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Some Properties of Associated Stirling Numbers

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Abstract

In this paper, we discuss the properties of associated Stirling numbers. By means of the method of coefficients, we establish a series of identities involving associated Stirling numbers, Bernoulli numbers, harmonic numbers, and the Cauchy numbers of the first kind. In addition, we give the asymptotic expansion of certain sums involving 2-associated Stirling numbers of the second kind by Darboux's method.

1 Introduction

Stirling numbers are generalized by many forms. See for instance [1,2,3,4,5] and [9]. In this paper, we are interested in associated Stirling numbers. The associated Stirling numbers of the first kind $s_2(n,k)$ [3] are given by

$$\sum_{n=k}^{\infty} s_2(n,k)\frac{t^n}{n!} = \frac{[\ln(1+t)-t]^k}{k!}$$

and the r-associated Stirling numbers of the second kind $S_r(n,k)$ [3] are given by

$$\sum_{n=k}^{\infty} S_r(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left[e^t - \sum_{j=0}^{r-1}\frac{t^j}{j!}\right]^k,$$

where k and r are positive integers. It is clear that

$$\begin{aligned}\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} &= \frac{(e^t-1-t)^k}{k!},\\ \sum_{n=k}^{\infty} S_3(n,k)\frac{t^n}{n!} &= \frac{(e^t-1-t-t^2/2)^k}{k!}.\end{aligned}$$

Like the ordinary Stirling numbers, the associated Stirling numbers also play important roles in combinatorics. For example, $|s_2(n,k)|$ equals the number of derangements of a set N

($|N| = n$), with k orbits, and $S_r(n,k)$ is the number of partitions of the set N ($|N| = n$), into k blocks, all of cardinality $\geq r$. It is clear that $S_1(n,k)$ is the Stirling number of the second kind $S(n,k)$. Therefore, associated Stirling numbers deserve to be investigated. The aim of this paper is to investigate the properties of associated Stirling numbers by making use of the method of coefficients [7]. We establish a series of identities relating associated Stirling numbers with Bernoulli, harmonic, and Cauchy numbers of the first kind. In addition, we give the asymptotic expansion of certain sums involving r-associated Stirling numbers by Darboux's method.

The paper is organized as follows. In Section 2, we establish a series of identities involving associated Stirling, Bernoulli, harmonic and Cauchy numbers of the first kind. In Section 3, we give the asymptotic expansion of certain sums involving r-associated Stirling numbers by Darboux's method.

For convenience, we recall some definitions of combinatorial numbers involved in the paper. Throughout, we denote Stirling numbers of the first kind by $s(n,k)$, and let B_n, $B_n^{(k)}$, and E_n stand for Bernoulli, generalized Bernoulli, and Euler numbers respectively. That is,

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{\ln^k(1+t)}{k!}, \qquad \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = \frac{t}{e^t-1},$$
$$\sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!} = \frac{t^k}{(e^t-1)^k} \quad (k \geq 1), \quad \sum_{n=0}^{\infty} E_n\frac{t^n}{n!} = \frac{2}{e^t+e^{-t}}.$$

The Cauchy numbers of the first kind a_n are given by

$$\sum_{n=0}^{\infty} a_n\frac{t^n}{n!} = \frac{t}{\ln(1+t)}.$$

The harmonic numbers H_n are given by

$$\sum_{n=1}^{\infty} H_n t^n = -\frac{\ln(1-t)}{1-t}.$$

In this paper, $[t^n]f(t)$ denotes the coefficient of t^n in $f(t)$, where

$$f(t) = \sum_{n=0}^{\infty} f_n t^n.$$

The expression $[t^n]$ is called the *"coefficient of" functionals* [7]. If $f(t)$ and $g(t)$ are formal power series, the following relations hold [7]:

$$[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n]g(t), \tag{1.1}$$
$$[t^n]tf(t) = [t^{n-1}]f(t), \tag{1.2}$$
$$[t^n]f'(t) = (n+1)[t^{n+1}]f(t), \tag{1.3}$$
$$[t^n]f(t)g(t) = \sum_{k=0}^{n}([y^k]f(y))[t^{n-k}]g(t). \tag{1.4}$$

In Section 2, we obtain a series of identities related to associated Stirling numbers by using (1.1)-(1.4).

2 Identities involving associated Stirling, Bernoulli, and harmonic numbers

Bernoulli numbers and harmonic numbers are important in combinatorics, and Stirling numbers are related to them. From [3], we know that Stirling numbers and Bernoulli numbers satisfy

$$\sum_{j=0}^{n} \frac{(-1)^j j! S(n,j)}{j+1} = B_n, \quad \sum_{j=0}^{n} s(n,j) B_j = \frac{(-1)^n n!}{n+1}.$$

By the generating functions of $S_2(n,k)$, $S(n,k)$, and B_n, we observe that $S_2(n,k)$ is also related to B_n, and we have

Theorem 2.1. *For $n \geq 1$ and $k \geq 1$, $S_2(n,k)$, B_n, and $S(n,k)$ satisfy the equations*

$$\begin{aligned} \sum_{j=0}^{n} S_2(n-j+k,k) \binom{n+k}{j} B_j &= (n+k) \sum_{j=1}^{k} \frac{(-1)^{k-j}}{j} \binom{n+k-1}{k-j} S(n+j-1,j-1) \\ &\quad + (-1)^k \binom{n+k}{k} B_n, \end{aligned} \tag{2.1}$$

$$\sum_{j=0}^{n} \binom{n+k-1}{j} S_2(n-j+k,k) B_j = (n+k-1) S_2(n+k-2,k-1) \quad k \geq 2. \tag{2.2}$$

Proof. By the definitions of $S_2(n,k)$, B_n, and $S(n,k)$, we have

$$\begin{aligned} \sum_{j=0}^{n} S_2(n-j+k,k) \binom{n+k}{j} B_j &= (n+k)! \sum_{j=0}^{n} \frac{S_2(n-j+k,k)}{(n-j+k)!} \cdot \frac{B_j}{j!} \\ &= (n+k)! \sum_{j=0}^{n} [t^{n-j+k}] \frac{(e^t-1-t)^k}{k!} [t^j] \frac{t}{e^t-1} \\ &= (n+k)! \sum_{j=0}^{n} [t^{n-j}] \frac{(e^t-1-t)^k}{k! t^k} [t^j] \frac{t}{e^t-1} \\ &= (n+k)! [t^n] \frac{(e^t-1-t)^k t}{k! t^k (e^t-1)} \\ &= (n+k)! [t^n] \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(e^t-)^{j-1} t^{-j+1}}{k!} \\ &= \frac{(-1)^k (n+k)!}{k!} [t^n] \frac{t}{e^t-1} + (n+k)! \sum_{j=1}^{k} [t^n] \frac{(-1)^{k-j} (e^t-1)^{j-1}}{j(k-j)!(j-1)! t^{j-1}} \\ &= (-1)^k \binom{n+k}{k} B_n + (n+k)! \sum_{j=1}^{k} \frac{(-1)^{k-j} S(n+j-1,j-1)}{j(k-j)!(n+j-1)!}. \end{aligned}$$

Then (2.1) holds.

Now we give the proof of (2.2).

$$\begin{aligned}
\sum_{j=0}^{n}\binom{n+k-1}{j}S_2(n-j+k,k)B_j &= (n+k-1)!\sum_{j=0}^{n}\frac{S_2(n-j+k,k)}{(n-j+k-1)!}\cdot\frac{B_j}{j!}\\
&= (n+k-1)!\sum_{j=0}^{n}\frac{(n-j+k)S_2(n-j+k,k)}{(n-j+k)!}\cdot\frac{B_j}{j!}\\
&= (n+k-1)!\sum_{j=0}^{n}(n-j+k)[t^{n-j+k}]\frac{(e^t-1-t)^k}{k!}[t^j]\frac{t}{e^t-1}\\
&= (n+k-1)!\sum_{j=0}^{n}[t^{n-j+k-1}]\frac{(e^t-1-t)^{k-1}(e^t-1)}{(k-1)!}[t^j]\frac{t}{e^t-1}\\
&= (n+k-1)!\sum_{j=0}^{n}[t^{n-j}]\frac{k(e^t-1-t)^{k-1}(e^t-1)}{t^{k-1}k!}[t^j]\frac{t}{e^t-1}\\
&= (n+k-1)![t^n]\frac{(e^t-1-t)^{k-1}}{t^{k-2}(k-1)!}\\
&= (n+k-1)![t^{n+k-2}]\frac{(e^t-1-t)^{k-1}}{(k-1)!}\\
&= (n+k-1)S_2(n+k-2,k-1).
\end{aligned}$$

This completes the proof. □

Formula (2.1) relates associated Stirling, Bernoulli, and Stirling numbers of the second kind.

The generating functions of generalized Bernoulli numbers $B_n^{(k)}$ implies that they are related to associated Stirling numbers. For $S_2(n,k)$ and $B_n^{(k)}$, we get

Corollary 2.1. *For $n \geq 1$ and $k \geq 1$, 2-associated Stirling numbers $S_2(n,k)$ and generalized Bernoulli numbers $B_n^{(k)}$ satisfy*

$$\sum_{j=0}^{n}S_2(n-j+k,k)\binom{n+k}{j}B_j^{(k)} = \binom{n+k}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}B_n^{(k-j)}. \tag{2.3}$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{n} S_2(n-j+k,k)\binom{n+k}{j}B_j^{(k)} &= (n+k)!\sum_{j=0}^{n}\frac{S_2(n-j+k,k)}{(n-j+k)!}\cdot\frac{B_j^{(k)}}{j!} \\
&= (n+k)!\sum_{j=0}^{n}[t^{n-j+k}]\frac{(e^t-1-t)^k}{k!}[t^j]\frac{t^k}{(e^t-1)^k} \\
&= \frac{(n+k)!}{k!}\sum_{j=0}^{n}[t^{n-j}]\frac{(e^t-1-t)^k}{t^k}[t^j]\frac{t^k}{(e^t-1)^k} \\
&= \frac{(n+k)!}{k!}[t^n]\frac{(e^t-1-t)^k}{(e^t-1)^k} \\
&= \frac{(n+k)!}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{B_n^{(k-j)}}{n!}.
\end{aligned}$$

Hence (2.3) holds. □

For $S_3(n,k)$ and Bernoulli numbers B_n, we have

Theorem 2.2. *For $n \geq k$ and $k \geq 1$, $S_3(n,k)$ and B_n satisfy*

$$\begin{aligned}
\sum_{j=0}^{n}\binom{n+k}{j+k}S_3(j+k,k)B_{n-j} &= (n+k)!\sum_{j=1}^{k}\frac{(-1)^{k-j}}{j(k-j)!}\sum_{j_1=0}^{k-j}\binom{k-j}{j_1}\frac{S(n-j_1+j-1,j-1)}{2^{j_1}(n-j_1+j-1)!} \\
&\quad +\frac{(-1)^k(n+k)!}{k!}\sum_{j=0}^{k}\frac{B_{n-j}}{2^j(n-j)!}\binom{k}{j}. \qquad (2.4)
\end{aligned}$$

Proof. From the generating functions of $S_3(n,k)$ and B_n, we have

$$
\begin{aligned}
\sum_{j=0}^{n}\binom{n+k}{j+k}S_3(j+k,k)B_{n-j} &= (n+k)!\sum_{j=0}^{n}\frac{S_3(j+k,k)}{(j+k)!}\frac{B_{n-j}}{(n-j)!}\\
&= (n+k)!\sum_{j=0}^{n}[t^{j+k}]\frac{(e^t-1-t-t^2/2)^k}{k!}[t^{n-j}]\frac{t}{e^t-1}\\
&= (n+k)!\sum_{j=0}^{n}[t^{j}]\frac{(e^t-1-t-t^2/2)^k}{t^k k!}[t^{n-j}]\frac{t}{e^t-1}\\
&= (n+k)![t^n]\frac{(e^t-1-t-t^2/2)^k t}{k!t^k(e^t-1)}\\
&= (n+k)![t^n]\left(\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(e^t-1)^j(t+t^2/2)^{k-j}t}{k!(e^t-1)t^k}\right)\\
&= (n+k)![t^n]\frac{(-1)^k(1+t/2)^k t}{k!(e^t-1)}\\
&\quad +[t^n]\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\frac{(e^t-1)^{j-1}(t+t^2/2)^{k-j}t}{k!t^k}\\
&= (-1)^k(n+k)!\sum_{j=0}^{k}\binom{k}{j}\frac{B_{n-j}}{2^j(n-j)!k!}\\
&\quad +\frac{(n+k)!}{k!}[t^n]\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}(e^t-1)^{j-1}\sum_{j_1=0}^{k-j}\binom{k-j}{j_1}\frac{t^{j_1-j+1}}{2^{j_1}}\\
&= \frac{(-1)^k(n+k)!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{B_{n-j}}{2^j(n-j)!}\\
&\quad +(n+k)!\sum_{j=1}^{k}\frac{(-1)^{k-j}}{j(k-j)!}\sum_{j_1=0}^{k-j}\binom{k-j}{j_1}\frac{S(n-j_1+j-1,j-1)}{2^{j_1}(n-j_1+j-1)!}
\end{aligned}
$$

Then (2.4) holds. $\square$

There are many identities relating Stirling numbers of the first kind and harmonic numbers in [3]. For example,

$$
\begin{aligned}
(-1)^{n+1}s(n+1,2) &= n!H_n,\\
(-1)^{n}s(n+1,3) &= \frac{n!}{2}(H_n^2-H_n^{(2)}),\\
(-1)^{n+1}s(n+1,4) &= \frac{n!}{6}(H_n^2-3H_nH_n^{(2)}+2H_n^{(3)}),
\end{aligned}
$$

where $H_n^{(s)}=1+2^{-s}+3^{-s}+\cdots+n^{-s}$.

For associated Stirling numbers of the first kind and harmonic numbers, we can prove

Theorem 2.3. *For $n \geq 1$ and $k \geq 1$, we have*

$$\sum_{j=0}^{n} \frac{(-1)^j H_{j+1} s_2(n-j+k,k)}{(j+2)(n-j+k)!} = \frac{(-1)^k}{2} \sum_{j=0}^{k} \frac{(-1)^j (j+1)(j+2) s(n+j+2,j+2)}{(k-j)!(n+j+2)!}. \tag{2.5}$$

Proof. By integrating the generating function for H_n we have

$$\sum_{n=0}^{\infty} \frac{H_{n+1} t^n}{n+2} = \frac{\ln^2(1-t)}{2t^2}.$$

One can verify that

$$\frac{[\ln(1-t)+t]^k \ln^2(1-t)}{2(-1)^k k! t^{k+2}} = \frac{(-1)^k}{2k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\ln^{j+2}(1-t)}{t^{j+2}}.$$

Then

$$\begin{aligned}
[t^n] \frac{[\ln(1-t)+t]^k \ln^2(1-t)}{2(-1)^k k! t^{k+2}} &= \sum_{j=0}^{n} [t^{n-j+k}] \frac{[\ln(1-t)+t]^k}{(-1)^k k!} [t^j] \frac{\ln^2(1-t)}{2t^2} \\
&= \sum_{j=0}^{n} \frac{(-1)^{n-j} s_2(n-j+k,k) H_{j+1}}{(n-j+k)!(j+2)} \\
&= \frac{(-1)^k}{2k!} \sum_{j=0}^{k} \binom{k}{j} [t^n] \frac{\ln^{j+2}(1-t)}{t^{j+2}},
\end{aligned}$$

$$\sum_{j=0}^{n} \frac{(-1)^{n-j} H_{j+1} s_2(n-j+k,k)}{(j+2)(n-j+k)!} = \frac{(-1)^{n+k}}{2} \sum_{j=0}^{k} \frac{(-1)^j (j+1)(j+2) s(n+j+2,j+2)}{(k-j)!(n+j+2)!}.$$

Hence (2.5) holds. □

There are some identities involving Stirling numbers and Cauchy numbers of the first kind. For example

$$\sum_{j=0}^{n} a_j S(n,j) = \frac{1}{n+1}, \quad a_n = \sum_{j=0}^{n} \frac{s(n,j)}{j+1}.$$

See [3, 6] for more details. For associated Stirling numbers of the first kind and the Cauchy numbers of the first kind, we have

Theorem 2.4. *For $n \geq 1$ and $k \geq 1$, $s_2(n,k)$ and a_n satisfy*

$$\begin{aligned}
\sum_{j=0}^{n} s_2(n-j+k,k) \binom{n+k}{j} a_j &= (n+k) \sum_{j=1}^{k} \frac{(-1)^{k-j}}{j} \binom{n+k-1}{k-j} s(n+j-1,j-1) \\
&\quad + (-1)^k \binom{n+k}{k} a_n.
\end{aligned} \tag{2.6}$$

The proof of (2.6) is similar to that of (2.1) and is omitted here.

Formula (2.6) relates associated Stirling numbers and Cauchy numbers.

3 Asymptotic Expansion of Certain Sums Involving 2-associated Stirling numbers of the second kind, Bernoulli numbers, and Euler Numbers

We know that it is difficult to compute the accurate values of certain sums involving r-associated Stirling numbers. However, sometimes we can give their asymptotic expansion. In this section, we give asymptotic expansion of certain sums for 2-associated Stirling numbers of the second kind, Bernoulli numbers, and Euler numbers by Darboux's method. We first recall a lemma (see [8]):

Lemma: Assume that $f(t)=\sum_{n\geq 0}a_nt^n$ is an analytic function for $|t|<r$ and with a finite number of algebraic singularities on the circle $|t|=r$. $\alpha_1, \alpha_2, \cdots, \alpha_l$ are singularities of order ω, where ω is the highest order of all singularities. Then

$$a_n=(n^{\omega-1}/\Gamma(\omega))\times\left(\sum_{k=1}^{l}g_k(\alpha_k)\alpha_k^{-n}+\mathrm{o}(r^{-n})\right), \tag{3.1}$$

where $\Gamma(\omega)$ is the gamma function, and

$$g_k(\alpha_k)=\lim_{t\to\alpha_k}(1-(t/\alpha_k))^{\omega}f(t).$$

By using (3.1), we obtain

Theorem 3.1. *Suppose that $n\geq 1$ and $k\geq 1$, where k is fixed. When $n\to\infty$, we have*

$$\sum_{p+q=2n}\frac{S_2(p+k,k)B_q}{(p+k)!q!} \sim \frac{2(-1)^{n+k+1}}{(2\pi)^{2n}k!}, \tag{3.2}$$

$$\sum_{p+q=n}\frac{S_2(p+k,k)E_q}{(p+k)!q!} \sim \frac{2^{n+1}[(2+2i-\pi)^k i^{-n}+(2-2i-\pi)^k(-i)^{-n}]}{\pi^{n+k+1}k!}. \tag{3.3}$$

Proof. Because the proof of (3.3) is similar to that of (3.2), we only prove that (3.2) holds. It is clear that

$$\sum_{p=0}^{\infty}S_2(p+k,k)\frac{t^p}{(p+k)!}\sum_{q=0}^{\infty}B_q\frac{t^q}{q!} = \frac{(e^t-1-t)^k}{k!t^{k-1}(e^t-1)}.$$

Let

$$f(t)=\frac{(e^t-1-t)^k}{k!t^{k-1}(e^t-1)}.$$

Then $f(t)$ is analytic for $|t|<2\pi$ and with two algebraic singularities on the circle $|t|=2\pi$. $\alpha_1=2\pi i$ and $\alpha_2=-2\pi i$ are singularities of order 1. One can compute that

$$\begin{aligned}\lim_{t\to 2\pi i}\left(1-\frac{t}{2\pi i}\right)f(t) &= \lim_{t\to -2\pi i}\left(1+\frac{t}{2\pi i}\right)f(t)\\ &= \frac{(-1)^{k+1}}{k!}.\end{aligned}$$

It follows from (3.1) that

$$\sum_{p+q=n} \frac{S_2(p+k,k)B_q}{(p+k)!q!} = \frac{1}{\Gamma(1)}\left\{\frac{(-1)^{k+1}[(2\pi i)^{-n}+(-2\pi i)^{-n}]}{k!} + \mathrm{o}((2\pi)^{-n})\right\}.$$

Then we have

$$\sum_{p+q=2n} \frac{S_2(p+k,k)B_q}{(p+k)!q!} \sim \frac{(-1)^{k+1}[i^{2n}+(-i)^{2n}]}{(2\pi)^{2n}k!}$$

Hence (3.2) holds. □

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References

[1] A. Z. Broder, The r-Stirling numbers, *Discrete Math.* **49** (1984), 241–259.

[2] L. Carlitz, On some polynomials of Tricomi, *Boll Un. M. Ital.* **13** (1958), 58–64.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, 1974.

[4] R. Ehrenborg, Determinants involving q-Stirling numbers, *Adv. App. Math.* **31** (2003), 630–642.

[5] H. W. Gould, The q-Stirling numbers of the first and second kinds, *Duke Math. J.* **28** (1961), 281–289.

[6] D. Merlini, R. Sprugnoli, and M. C. Verri, The Cauchy numbers, *Discrete Math.* **306** (2006), 1906–1920.

[7] D. Merlini, R. Sprugnoli, and M. C. Verri, The method of coefficients, *Amer. Math. Monthly* **114** (2007), 40–57.

[8] G. Szegö, *Orthogonal polynomials, Amer. Math. Soc. Coll. Pub.*, Vol. 23, rev. ed., 1959.

[9] M. Wachs, D. White, p,q-Stirling numbers and set partion statistics, *J. Combin. Theory Ser. A* **56** (1990), 27–46.

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