 0 & 0 & 0 & -25\,\text{sech}^2(t) & -11\tanh(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This verifies that $P_n(x)$ is indeed a family of orthogonal polynomials, for which

$$u_n(t) = -n^2\text{sech}^2(t), \quad b_n(t) = -(2n+1)\tanh(t).$$

It is easy now to verify that, with these values, (u_n, b_n) satisfies the Toda chain equations Eq. (1). $\square$

We may describe the moments m_n of this family of polynomials by

$$m_n = n![x^n]\frac{\text{sech}(x+t)}{\text{sech}(t)} = \frac{1}{\text{sech}(t)}\frac{d^n}{dt^n}\text{sech}(t). \tag{12}$$

The Hankel transform of m_n is then given by

$$h_n = (-1)^{\binom{n+1}{2}}\text{sech}(t)^{n(n+1)}\prod_{k=0}^{n}(k!)^2.$$

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