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Sets, Lists and Noncrossing Partitions

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Abstract

Partitions of $[n] = \{1, 2, \ldots, n\}$ into sets of lists (A000262) are somewhat less numerous than partitions of $[n]$ into lists of sets (A000670). Here we observe that the former are actually equinumerous with partitions of $[n]$ into lists of *noncrossing* sets and give a bijective proof. We show that partitions of $[n]$ into sets of noncrossing lists are counted by A088368 and generalize this result to introduce a transform on integer sequences that we dub the "noncrossing partition" transform. We also derive recurrence relations to count partitions of $[n]$ into lists of noncrossing lists.

1 Introduction

A *partition* of $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty disjoint sets, called *blocks*, whose union is $[n]$. The notion of partition can be generalized by taking into account the order of the elements within each block or the order of the blocks themselves or both. To distinguish cases we use the terms list and set with their usual connotations of ordered and unordered respectively. Thus there are four cases: sets of sets (ordinary set partitions), sets of lists, lists of sets, and lists of lists. For unrestricted partitions the four counting sequences are respectively the Bell numbers (A000110), A000262, A000670, and A002866.

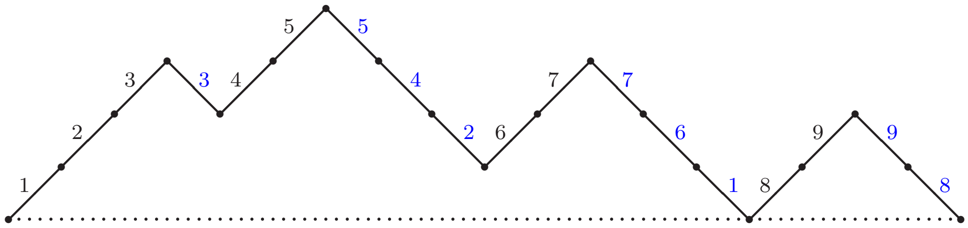
A partition is *noncrossing* if there do not exist four distinct elements $a < b < c < d$ with a, c both in one block and b, d both in another. It is well known that noncrossing partitions of $[n]$ (sets of noncrossing sets) are counted by the Catalan number C_n (A000108).

In §2 we show that partitions of $[n]$ into lists of noncrossing sets are equinumerous with partitions of $[n]$ into arbitrary sets of lists. In §3 we show that the "set of noncrossing lists" case has a generating function $A(x) = 1 + x + 3x^2 + 13x^3 + 69x^4 + \cdots$ that satisfies $A(x) = \sum_{k=0}^{\infty} k!\big(xA(x)\big)^k$ and hence is given by A088368, and we deduce a moderately efficient recurrence relation. In §4 we define the noncrossing partition transform on integer sequences and give some examples. In §5 we adapt the method of §3 to obtain an analogous recurrence for the "list of noncrossing lists" case.

2 Lists Of Noncrossing Sets $\longleftrightarrow$ Sets Of Lists

It is easy to count partitions of $[n]$ into sets of k lists: start with all $n!$ permutations of $[n]$; then for each one choose $k-1$ of the $n-1$ spaces between its entries to split it into a list of k nonempty lists. This yields all partitions of $[n]$ into lists of k lists and shows that there are $n!\binom{n-1}{k-1}$ of them. Finally, to count partitions $[n]$ into sets (rather than lists) of k lists, divide by $k!$. The result is $\frac{n!}{k!}\binom{n-1}{k-1}$, the so-called Lah number $L(n,k)$ (A105278).

To count the lists of noncrossing sets, first recall the well known bijection (essentially due to Prodinger [1]) from Dyck n-paths to noncrossing partitions of $[n]$ illustrated below.



number the upsteps left to right,
label each downstep with the number on its matching upstep,
form the partition of $[n]$ whose blocks are the labels on the descents.

The Dyck path shown thus corresponds to the noncrossing partition 3-542-761-98 (in a standard form: entries decreasing in each block and blocks listed in increasing order of their first entries). This bijection sends # peaks in the Dyck path to # blocks in the partition. Since a noncrossing partition of $[n]$ with k blocks gives rise to $k!$ lists of sets, partitions of $[n]$ into lists of noncrossing sets correspond to peak-labeled Dyck n-paths where peak-labeled means the peaks are labeled $1, 2, 3, \ldots$ in some order. Now the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ (A001263) is known to count Dyck n-paths with k peaks, and so the number of peak-labeled Dyck n-paths with $n+1-k$ peaks is $(n+1-k)!N(n,k)$, which simplifies to the Lah number $L(n,k)$ mentioned above.

Summing over k in the two preceding paragraphs yields the equivalence of the section title. However, we wish to show this equivalence directly by giving a bijection from peak-labeled Dyck n-paths with k peaks to partitions of $[n]$ into sets of $n+1-k$ lists. Using the Dyck path above as a working example (with $n = 9$ and $k = 4$), begin by *prepending an upstep*. Record the peak labels, ascent lengths, and descent lengths in left to right order as shown on the left below.

$$
\begin{array}{c|cccc}
\text{labels} & 3 & 1 & 4 & 2 \\
\hline
\text{ascents} & 4 & 2 & 2 & 2 \\
\text{descents} & 1 & 3 & 3 & 2
\end{array}
\xrightarrow[\text{columns}]{\text{rotate}}
\begin{array}{cccc}
2 & 3 & 1 & 4 \\
\hline
2 & 4 & 2 & 2 \\
2 & 1 & 3 & 3
\end{array}
\xrightarrow[\text{last col}]{\text{delete}}
\begin{array}{ccc}
2 & 3 & 1 \\
\hline
2 & 4 & 2 \\
2 & 1 & 3
\end{array}
\xrightarrow[\text{sums}]{\text{partial}}
\begin{array}{ccc}
2 & 3 & 1 \\
\hline
2 & 6 & 8 \\
2 & 3 & 6
\end{array}
$$

The arrows illustrate the following steps: (i) cyclically rotate the columns so that the largest peak label is last, (ii) drop the last column, and (iii) form partial sums of the bottom two rows. Now form $[n] \setminus \{\text{middle row}\} = [9] \setminus \{2, 6, 8\} = \{1, 3, 4, 5, 7, 9\}$—these numbers will be the first entries of the lists—and $[n] \setminus \{\text{bottom row}\} = [9] \setminus \{2, 3, 6\} = \{1, 4, 5, 7, 8, 9\}$ and apply the difference operator (leaving the first entry intact) to get $\{1, 3, 1, 2, 1, 1\}$—these numbers will be the lengths of the lists. From their lengths and first entries, we now have partial lists

$$\boxed{1} \quad \boxed{3}\boxed{}\boxed{} \quad \boxed{4} \quad \boxed{5}\boxed{} \quad \boxed{7} \quad \boxed{9}$$

and all that remains is to fill in the blanks. This is done by arranging the missing numbers in the order of their associated labels as in the table following the last arrow above—thus 2,6,8 in the order 2,3,1 is 6,8,2—and then inserting them left to right in the blank squares. The final result is

$$\boxed{1} \quad \boxed{3}\boxed{6}\boxed{8} \quad \boxed{4} \quad \boxed{5}\boxed{2} \quad \boxed{7} \quad \boxed{9}$$

giving the lists in increasing order of their first entries. We leave the interested reader to verify that the mapping is invertible; an appeal to the cycle lemma (see e.g., [2, pp. 359–360]) will be needed to determine the appropriate cyclic rotation.

3 Sets Of Noncrossing Lists

Sequence A088368 is defined by the generating function equation $A(x) = \sum_{k=0}^{\infty} k!\big(xA(x)\big)^k$. We will show that this sequence counts partitions of $[n]$ into sets of noncrossing lists. Let $\mathcal{U}(n)$ denote this set of partitions and $\mathcal{U}(n, k)$ the subset for which n occurs in a list of length k. Set $u(n) = |\mathcal{U}(n)|$ and $u(n, k) = |\mathcal{U}(n, k)|$; thus $u(n) = \sum_{k=1}^{n} u(n, k)$. For a partition in $\mathcal{U}(n, k)$ the entries in the list containing n split $[n]$ into a sequence of subintervals $I_1, I_2, \ldots, I_k$ of lengths, say, $a_1, a_2, \ldots, a_k$ ($a_i \ge 1,\ 1 \le i \le k,\ \sum_{i=1}^{k} a_i = n$). Thus with $n = 8$ the list 3,8,4 yields $[1, 2, 3]$, $[4]$, $[5, 6, 7, 8]$. Set $J_i = I_i \backslash \{a_i\},\ 1 \le i \le k$. The remaining lists are formed from entries of the J_i's and since no crossovers are allowed between these lists (the noncrossing property would be violated), we are restricted to partitioning each J_i into a set of noncrossing lists. This can be done in $u(b_1)u(b_2)\cdots u(b_k)$ ways where $b_i = a_i - 1,\ 1 \le i \le k$. Clearly, $b_i \ge 0$ and $\sum_{i=1}^{k} b_i = n - k$. Thus,

$$u(n, k) = k! \sum_{(b_1,\ldots,b_k)} u(b_1)u(b_2)\cdots u(b_k) \tag{1}$$

where the sum is taken over all nonnegative k-tuples $(b_1, b_2, \ldots, b_k)$ whose sum is $n - k$ (weak compositions of $n - k$ into k parts), and the $k!$ factor serves to order the block containing n. Let $U(x) = \sum_{n \ge 0} u(n)x^n$ with $u(0) := 1$. Then the right hand side in (1) is $[x^{n-k}]k!\big(U(x)\big)^k = [x^n]k!\big(xU(x)\big)^k$. Multiply by x^n and sum over n and k to get $U(x) = \sum_{k=0}^{\infty} k!\big(xU(x)\big)^k$, as claimed.

The recurrence (1) is not efficient: there are $\binom{n-1}{k-1}$ nonnegative k-tuples whose sum is $n-k$. Thus to compute $u(n)$ using (1) involves a sum over $\sum_{k=1}^{n}\binom{n-1}{k-1}=2^{n-1}$ terms. It is easy, however, to reduce it to a sum over (integer) partitions of n, a set whose size, turning the famous Hardy-Rademacher-Ramanujan formula into round figures, is approximately $\frac{1}{7n}13^{\sqrt{n}} \ll 2^{n-1}$. Count frequencies in a weak composition $(b_1, b_2, \ldots, b_k)$ of $n-k$ indexing a summand in (1) to get a weak partition $0^{p_0}\,1^{p_1}\,2^{p_2}\,\ldots\,(n-1)^{p_{n-1}}$ (in frequency-of-parts form) of $n-k$ into k parts; thus $\sum_{i=0}^{n-1}p_i = k$ and $\sum_{i=0}^{n-1}ip_i = n-k$. Each such weak partition of $n-k$ comes from $\binom{p_0+p_1+\cdots+p_{n-1}}{p_0,\,p_1,\,\ldots,\,p_{n-1}}$ weak compositions of $n-k$, all of which make the same contribution $k!\prod_{i=1}^{n-1}u(i)^{p_i} = (p_0+p_1+\cdots+p_{n-1})!\prod_{i=1}^{n-1}u(i)^{p_i}$ to the sum $u(n,k)$. Furthermore, all such weak partitions (regardless of k) arise by subtracting 1 from each part of a partition of n. These observations translate into a faster recurrence for $u(n)$:

$$
\begin{aligned}
u(0) &= 1, \quad \text{and for } n \geq 1 \\
u(n) &= \sum (p_1+\cdots+p_n)!\binom{p_1+\cdots+p_n}{p_1,\,\ldots,\,p_n}u(0)^{p_1}u(1)^{p_2}\cdots u(n-1)^{p_n}
\end{aligned}
\tag{2}
$$

where the sum is over all partitions $1^{p_1}\,2^{p_2}\,\ldots\,n^{p_n}$ of n.

4 The "Noncrossing Partition" Transform

A closer look at the previous section suggests a transform on integer sequences and a combinatorial interpretation of it. For a sequence $(a_k)_{k\geq 0}$ with $a_0=1$, define $(b_k)_{k\geq 0}$ by

$$B(x) = \frac{1}{x}\left(\frac{x}{A(x)}\right)^{\langle -1\rangle} \tag{3}$$

where A and B are the ordinary generating functions for (a_k) and (b_k) respectively, and $^{\langle -1\rangle}$ denotes compositional inverse (reversion of series). Equivalently, $B(x)$ is the unique power series satisfying

$$\sum_{k\geq 0} a_k\big(xB(x)\big)^k = B(x). \tag{4}$$

The case $a_k = k!$ was treated in the previous section, where b_k was then shown to count partitions of $[k]$ into noncrossing lists. The argument readily generalizes, however, from $k!$ to arbitrary a_k (subject to $a_0 = 1$) to establish the following interpretation for b_k. If a_k counts a class of combinatorial configurations, say A-structures, on k-sets, then b_k counts the configurations obtained thusly:

Partition the set $[k]$ into noncrossing blocks and then put an A-structure on each block.

For this reason, we call the transform $(a_k) \to (b_k)$ defined by (3) the *noncrossing partition* transform. Note that, necessarily, $b_0 = 1$ and $b_1 = a_1$. Here are a few examples (in all cases,

$a_0 = 1$ and $b_0 = 1$).

$(a_k)_{k\geq1}$	$(b_k)_{k\geq1}$
1	C_k (A000108)
2^k	$\frac{2^k}{k+1}\binom{2k}{k}$
$\frac{1}{k+1}\binom{2k}{k}$	$\frac{1}{2k+1}\binom{3k}{k}$
$\frac{1}{2}\binom{2k}{k}$	$2^{k-1}C_k$

$(a_k)_{k\geq1}$	$(b_k)_{k\geq1}$
F_{k-1} (A000045)	$\triangle$-free dissections (A046736)
2^{k-1}	little Schröder # (A001003)
little Schröder # (A001003)	"blobs" (A003168)
C_{k-1}	big Schröder # (A006318)

The noncrossing partition transform $(a_k) \to (b_k)$

In particular, if (a_k) counts permutations of $[k]$ with some property, then b_k counts partitions of $[k]$ into noncrossing lists, each of which has the property in question. For example, since 321-avoiding permutations are counted by the Catalan numbers, we see from the table above that the number of partitions of $[k]$ into noncrossing 321-avoiding permutations is $\frac{1}{2k+1}\binom{3k}{k}$.

5 Lists Of Noncrossing Lists

It is possible to use the decomposition of the block containing n as in the Section 3 to obtain recurrence relations for the number of partitions of $[n]$ into lists of noncrossing lists. (We continue to use the descriptive term "block" but now it means a list rather than a set.) Here, however, the factor $u(i)$ in the product on the right side of (2) will count *lists* of blocks, and so it will be necessary to remove the order on each such list of blocks, throw the block containing n into the mix, and then re-order the whole lot. This requires keeping track of the number of blocks. So let $u(n, j)$ denote the number of partitions of $[n]$ into a list of j noncrossing lists.

In a partition $\mathbf{\Pi}$ of $[n]$ into lists of noncrossing lists, let k denote the length of the block containing n. As in §3, this block induces a decomposition of $[n]$ into intervals $I_1, I_2, \ldots, I_k$ whose terminal points form the block. Deleting the endpoints gives a list of intervals $J_1, J_2, \ldots, J_k$ each of which is a union of blocks in $\mathbf{\Pi}$. Let $(a_i)_{i=1}^k$ denote the lengths of the I_i taken in decreasing order so that $\pi = (a_i)_{i=1}^k$ is an integer partition of n. Set $b_i = a_i - 1$ and suppose the first r b_i's are positive. Then for $i = 1, 2, \ldots, r$ the J interval corresponding to b_i is the union of some number c_i $(1 \leq c_i \leq b_i)$ of blocks in the original partition $\mathbf{\Pi}$. The total number of blocks in $\mathbf{\Pi}$ is then $1 + \sum_{i=1}^r c_i$. Write the integer partition π of n in frequency-of-parts form $1^{p_1}\, 2^{p_2} \ldots (n)^{p_n}$; thus $\sum ip_i = n$ and $\sum p_i = k$.

Thus summands in the recursive sum for $u(n, j)$ are indexed by configurations of the form

$$\begin{matrix} b_1 & b_2 & \ldots & b_r & 0 & \ldots & 0 \\ c_1 & c_2 & \ldots & c_r & & & \end{matrix} \tag{5}$$

where $1 \leq c_i \leq b_i$ for $1 \leq i \leq r$, and the top row is $\pi - 1$ (entrywise) for some integer partition $\pi = (a_i)_{i=1}^k$ of n, and $1 + \sum_{i=1}^r c_i = j$.

Then, with the sum taken over these configurations,

$$u(n,j) = \sum \begin{array}{cl} k! & [\,\text{permute entries of block containing } n\,] \times \\ \binom{p_1+\ldots+p_n}{p_1,\ldots,p_n} & [\,\text{permute the lengths } |I_1|,\ldots,|I_k|\,] \times \\ j! & [\,\text{permute blocks in } \mathbf{\Pi}\,] \times \\ \frac{u(b_1,c_1)}{c_1!}\frac{u(b_2,c_2)}{c_2!}\cdots\frac{u(b_r,c_r)}{c_r!} & [\,\text{the denominators eliminate the inter-block} \\ & \ \text{order captured by } u\,] \end{array}$$

Equivalently,

$$u(n,j) = \sum (p_1+\cdots+p_n)!\binom{p_1+\cdots+p_n}{p_1,\ \ldots,\ p_n}\binom{1+c_1+\cdots+c_r}{1, c_1,\ \ldots,\ c_r} u(b_1,c_1)\cdots u(b_r,c_r) \qquad (6)$$

and $u(n) := \sum_{j=1}^{n} u(n,j)$ gives the number of partitions of $[n]$ into lists of noncrossing lists.

The sequence $\big(u(n)\big)_{n\ge1}$ begins $(1,\ 4,\ 24,\ 184,\ 1680,\ 17592,\ 206472,\ 2674752,\ \ldots)$. The total number of terms $t(n)$ in the sum for $u(n)$ in (6) is $\sum b_1 b_2\cdots b_r$ taken over all the configurations in (5) above. Thus $t(n)$ is the sum of products of the nonzero entries in $\pi - 1$ taken over all partitions π of n. The generating function $\sum_{n\ge0} t(n)x^n$ with $t(0) := 1$ is given by

$$\frac{1}{1-x}\prod_{k\ge2}\frac{1}{1-(k-1)x^k} = 1 + x + 2\,x^2 + 4\,x^3 + 8\,x^4 + 14\,x^5 + 27\,x^6 + 45\,x^7 + 82\,x^8 + \ldots. \qquad (7)$$

(Cf. A006906 for the sum of products of the entries taken over all partitions of n.)

The number of terms in the sum for $u(n,j)$ can be somewhat further reduced by collecting equal $u(b_i,c_i)$ factors: if b occurs j times among the b_i, then collecting equal c's reduces the contribution of b to the number of terms from a factor of b^j to a factor of $\binom{b+j-1}{j}$. The generating function for the total number of terms thereby changes from (7) to

$$\frac{1}{1-x}\prod_{k\ge2}\frac{1}{(1-x^k)^{k-1}} = 1 + x + 2\,x^2 + 4\,x^3 + 8\,x^4 + 14\,x^5 + 26\,x^6 + 44\,x^7 + 77\,x^8 + \ldots.$$

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References

[1] Helmut Prodinger, A correspondence between ordered trees and noncrossing partitions, *Discrete Math.* **46** (1983), 205–206.

[2] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics* (2nd edition), Addison-Wesley, 1994.

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