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A Symbolic Operator Approach to Power Series Transformation-Expansion Formulas

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Abstract

In this paper we discuss a kind of symbolic operator method by making use of the defined Sheffer-type polynomial sequences and their generalizations, which can be used to construct many power series transformation and expansion formulas. The convergence of the expansions are also discussed.

1 Introduction

The closed form representation of series has been studied extensively. See, for examples, Comtet [1], Ch. Jordan [11], Egorechev [2], Roman-Rota [15], Sofo [17], Wilf [18], Petkovšek-Wilf-Zeilberger's book [13], and the author's recent work with Hsu, Shiue, and Toney [4]. This paper is a sequel to the work [4] and the paper with Hsu and Shiue [5], in which the main results are special cases of Theorem 2 shown below. The object of this paper is to make use of the following defined generalized Sheffer-type polynomial sequences and the classical operators Δ (difference), E (shift), and D (derivative) to construct a method for the summation of power series expansions that appears to have a certain wide scope of applications.

As an important tool in the calculus of finite differences and in combinatorial analysis, the operators E, Δ, D are defined by the following relations.

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$$Ef(t) = f(t+1), \quad \Delta f(t) = f(t+1) - f(t), \quad Df(t) = \frac{d}{dt} f(t).$$

Powers of these operators are defined in the usual way. In particular for any real numbers x, one may define $E^x f(t) = f(t+x)$. Also, the number 1 may be used as an identity operator, viz. $1f(t) \equiv f(t)$. Then it is easy to verify that these operators satisfy the formal relations (*cf.* [11])

$$E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1+\Delta).$$

Note that $E^k f(0) = \left[E^k f(t)\right]_{t=0} = f(k)$, so that $(xE)^k f(0) = f(k)x^k$. This means that $(xE)^k$ with x as a parameter may be used to generate a general term of the series $\sum_{k=0}^{\infty} f(k)x^k$.

Definition 1. Let $A(t)$, $B(t)$, and $g(t)$ be any formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $A(0) = 1$, $B(0) = 1$, $g(0) = 0$, and $g'(0) \neq 0$. Then the polynomials $p_n(x)$ $(n = 0, 1, 2, \cdots)$ defined by the generating function (GF)

$$A(t)B(xg(t)) = \sum_{n\geq 0} p_n(x)t^n \tag{1}$$

are called generalized Sheffer-type polynomials associated with $(A(t), B(t), g(t))$. Accordingly, $p_n(D)$ with $D \equiv d/dt$ is called Sheffer-type differential operator of degree n associated with $A(t)$, $B(t)$, and $g(t)$. In particular, $p_0(D) \equiv I$ is the identity operator due to $p_0(x) = 1$.

In Definition 1, if $B(t) = e^t$, then the defined $\{p_n(x)\}$ is a classical Sheffer-type polynomial sequence associated with $(A(t), g(t)) \equiv (A(t), exp(t), g(t))$. As examples, classical Sheffer-type polynomials include Bernoulli polynomials, Euler polynomials, and Laguerre polynomials generalized by $(A(t), g(t)) = (t/(e^t - 1), t)$, $(2/(e^t + 1), t)$, and $((1-t)^{-p}, t/(t-1))$ $(p > 0)$, respectively.

We call the infinite matrix $[d_{n,k}]_{n,k\geq 0}$ with real entries or complex entries a generalized Riordan matrix (the originally defined Riordan matrices need $g'(0) = 1$) if its kth column satisfies

$$\sum_{n\geq 0} d_{n,k}t^n = A(t)(g(t))^k; \tag{2}$$

that is,

$$d_{n,k} = [t^n]A(t)(g(t))^k,$$

the nth term of the expansion of $A(t)(g(t))^k$. The Riordan matrix is denoted by $(A(t), g(t))$ or $[d_{n,k}]$ described in (2). Then the generalized Sheffer-type polynomial sequence associated with $(A(t), B(t), g(t))$ is the result of the following matrix multiplication

$$[d_{n,k}]\begin{bmatrix}1\\ b_1x\\ b_2x^2\\ \vdots\\ b_nx^n\\ \vdots\end{bmatrix}.$$

If $\left[d_{n,k}=[t^n]A(t)(g(t))^k\right]$ and $\left[c_{n,k}=[t^n]C(t)(f(t))^k\right]$ are two Riordan matrices, and $\{p_n(x)\}$ and $\{q_n(x)\}$ are two corresponding generalized Sheffer-type polynomial sequences associated with $(A(t),B(t),g(t))$ and $(C(t),B(t),f(t))$, respectively, then we can define a umbral composition (*cf.* its special case of $B(t)=e^t$ is given in [14] and [15]) between $\{p_n(x)\}$ and $\{q_n(x)\}$, denoted by $\{p_n(x)\}\#\{q_n(x)\}$. The resulting sequence is the generalized Sheffer-type polynomial sequence associated with $(A(t)C(g(t)),\, B(t), f(g(t))$. Clearly, the sequence $\{p_n(x)\}\#\{q_n(x)\}$ is the result of the following matrix multiplication

$$[d_{n,k}][c_{n,k}]\begin{bmatrix}1\\ b_1x\\ b_2x^2\\ \vdots\\ b_nx^n\\ \vdots\end{bmatrix}.$$

A power series $B(t)=1+\sum_{k=1}^{\infty}b_kt^k$ is said to be *regular* if $b_k\neq 0$ for all $k\geq 1$. Under the composition operator $\#$, it can be proved that all generalized Sheffer-type polynomial sequences associated with a regular $B(t)$ form a group, called the generalized Sheffer group associated with $B(t)$. Its verification is analogous to the classical Sheffer group associated with e^t established in [14] (see also in [6]).

In [6], the author established the isomorphism between the classic Sheffer group and the Riordan group based on the following bijective mapping: $\theta:[d_{n,k}]\mapsto\{p_n(x)\}$ or $\theta:(A(t),g(t))\mapsto\{p_n(x)\}$, i.e.,

$$\theta([d_{n,k}]_{n\geq k\geq 0}):=\sum_{j=0}^{n}d_{n,j}x^j/j!=[d_{n,k}]_{n\geq k\geq 0}X, \tag{3}$$

for fixed n, where $X=(1,x,x^2/2!,\ldots)^T$, or equivalently,

$$\theta((A(t),g(t)):=[t^n]A(t)e^{xg(t)} \tag{4}$$

It is clear that $(1,t)$, the identity Riordan array, maps to the identity Sheffer-type polynomial sequence $\{p_n(x)\equiv x^n/(n!)\}_{n\geq 0}$. From the definitions (2) we immediately know that

$$p_n(x)=[t^n]A(t)e^{xg(t)}\ \text{ if and only if }\ d_{n,k}=[t^n]A(t)\left(g(t)\right)^k. \tag{5}$$

Therefore, the Riordan matrices from the sequences shown in the On-Line Encyclopedia of Integer Sequences (OLEIS) also present the coefficients of the corresponding classic Sheffer-type polynomials. For example, sequence A129652

$$1, 1, 1, 3, 2, 1, 13, 9, 3, 1, 73, 52, 18, 4, 1, \ldots$$

presents both Riordan array $(e^{x/(1-x)}, x)$ and the corresponding Sheffer-type polynomial sequence:

$$\begin{aligned}
&p_0(x) = 1\\
&p_1(x) = 1 + x\\
&p_2(x) = 3 + 2x + x^2/2!\\
&p_3(x) = 13 + 9x + 3x^2/2! + x^3/3! \ldots
\end{aligned}$$

The row sums of Riordan array $(e^{x/(1-x)}, x)$, i.e., the polynomial values at $x = 1$, are A052844, while diagonal sums of the array are A129653 (*cf.* [16]). Other examples, including, A000262, A084358, A133289, etc., can be also found in [16].

Suppose that $\Phi(t)$ is an analytic function of t or a formal power series in t, say

$$\Phi(t) = \sum_{k=0}^{\infty} c_k t^k, \qquad c_k = [t^k]\Phi(t), \tag{6}$$

where c_k can be real or complex numbers. Then, formally we have a sum of general form

$$\Phi(xE)f(0) = \sum_{k=0}^{\infty} c_k f(k) x^k. \tag{7}$$

In certain cases, $\Phi(\alpha + \beta)$ or $\Phi(\alpha\beta)$ can be decomposed into something having a power series in β as a part. Accordingly the operator $\Phi(xE) = \Phi(x + x\Delta) = \Phi(xe^D)$ can be expressed as some power series involving operators Δ^k or D^k's. Then it may be possible to compute the right-hand side of (7) by means of operator-series in Δ^k or D^k's. This idea could be readily applied to various elementary functions $\Phi(t)$. Therefore, we can obtain various transformation formulas as well as series expansion formulas for the series of the form (7).

It is well-known that the Eulerian fraction is a powerful tool to study the Eulerian polynomial, Euler function and its generalization, Jordan function (*cf.* [1]).

The classical Eulerian fraction can be expressed in the form

$$\alpha_m(x) = \frac{A_m(x)}{(1-x)^{m+1}} \quad (x \neq 1), \tag{8}$$

where $A_m(x)$ is the mth degree Eulerian polynomial of the form

$$A_m(x) = \sum_{j=0}^{m} j! S(m, j) x^j (1-x)^{m-j}, \tag{9}$$

$S(m,j)$ being Stirling numbers of the second kind, i.e., $j!S(m,j) = [\Delta^j t^m]_{t=0}$, which is also denoted by $\left\{ \begin{matrix} m \\ j \end{matrix} \right\}$. Evidently $\alpha_m(x)$ can be written in the form (*cf.* [4])

$$\alpha_m(x) = \sum_{j=0}^{m} \frac{j!S(m,j)x^j}{(1-x)^{j+1}}. \tag{10}$$

In order to express some new formulas for certain general types of power series, we need to introduce the following extension of Euler fraction, denoted by $\alpha_n(x, A(x), B(x), g(x))$, using the generalized Sheffer-type polynomials associated with analytic functions or power series $A(t)$, $B(t)$, and $g(t)$, which satisfy conditions in Definition 1.

$$\alpha_n(x, A(x), B(x), g(x)) := \sum_{j=0}^{n} \sum_{\ell=j}^{\infty} j! \binom{\ell}{j} p_\ell(x) S(n,j), \tag{11}$$

where $p_\ell(x)$ is the generalized Sheffer-type polynomial of degree ℓ defined in Definition 1. In particular, if $A(x) = 1$ and $g(x) = x$, then $p_\ell(x) = B^{(\ell)}(0)x^\ell/\ell!$, and the generalized Euler fraction is hence

$$\alpha_n(x, 1, B(x), x) = \sum_{j=0}^{n} S(n,j) B^{(j)}(x) x^j \tag{12}$$

because

$$\sum_{\ell=j}^{\infty} j! \binom{\ell}{j} p_\ell(x) = \sum_{\ell=j}^{\infty} j! \binom{\ell}{j} B^{(\ell)}(0) \frac{x^\ell}{\ell!} = B^{(j)}(x)x^j.$$

Obviously, $\alpha_m(x)$ defined by (8) can be presented as

$$\alpha_m(x) = A_m(x, 1, (1-x)^{-1}, x).$$

From (11), two kinds of generalized Eulerian fractions in terms of $A(x) = 1$, $g(x) = x$, $B(x) = (1+x)^a$ and $B(x) = (1-x)^{-a-1}$, respectively, with real number a as a parameter, can be introduced respectively, namely

$$A_n(x, 1, (1+x)^a, x) = \sum_{j=0}^{n} \binom{a}{j} \frac{j!S(n,j)x^j}{(1+x)^{j-a}} \tag{13}$$

for $x \neq -1$, and

$$A_n(x, 1, (1-x)^{-a-1}, x) = \sum_{j=0}^{n} \binom{a+j}{j} \frac{j!S(n,j)x^j}{(1-x)^{a+j+1}} \tag{14}$$

for $x \neq 1$. These two generalized Eulerian fractions were given in [5]. Another extension of the classical Eulerian fraction is presented in [10].

Two major transformation and expansion formulas and their applications will be displayed in next section, and the convergence of the series in the formulas is presented in 3.

2 Series transformation-expansion formulas

Theorem 2. *Let $\{f(k)\}$ be a sequence of numbers (in $\mathbb{R}$ or $\mathbb{C}$), and let $h(t)$ be infinitely differentiable. Assume $A(t)$, $B(t)$, and $g(t)$ are analytic functions in a disk centered at the origin or power series defined as in Definition 1 and $\{p_n(x)\}$ is the generalized Sheffer-type polynomial sequence associated with $A(x)$, $B(x)$, and $g(x)$, we have formally*

$$\sum_{n=0}^{\infty} f(n)p_n(x) = \sum_{n=0}^{\infty} \Delta^n f(0)\left(\sum_{\ell=n}^{\infty}\binom{\ell}{n}p_\ell(x)\right) \tag{1}$$

$$\sum_{n=0}^{\infty} h(n)p_n(x) = \sum_{n=0}^{\infty}\frac{1}{n!}h^{(n)}(0)\alpha_n(x,A(x),B(x),g(x)), \tag{2}$$

where $S(n,j)$ is the Stirling numbers of the second kind, and $\alpha_n(x,A(x),\ B(x),g(x)) = \sum_{j=0}^{n}\sum_{\ell=j}^{\infty} j!\binom{\ell}{j}p_\ell(x)S(n,j)$ is the generalized Eulerian fraction defined as in (11).

In particular, if $g(t)=t$, then the transformation and expansion formulas (1) becomes to

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{f(n)}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}A^{(n-\ell)}(0)B^{(\ell)}(0)x^\ell\\ = &\sum_{n=0}^{\infty}\frac{\Delta^n f(0)}{n!}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}A^{(n-\ell)}(1)B^{(\ell)}(x)x^\ell\right).\end{aligned} \tag{3}$$

Proof. Applying the operator $A(E)B(xg(E))$ to $f(t)$ at $t=0$, where E is the shift operator, we obtain the left-hand side of (1).

On the other hand, we have

$$\begin{aligned}&A(E)B(xg(E))f(t)|_{t=0} = A(1+\Delta)B(xg(1+\Delta))f(t)|_{t=0}\\ = &\sum_{\ell=0}^{\infty}p_\ell(x)\ (1+\Delta)^\ell f(t)\big|_{t=0} = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell}\binom{\ell}{n}p_\ell(x)\Delta^n f(0),\end{aligned}$$

which implies the double sum on the right-hand side of (1).

Similarly, for the infinitely differentiable function $h(t)$, we can present

$$\begin{aligned}&A(E)B(xg(E))h(t)|_{t=0} = A(e^D)B(xg(e^D))h(t)\big|_{t=0}\\ = &\sum_{\ell=0}^{\infty}p_\ell(x)e^{\ell D}\ h(t)|_{t=0} = \sum_{\ell=0}^{\infty}p_\ell(x)\sum_{n=0}^{\infty}\frac{\ell^n}{n!}h^{(n)}(0)\\ = &\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{\infty}p_\ell(x)\ell^n\right)\frac{h^{(n)}(0)}{n!}\end{aligned}$$

By applying (1) to the internal sum of the rightmost side of the above equation for $f(t)=t^k$ and noting $S(k,j)=\left(\Delta^j t^k\big|_{t=0}\right)/j!$, we obtain

$$\begin{aligned}
& A(E)B(xg(E))h(t)|_{t=0} \\
= & \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\sum_{\ell=j}^{\infty}\binom{\ell}{j} p_\ell(x)\Delta^j t^n\big|_{t=0}\right)\frac{h^{(n)}(0)}{n!} \\
= & \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{\ell=j}^{\infty}\binom{\ell}{j} p_\ell(x) j! S(n,j)\right)\frac{h^{(n)}(0)}{n!} \\
= & \sum_{n=0}^{\infty}\frac{h^{(n)}(0)}{n!}\alpha_n(x,A(x),B(x),g(x)).
\end{aligned}$$

This completes the proof of the theorem.

If $g(t)=t$, then we have formally $p_n(x)=\sum_{\ell=0}^{n} A^{(n-\ell)}(0)B^{(\ell)}(0)x^\ell/((n-\ell)!\ell!)$ and

$$\begin{aligned}
& A(E)B(xE)f(t)|_{t=0} \\
= & A(1+\Delta)B(x+x\Delta)f(t)|_{t=0} \\
= & \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\frac{A^{(n-\ell)}(1)}{(n-\ell)!}\frac{B^{(\ell)}(x)}{\ell!}(x\Delta)^\ell\right) f(t)\big|_{t=0},
\end{aligned}$$

which can be written as the double sum on the right-hand side of (3). □

Remark 3. When $f(t)$ and $h(t)$ are polynomials, the right-hand sides of (1) and (2) are finite sums, which can be considered as the closed forms of the corresponding left-hand side series. For this reason, we call formulas (1) and (2) the series transformation and expansion (or transformation-expansion) formulas. Thus, for the rth degree polynomial $\phi(t)$, from (1) and (2) we have two expansion formulas,

$$\sum_{n=0}^{\infty}\phi(n)p_n(x)=\sum_{n=0}^{r}\Delta^n\phi(0)\left(\sum_{\ell=n}^{\infty}\binom{\ell}{n}p_\ell(x)\right) \tag{4}$$

$$\sum_{n=0}^{\infty}\phi(n)p_n(x)=\sum_{n=0}^{r}\frac{1}{n!}\phi^{(n)}(0)\alpha_n(x,A(x),B(x),g(x)), \tag{5}$$

where the right-hand sides can be considered as the GF's of $\{\phi(n)p_n(x)\}$.

Corollary 4. *Let $\{\alpha_n(x,A(x),B(x),g(x))\}$ be the generalized Eulerian fraction sequence defined by (11). Then the exponential generating function of the sequence is $A(e^t)B(xg(e^t))$. In particular, the exponential generating function of sequence $\{\alpha_n(x,1,B(x),x)\}$ is $B(xe^t)$.*

Proof. The exponential GF of $\{\alpha_n(x,A(x),B(x),g(x))\}$ can be written as

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \alpha_n(x, A(x), B(x), g(x)) \frac{t^n}{n!} \\
=\ & \sum_{n=0}^{\infty} \sum_{j=0}^{n} S(n,j) \sum_{\ell=j}^{\infty} \binom{\ell}{j} p_\ell(x) \frac{t^n}{n!} \\
=\ & \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{j!}\, \Delta^j u^n \big|_{u=0} \sum_{\ell=j}^{\infty} \binom{\ell}{j} p_\ell(x) \right) \frac{t^n}{n!}.
\end{aligned}
$$

Applying formula (1) for $f(j) = j^n$ into the double sum in the above parentheses yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \alpha_n(x, A(x), B(x), g(x)) \frac{t^n}{n!} \\
=\ & \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} j^n p_j(x) \right) \frac{t^n}{n!} \\
=\ & \sum_{j=0}^{\infty} p_j(x) e^{jt} = A(e^t) B(x g(e^t)).
\end{aligned}
$$

Here, the last step is due to Definition 1. $\square$

We now give two special cases of Theorem 2.

Corollary 5. *Let $\{f(k)\}$ be a sequence of numbers (in $\mathbb{R}$ or $\mathbb{C}$), and let $B(t)$ and $g(t)$ be infinitely differentiable on $[0, \infty)$. Then we have formally*

$$
\sum_{n=0}^{\infty} f(n) B^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \Delta^n f(0) B^{(n)}(x) \frac{x^n}{n!} \tag{6}
$$

$$
\sum_{n=0}^{\infty} g(n) B^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \alpha_n(x, 1, B(x), x), \tag{7}
$$

where $\alpha_n(x, 1, B(x), x)$ is the generalized Eulerian fraction defined by (12).

Proof. By setting $A(t) = 1$ and $g(t) = t$ into transformation-expansion formulas (1) and (2), we obtain formally $p_\ell(x) = B^{(\ell)}(0) x^\ell / \ell!$. Thus, the modified formulas (1) and (2) are respectively (6) and (7). $\square$

Example 6. Setting respectively $B(t) = (1-t)^{-m-1}$ $(t \neq 1)$ and $B(t) = (1+t)^m$ $(t \neq -1)$ into (6) and (7) yield the transformation-expansion formulas

$$\sum_{k=0}^{\infty}\binom{m+k}{k}f(k)x^k=\sum_{k=0}^{\infty}\binom{m+k}{k}\frac{x^k}{(1-x)^{m+k+1}}\Delta^k f(0) \tag{8}$$

$$\sum_{k=0}^{\infty}\binom{m+k}{k}h(k)x^k=\sum_{k=0}^{\infty}\frac{\alpha_k(x,1,(1-x)^{-m-1},x)}{k!}D^k h(0) \tag{9}$$

and

$$\sum_{k=0}^{\infty}\binom{m}{k}f(k)x^k=\sum_{k=0}^{\infty}\binom{m}{k}\frac{x^k}{(1+x)^{k-m}}\Delta^k f(0) \tag{10}$$

$$\sum_{k=0}^{\infty}\binom{m}{k}h(k)x^k=\sum_{k=0}^{\infty}\frac{\alpha_k(x,1,(1+x)^{m},x)}{k!}D^k h(0), \tag{11}$$

respectively, where $\alpha_k(x,1,(1-x)^{-m-1},x)$ and $\alpha_k(x,1,(1+x)^m,x)$ are defined in (14) and (13), respectively.

By substituting $m=0$ in formulas (8) and (9) or applying transform $x\mapsto -x$ and staking $m=-1$ in formulas (10) and (11), we obtain

$$\sum_{k=0}^{\infty}f(k)x^k=\sum_{k=0}^{\infty}\frac{x^k}{(1-x)^{k+1}}\Delta^k f(0) \tag{12}$$

$$\sum_{k=0}^{\infty}h(k)x^k=\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{k!}D^k h(0), \tag{13}$$

where $\alpha_k(x)$ is defined by (8) and (9). (12) and (13) were shown as in [4]. And (12) is an extension of the following well-known Euler series transform that can be found by setting $x=-1$ into (12):

$$\sum_{n=0}^{\infty}(-1)^n f(n)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\Delta^n f(0).$$

By applying operator E^m and multiplying x^m on the both sides of formula (8), we obtain its alternative form as follows:

$$\sum_{k=m}^{\infty}\binom{k}{m}f(k)x^k=\sum_{k=0}^{\infty}\binom{m+k}{m}\frac{x^{k+m}}{(1-x)^{m+k+1}}\Delta^k f(m) \tag{14}$$

Example 7. Let λ and θ be any real numbers. The generalized falling factorial $(t+\lambda|\theta)_p$ is usually defined by

$$(t+\lambda|\theta)_p=\Pi_{j=0}^{p-1}(t+\lambda-j\theta),\ (p\geq 1),\ (t+\lambda|\theta)_0=1.$$

It is known that Howard's degenerate weighted Stirling numbers (*cf.* [8]) may be defined by the finite differences of $(t+\lambda|\theta)_p$ at $t=0$:

$$S(p,k,\lambda|\theta) := \frac{1}{k!}\left[\Delta^k (t+\lambda|\theta)_p\right]_{t=0}.$$

Then, using (14), (8), and (10) with $f(t) = (t+\lambda|\theta)_p$, we get

$$\sum_{k=m}^{\infty} \binom{k}{m} (k+\lambda|\theta)_p x^k = \sum_{k=0}^{p} \binom{m+k}{k} \frac{k! S(p,k,\lambda|\theta) x^{k+m}}{(1-x)^{m+k+1}}, \tag{15}$$

$$\sum_{k=0}^{\infty} \binom{m+k}{k} (k+\lambda|\theta)_p x^k = \sum_{k=0}^{p} \binom{m+k}{k} \frac{k! S(p,k,\lambda|\theta) x^{k}}{(1-x)^{m+k+1}}, \tag{16}$$

$$\sum_{k=0}^{\infty} \binom{m}{k} (k+\lambda|\theta)_p x^k = \sum_{k=0}^{p} \binom{m}{k} \frac{k! S(p,k,\lambda|\theta) x^{k}}{(1+x)^{k-m}}. \tag{17}$$

The particular case of (17) with $x=1$, namely,

$$\sum_{k=0}^{m} \binom{m}{k} (k+\lambda|\theta)_p = \sum_{k=0}^{p} \binom{m}{k} 2^{m-k} k! S(p,k,\lambda|\theta),$$

was given in formula (35) of [10], and the particular case of (16) with $m=0$ was considered in [9]. It is also obvious that the classical Euler's summation formula for the arithmetic-geometric series (*cf.* for example, [3, Lemma 2.7]) is implied by (16) with $\lambda=\theta=0$ and $m=0$, or by (17) with $\lambda=\theta=0$, $m=-1$, $x\mapsto -x$.

Some other series transformation-expansion formulas can be constructed formally from the above formulas by using integration or differentiation. For instance, taking the integral on the bother sides of (12) we obtain

$$\sum_{k=1}^{\infty} \frac{f(k)x^k}{k} = -f(0)\ln(1-x) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{1-x}\right)^k \Delta^k f(0), \tag{18}$$

which can also be considered as a special case of (6) for $B(t) = -\ln(1-t)$.

Using the substituting rule $t\mapsto D$ into (6) and applying the resulting operator to an infinitely differentiable function f with the similar argument shown in Theorem 2, we have

$$A(D)B(xg(D))f(t)|_{t=0} = \sum_{n=0}^{\infty} p_n(x) f^{(n)}(0). \tag{19}$$

We now specify A, B and g in (19) to establish the following corollary.

Corollary 8. *If* $(A(t), B(t), g(t)) = (t/(e^t-1), e^t, t)$, $(2/(e^t+1), e^t, t)$, $(t/(\ln(t+1)), e^t, \ln(t+1))$, *then from (19) we have*

$$Df(x+y) = \sum_{n=0}^{\infty} \phi_n(x) D^n \Delta f(y) \tag{20}$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \Delta^n f(x) = \sum_{n=0}^{\infty} E_n(x) D^n f(0) \tag{21}$$

$$\Delta f(x+y) = \sum_{n=0}^{\infty} \psi_n(x) \Delta^n D f(y), \tag{22}$$

where $\phi(x)$ and $\psi(x)$ are Bernoulli polynomials of the first and second kind, respectively, and $E_n(x)$ are Euler polynomials.

Proof. If $(A(t), B(t), g(t)) = (t/(e^t - 1), e^t, t)$, $(2/(e^t + 1), e^t, t)$, $(t/(\ln(t+1)), e^t, \ln(t+1))$, then the corresponding Sheffer-type polynomials are $p_n(x) = \phi_n(x)$, $E_n(x)$, and $\psi_n(x)$, respectively (*cf.* [11, pp. 250, 309, 279]), and the corresponding operators on the left-hand side of (19) for the different $(A(t), B(t), g(t))$ become respectively $(De^{xD}/(e^D - 1) = DE^x/\Delta$,

$$\frac{2e^{xD}}{e^D + 1} = \frac{2E^x}{\Delta + 2} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Delta}{2}\right)^n,$$

and $\Delta(\Delta+1)^x/(\ln(\Delta+1)) = \Delta E^x/D$. Hence, the proof of the theorem is complete. $\square$

The results in Corollary 8 were given in [7] by using different treatment for each individual formula while our method described here can be considered as a uniform approach, which can be used to find more transformation and expansion formulas.

It is obvious that for $x = 0$, formulas (20)-(22) are specified as

$$Df(y) = \sum_{n=0}^{\infty} \frac{B_n^{(1)}}{n!} D^n \Delta f(y) \tag{23}$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \Delta^n f(0) = \sum_{n=0}^{\infty} e_n D^n f(0) \tag{24}$$

$$\Delta f(y) = \sum_{n=0}^{\infty} b_n \Delta^n D f(y), \tag{25}$$

where $B_n^{(1)} = n!\phi_n(0)$ is the first order generalized Bernoulli number, and $e_n = E_n(0)$, and $b_n = \psi_n(0)$.

3 Convergence of the series transformation-expansions

As may be conceived, various formulas displayed in the list in Section 2 may be employed to construct some summation formulas with estimable remainders (*cf.* the proof of Theorem

11 below). In what follows convergence problems related to the series expansions in Section 2 will be investigated.

We now establish convergence conditions for the series expansions in (6) and (7). Suppose that $\{f(k)\}$ and $\{h(k)\}$ are bounded sequences (say $|f(k)| < M$ and $|h(k)| < M$ for all k), and that $g(z)$ is analytic for $|z| < \rho$. Then it is follows that the left-hand sides of (6) and (7) as well as the right-hand sides of (6) and (7) are absolutely convergent series for $|x| < \rho$. Hence, we have the following convergence theorem.

Theorem 9. *If $\{f(k)\}$ and $\{g(k)\}$ are bounded sequences, and that $B(z)$ is analytic for $|z| < \rho$ for some positive real number ρ, then the series expansions in (6) and (7) converge absolutely for all $|x| < \rho$.*

The convergence on the general case where $\{f(k)\}$ is not bounded presents some complicated situation. The next theorem gives a discussion for the series transformation-expansion formulas shown in (12), (13), and (18), and general way may be developed through it, which is left for the interested reader to consider.

Theorem 10. *Let $\{f(k)\}$ be a sequence of numbers (in $\mathbb{R}$ or $\mathbb{C}$), and denote $\theta := \overline{\lim}_{k\to\infty} |f(k)|^{1/k}$. Then the series expansions in (12), (13), and (18) are convergent for all nonzero x satisfying $|x|\theta < 1$.*

Proof. Substituting the expression of $\alpha_k(x)$ defined by (10) into (12) and noting $j! \sum_{k=j}^{\infty} S(k,j) \frac{D^k}{k!} = (e^D - 1)^j = \Delta^j$ (*cf.* [4]) yields (13). More precisely,

$$\begin{aligned}
\sum_{k=0}^{\infty} g(k) x^k &= \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} D^k g(0) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{j!}{k!} S(k,j) \frac{x^j}{(1-x)^{j+1}} D^k g(0) \\
&= \sum_{j=0}^{\infty} \frac{x^j}{(1-x)^{j+1}} \left(j! \sum_{k=j}^{\infty} S(k,j) \frac{D^k}{k!} \right) g(0) \\
&= \sum_{j=0}^{\infty} \frac{x^j}{(1-x)^{j+1}} \Delta^j g(0).
\end{aligned}$$

Hence, we only need to show the convergence of expansions in (12) and (18).

In accordance with Cauchy's root test, the convergence of the series on the left-hand side of (12) and (18) is obvious because of the condition $|x|\theta < 1$. To prove the convergence of the series expansion on the right-hand side of (12), we choose $\rho > \theta$ such that $\theta|x| < \rho|x| < 1$. Thus for large k we have $|f(k)|^{\frac{1}{k}} < \rho$. Consequently,

$$\left|\Delta^k f(0)\right|^{\frac{1}{k}} \le \left(\sum_{j=0}^{k} \binom{k}{j} |f(j)| \right)^{\frac{1}{k}} < (2)^{\frac{1}{k}} \rho \to \rho$$

as $k \to \infty$. Therefore, for every $x \in (-1/\theta, 0)$

$$\begin{aligned} & \overline{\lim_{k\to\infty}} \left| \frac{1}{k} \left(\frac{x}{1-x} \right)^k \Delta^k f(0) \right|^{\frac{1}{k}} \\ = & \overline{\lim_{k\to\infty}} \left| \frac{x}{1-x} \right| \left| \Delta^k f(0) \right|^{\frac{1}{k}} \leq \rho \left| \frac{x}{1-x} \right| < \rho |x| < 1. \end{aligned}$$

Hence, from the root test, the series expansion on the right-hand side of (18) is convergent. Similarly, the expansion on the right-hand side of (12) converges as well. This completes the proof of the theorem. □

To extend the convergence intervals of the series expansions in (12) and (13), we need more precise estimation as follows.

Theorem 11. *Let $\{f(k)\}$ be a sequence of numbers (in $\mathbb{R}$ or $\mathbb{C}$), and let $\theta = \overline{\lim_{k\to\infty}} |f(k)|^{1/k}$. Then for any given x with $x \neq 0$ we have the convergent expressions (12) and (13) provided that $|x|\theta < 1$.*

Proof. As we mentioned in the proof of Theorem 10, it is sufficient to show the convergence of (12). For this purpose, we now find a remainder of the expansion of (12) as follows. Formally, we have

$$\begin{aligned} & (1 - xE)^{-1} = (1 - x - x\Delta)^{-1} \\ = & (1-x)^{-1} \left(1 - \frac{x}{1-x} \Delta \right)^{-1} \\ = & (1-x)^{-1} \left\{ \sum_{\ell=0}^{n-1} \left(\frac{x}{1-x} \right)^{\ell} \Delta^{\ell} + \frac{\left(\frac{x}{1-x} \Delta \right)^n}{1 - \left(\frac{x}{1-x} \Delta \right)} \right\} \\ = & \sum_{\ell=0}^{n-1} \frac{x^{\ell}}{(1-x)^{\ell+1}} \Delta^{\ell} + \left(\frac{x}{1-x} \right)^n \frac{\Delta^n}{1 - xE} \\ = & \sum_{\ell=0}^{n-1} \frac{x^{\ell}}{(1-x)^{\ell+1}} \Delta^{\ell} + \left(\frac{x}{1-x} \right)^n \sum_{\ell=0}^{\infty} x^{\ell} E^{\ell} \Delta^n. \end{aligned}$$

Since $E^{\ell} \Delta^n f(0) = \Delta^n E^{\ell} f(0) = \Delta^n f(\ell)$, applying operator $(1 - xE)^{-1}$ and the rightmost operator shown above to $f(t)|_{t=0}$, respectively, yields

$$\begin{aligned} & (1 - xE)^{-1} f(t)\big|_{t=0} = \sum_{k=0}^{\infty} f(k) x^k \\ = & \sum_{k=0}^{n-1} \frac{x^k}{(1-x)^{k+1}} \Delta^k f(0) + \frac{x^n}{(1-x)^n} \sum_{\ell=0}^{\infty} x^{\ell} \Delta^n f(\ell). \end{aligned} \tag{1}$$

Since $|x|\theta < 1$ ($x \neq 0$), the convergence of the series expansion on the left-hand side of (12) or (1) is obtain. To prove the convergence of the right-hand side of (1), i.e., the remainder form of (12), it is sufficient to show that $\sum_{\ell=0}^{\infty} x^\ell \Delta^n f(\ell)$ is absolutely convergent. Choose $\rho > \theta$ such that

$$\theta|x| < \rho|x| < 1.$$

Thus, for large k we have $|f(k)|^{1/k} < \rho$, i.e., $|f(k)| < \rho^k$. Consequently we have, for large ℓ

$$\begin{aligned} |\Delta^n f(\ell)|^{1/\ell} &\leq \left(\sum_{j=0}^{n}\binom{n}{j}|f(\ell+j)|\right)^{1/\ell} \leq \left(\sum_{j=0}^{n}\binom{n}{j}\rho^{\ell+j}\right)^{1/\ell} \\ &= \rho(1+\rho)^{n/\ell} \to \rho \end{aligned}$$

as $\ell \to \infty$. Thus

$$\overline{\lim_{\ell\to\infty}} \left|x^\ell \Delta^n f(\ell)\right|^{1/\ell} \leq \rho|x| < 1,$$

so that the series on the right-hand side of (1) or (12) is also convergent absolutely under the given conditions. □

A sequence $\{a_n\}$ is called a null sequence if for any given positive number ϵ, there exists and integer N such that every $n > N$ implies $|a_n| < \epsilon$. [12] (*cf.* Theorem 4 in Section 43) pointed out that a linear combination of $\{a_n\}$, denoted by $\{a'_n = \sum_{k=0}^{n} c_{n,k}a_k\}$, is also a null sequence if the coefficient set $\{c_{n,k}\}_{0\leq k\leq n}$ ($n = 0, 1, 2, \ldots$) satisfies the following two conditions:

(i) Every column contains a null sequence, i.e., for fixed $k \geq 0$, $c_{n,k} \to 0$ when $n \to \infty$.

(ii) There exists a constant K such that the sum $|a_{n,0}| + |a_{n,1}| + \cdots + |a_{n,n}| < K$ for every n.

By using this claim of the null sequence, we can have the following convergence result of the series expansions in (12) and (13).

Theorem 12. *Suppose that $\{f(n)\}$ is a given sequence of numbers (real or complex) such that $\sum_{n=0}^{\infty} f(n)x^n$ is convergent for every $x \in \Omega$ with $\Omega \cap (-\infty, 0) \neq \phi$. Then the series expressions on the right-hand sides of (12) and (13) converge for every $x \in \Omega \cap (-\infty, 0)$.*

Proof. We write the remainder of expression (1) as follows.

$$
\begin{aligned}
R_n &:= \frac{x^n}{(1-x)^n}\sum_{\ell=0}^{\infty} x^\ell \Delta^n f(\ell)\\
&= \frac{x^n}{(1-x)^n}\sum_{\ell=0}^{\infty}\sum_{j=0}^{n}(-1)^{n-j}x^\ell\binom{n}{j}f(j+\ell)\\
&= \frac{(-x)^n}{(1-x)^n}\sum_{j=0}^{n}\binom{n}{j}\sum_{\ell=0}^{\infty}(-1)^j x^\ell\binom{n}{j}f(j+\ell)\\
&= \frac{(-x)^n}{(1-x)^n}\sum_{j=0}^{n}(-x)^{-j}\binom{n}{j}\sum_{\ell=j}^{\infty}x^\ell f(\ell) = \frac{(-x)^n}{(1-x)^n}\sum_{j=0}^{n}(-x)^{-j}\binom{n}{j}x_j,
\end{aligned}
$$

where $x_j = \sum_{\ell=j}^{\infty} x^\ell f(\ell)$ $(0 \le j \le n)$. Since $\sum_{\ell=0}^{\infty} x^\ell f(\ell)$ converges, x_j is the term of a null sequence, applying the result on the linear combination of a null sequence shown above, we find the coefficients of x_j in the linear combination of the rightmost sum,

$$
c_{n,j} := \frac{(-x)^n}{(1-x)^n}(-x)^{-j}\binom{n}{j}
$$

satisfy the following two conditions: (1) If j is fixed, we have $c_{n,j} \to 0$ as $n \to \infty$ because

$$
|c_{n,j}| = \frac{|x|^{n-j}}{(1-x)^n}\binom{n}{j} < \frac{n^j}{(1-x)^n}
$$

and $1/(1-x) < 1$ for every $x \in \Omega \cap [-1, 0)$; and

$$
|a_{n,j}| = \frac{|x|^{n-j}}{(1-x)^n}\binom{n}{j} < \left(\frac{|x|}{1-x}\right)^n n^j
$$

and $|x/(1-x)| < 1$ for every $x \in \Omega \cap (-\infty, -1)$. (2) For every n and for every $x \in \Omega \cap (-\infty, 0)$ we have

$$
\sum_{j=0}^{n} |a_{n,j}| = \frac{1}{(1-x)^n}\sum_{j=0}^{n}(-x)^{n-j}\binom{n}{j} = 1.
$$

Therefore, Theorem 4 in Section 43 of [12] shows that R_n is also the term of a null sequence, so the series on the right-hand side of (12) converges for every $x \in \Omega \cap (-\infty, 0)$. In addition, the convergence of the right-hand series expansion of (13) is followed. □

We now discuss the convergence of the series expansions in (20)-(22). Actually, we may sort the series transformation-expansion formulas associated with $(A(t), B(t), t)$ into two classes. The first class includes only either the sum $\sum \beta_k D^k f$ or the sum $\sum \gamma_k E^k f$ in the formulas such as (20) and (22). The second class includes the sums $\sum \beta_k D^k f$ and/or $\sum \gamma_k E^k f$ on both sides of the transformation-expansion formulas like (21). We may establish the following convergence theorem.

Theorem 13. *For the first class series expansions associated with $\sum \beta_k\, D^k f$ (or $\sum \gamma_k E^k f$) defined above, their absolute convergence are ensured if $\overline{\lim}_{k\to\infty} \left|D^k f\right|^{1/k} < 1$ (or $\overline{\lim}_{k\to\infty} \left|E^k f\right|^{1/k} < 1$) and $|\beta_k| \le 1$ (or $|\gamma_k| \le 1$).*

The second class series expansions defined above absolutely converge if $\overline{\lim}_{k\to\infty} \left|D^k f\right|^{1/k} < 1$, $|\beta_k| \le 1$, and $|\gamma_k| \le (1/(e-1))^k$.

Proof. The first half of the theorem is easy to be verified by using the root test.

To prove the second half, we need the following statement: If $f \in C^\infty$, then $\overline{\lim}_{k\to\infty} \left|D^k f(y)\right|^{1/k} < a$, a positive real number, implies

$$\overline{\lim_{k\to\infty}} \left|\Delta^k f(y)\right|^{1/k} < e^a - 1. \tag{2}$$

In fact, if we denote $\overline{\lim}_{n\to\infty} |D^n f(y)|^{1/n} = \theta$, then there exists a number γ such that $\theta < \gamma < a$. Thus for large enough n we have $|D^n f(y)|^{1/n} < \gamma$ or $|D^n f(y)| < \gamma^n$.

Noting $S(n,m) \ge 0$ and $|D^n f(y)| < \gamma^n$, we obtain

$$\begin{aligned} \left|\Delta^k f(y)\right| &= \left|\sum_{n\ge k} \frac{k!}{n!} S(n,k) D^n f(y)\right| \le \sum_{n\ge k} \frac{k!}{n!} S(n,k)\, |D^n f(y)| \\ &\le \sum_{n\ge k} \frac{k!}{n!} S(n,k)\gamma^n = (e^\gamma - 1)^k < (e^a - 1)^k. \end{aligned}$$

Here the rightmost equality is from Jordan [11, p. 176].

Therefore, $\overline{\lim}_{k\to\infty} \left|D^k f\right|^{1/k} < 1$ implies that $\overline{\lim}_{k\to\infty} \left|\Delta^k f(y)\right|^{1/k} < e - 1$. Those two inequalities and the conditions for the coefficients $\{\beta_k\}$ and $\{\gamma_k\}$ confirm the absolute convergence of the second class series expansions with the root test. □

As a corollary of Theorem 13, we now establish the convergence results of the series expansions in (20)-(22).

Corollary 14. *For given $f \in C^\infty$ and $x, y \in \mathbb{R}$, the absolute convergence of the series expansion (20) is ensured by the condition*

$$\overline{\lim_{k\to\infty}} \left|\Delta D^k f(y)\right|^{1/k} < 1. \tag{3}$$

Similarly, the absolute convergence of the series expansion (21) and (22) are ensured by the conditions

$$\overline{\lim_{k\to\infty}} \left|\Delta^k D f(y)\right|^{1/k} < 1 \tag{4}$$

and

$$\overline{\lim_{k\to\infty}} \left|D^k f(y)\right|^{1/k} < 1, \tag{5}$$

respectively.

Proof. From Theorem 13, it is sufficient to show that

$$\varlimsup_{k\to\infty} |\phi_k(x)|^{1/k} \le 1, \tag{6}$$

$$\varlimsup_{k\to\infty} |E_k(x)|^{1/k} \le 1, \tag{7}$$

and

$$\varlimsup_{k\to\infty} |\psi_k(x)|^{1/k} \le 1, \tag{8}$$

which will be proved below from the basic properties of $\phi_k(x)$, $E_k(x)$, and $\psi_k(x)$ shown as in Jordan [11].

Write the Bernoulli polynomials of the first kind, $\phi_k(x)$, as (*cf.* Jordan [11, p. 321])

$$\phi_k(x) = \sum_{j=0}^{k} \frac{x^{k-j}}{(k-j)!}\alpha_j,$$

where $\alpha_j = B_j^{(1)}/j!$, and $B_j^{(1)}$ are ordinary Bernoulli numbers. Note that $\alpha_0 = 1$, $\alpha_1 = -1/2$, $\alpha_{2m+1} = 0$ $(m \in \mathbb{N})$ and (*cf.* [11, p. 245])

$$|\alpha_{2m}| \le \frac{1}{12(2\pi)^{2m-2}},\ (m = 0, 1, 2, \ldots).$$

Thus for $k \ge 2$

$$\begin{aligned} |\phi_k(x)| &\le \frac{|x|^k}{k!} + \frac{|x|^{k-1}}{2(k-1)!} + \sum_{j=2}^{k}\left(\frac{1}{12(2\pi)^{j-2}}\right)\frac{|x^{k-j}|}{(k-j)!} \\ &< \sum_{j=0}^{k} \frac{|x|^j}{j!} \le e^{|x|}. \end{aligned}$$

It follows that $|\phi_k(x)|^{1/k} < exp(|x|/k) \to 1$ as $k \to \infty$, which implies (6).

Secondly, note that Euler polynomial $E_k(x)$ can be written in the form

$$E_k(x) = \sum_{j=0}^{k} e_j \frac{x^{k-j}}{(k-j)!},\ (e_0 = 1), \tag{9}$$

where $e_j = E_j(0)$, $e_{2m} = 0$ $(m = 1, 2, \ldots)$, and e_{2m-1} satisfies the inequality (*cf.* [11, p. 302]).

$$|e_{2m-1}| < \frac{2}{3\pi^{2m-2}} < 1\ (m = 1, 2, \ldots). \tag{10}$$

Thus we have the estimation

$$|E_k(x)| \le \frac{|x|^k}{k!} + \sum_{j=1}^{k} |e_j| \frac{|x|^{k-j}}{(k-j)!} \le \frac{|x|^k}{k!} + \sum_{j=1}^{k} \frac{|x|^{k-j}}{(k-j)!} < e^{|x|}.$$

Consequently we get

$$\overline{\lim_{k\to\infty}} |E_k(x)|^{1/k} \le \lim_{k\to\infty} \left(e^{|x|}\right)^{1/k} = 1.$$

Hence (7) is verified.

Finally, from [11, p. 268], we have an integral representation of $\psi_k(x)$, the Bernoulli polynomials of the second kind, namely

$$\psi_k(x) = \int_0^1 \binom{x+t}{k} dt. \tag{11}$$

For $t \in [0,1]$ and for large k we have the estimation

$$\left|\binom{x+t}{k}\right| = \frac{|(x+t)_k|}{k!} = \frac{|(k-x-t-1)_k|}{k!} = o\left(\frac{k+[|x|])_k}{k!}\right) = o\left(k^{[|x|]}\right).$$

This means that there is a constant $M > 0$ such that

$$\max_{0\le t\le 1} \left|\binom{x+t}{k}\right| < Mk^{[|x|]}.$$

Thus it follows that

$$\overline{\lim_{k\to\infty}} |\psi_k(x)|^{1/k} \le \overline{\lim_{k\to\infty}} \left(\int_0^1 \left|\binom{x+t}{k}\right| dt\right)^{1/k} \le \overline{\lim_{k\to\infty}} \left(Mk^{[|x|]}\right)^{1/k} = 1.$$

This is a verification of (8), and corollary is proved. □

Remark 15. The convergence conditions given in Theorem 10 can be improved by restricting f and using similar techniques shown in [4].

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