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# New Ramanujan-Type Formulas and Quasi-Fibonacci Numbers of Order 7

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**Abstract**

We give applications of the quasi-Fibonacci numbers of order 7 and the so-called sine-Fibonacci numbers of order 7 and many other new kinds of recurrent sequences to the decompositions of some polynomials. We also present the characteristic equations, generating functions and some properties of all these sequences. Finally, some new Ramanujan-type formulas are generated.

# 1 Introduction

The scope of the paper is the generalization of the following decompositions of polynomials [2, 3, 4, 8, 9]:

$$\big(\mathbb{X}-2\sin(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-2\sin(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-2\sin(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-\sqrt{7}\,\mathbb{X}^2+\sqrt{7}, \tag{1.1}$$

$$\big(\mathbb{X}-4\sin^2(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-4\sin^2(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-4\sin^2(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-7\,\mathbb{X}^2+14\,\mathbb{X}-7, \tag{1.2}$$

$$\big(\mathbb{X}-8\sin^3(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-8\sin^3(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-8\sin^3(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-4\sqrt{7}\,\mathbb{X}^2+21\,\mathbb{X}+7\sqrt{7}, \tag{1.3}$$

$$\big(\mathbb{X}-2\cos(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-2\cos(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-2\cos(\tfrac{8\pi}{7})\big)=\mathbb{X}^3+\mathbb{X}^2-2\,\mathbb{X}-1, \tag{1.4}$$

$$\big(\mathbb{X}-4\cos^2(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-4\cos^2(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-4\cos^2(\tfrac{8\pi}{7})\big)=\mathbb{X}^3-5\,\mathbb{X}^2+6\,\mathbb{X}-1, \tag{1.5}$$

$$\big(\mathbb{X}-8\cos^3(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-8\cos^3(\tfrac{4\pi}{7})\big)\big(\mathbb{X}-8\cos^3(\tfrac{8\pi}{7})\big)=\mathbb{X}^3+4\,\mathbb{X}^2-11\,\mathbb{X}-1, \tag{1.6}$$

$$\begin{aligned}&\big(\mathbb{X}-8\sin(\tfrac{2\pi}{7})\,\cos^2(\tfrac{8\pi}{7})\big)\big(\mathbb{X}-8\sin(\tfrac{4\pi}{7})\,\cos^2(\tfrac{2\pi}{7})\big)\big(\mathbb{X}-8\sin(\tfrac{8\pi}{7})\,\cos^2(\tfrac{4\pi}{7})\big)=\\ &\quad=\mathbb{X}^3-3\sqrt{7}\,\mathbb{X}^2+14\,\mathbb{X}+\sqrt{7},\end{aligned} \tag{1.7}$$

*etc.*

The main incentive for generating the decompositions of these polynomials is provided by the properties of the so-called quasi-Fibonacci numbers of order 7, $A_n(\delta)$, $B_n(\delta)$ and $C_n(\delta)$, $n\in\mathbb{N}$, described in [10] by means of the relations

$$(1+\delta\,(\xi^k+\xi^{6k}))^n=A_n(\delta)+B_n(\delta)\,(\xi^k+\xi^{6k})+C_n(\delta)\,(\xi^{2k}+\xi^{5k}) \tag{1.8}$$

for $k=1,2,3$, where $\xi\in\mathbb{C}$ is a primitive root of unity of order 7 (i.e., $\xi^7=1$ and $\xi\neq 1$), $\delta\in\mathbb{C}$, $\delta\neq 0$. Besides, an essential rôle in the decompositions of polynomials discussed in the paper is played by related numbers ($\delta\in\mathbb{C}$, $n\in\mathbb{N}$):

$$\begin{aligned}\mathcal{A}_n(\delta)&:=3\,A_n(\delta)-B_n(\delta)-C_n(\delta)=\\ &=\big(1+\delta\,(\xi+\xi^6)\big)^n+\big(1+\delta\,(\xi^2+\xi^5)\big)^n+\big(1+\delta\,(\xi^3+\xi^4)\big)^n\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\mathcal{B}_n(\delta)&:=\frac{1}{2}\big(\big(\mathcal{A}_n(\delta)\big)^2-\mathcal{A}_{2n}(\delta)\big)=\\ &=\Big(\big(1+2\,\delta\,\cos(\tfrac{2\pi}{7})\big)\big(1+2\,\delta\,\cos(\tfrac{4\pi}{7})\big)\Big)^n+\\ &\quad+\Big(\big(1+2\,\delta\,\cos(\tfrac{2\pi}{7})\big)\big(1+2\,\delta\,\cos(\tfrac{8\pi}{7})\big)\Big)^n+\\ &\quad+\Big(\big(1+2\,\delta\,\cos(\tfrac{4\pi}{7})\big)\big(1+2\,\delta\,\cos(\tfrac{8\pi}{7})\big)\Big)^n=\\ &=3\,\big(A_n(\delta)\big)^2-2\,A_n(\delta)\,B_n(\delta)-2\,A_n(\delta)\,C_n(\delta)+\\ &\quad+3\,B_n(\delta)\,C_n(\delta)-2\,\big(B_n(\delta)\big)^2-2\,\big(C_n(\delta)\big)^2=\\ &=A_n(\delta)\,\big(2\,\mathcal{A}_n(\delta)-3A_n(\delta)\big)-2\,\big(B_n(\delta)-C_n(\delta)\big)^2-B_n(\delta)\,C_n(\delta).\end{aligned} \tag{1.10}$$

Furthermore, to simplify the formulas, we will write

$$\mathcal{A}_n=\mathcal{A}_n(1),\quad \mathcal{B}_n=\mathcal{B}_n(1),\quad A_n=A_n(1),\quad B_n=B_n(1)\ \text{ and }\ C_n=C_n(1), \tag{1.11}$$

for every $n \in \mathbb{N}$. We note that the tables of values of these numbers can be found in the article [10].

## 2 Basic decompositions

Wituła et al. [10] determined the following two formulas:

$$\big(\mathbb{X} - \big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X} - \big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - \big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - \mathcal{B}_n\,\mathbb{X}^2 + (-1)^n\,\mathcal{A}_n\,\mathbb{X} - 1 \quad (2.1)$$

and

$$\big(\mathbb{X} - \big(1 + 2\,\delta\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X} - \big(1 + 2\,\delta\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - \big(1 + 2\,\delta\,\cos(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - \mathcal{A}_n(\delta)\,\mathbb{X}^2 + \mathcal{B}_n(\delta)\,\mathbb{X} - \big(1 - \delta - 2\,\delta^2 + \delta^3\big)^n. \quad (2.2)$$

From (2.2) three special formulas follow:

$$\big(\mathbb{X} - \big(1 + \cos^2(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X} - \big(1 + \cos^2(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - \big(1 + \cos^2(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - \big(\tfrac{3}{2}\big)^n\,\mathcal{A}_n\big(\tfrac{1}{6}\big)\,\mathbb{X}^2 + \big(\tfrac{3}{2}\big)^{2n}\,\mathcal{B}_n\big(\tfrac{1}{6}\big)\,\mathbb{X} - \big(\tfrac{13}{8}\big)^{2n}, \quad (2.3)$$

$$\big(\mathbb{X} - \big(2\,\sin(\tfrac{2\pi}{7})\big)^{2n}\big)\big(\mathbb{X} - \big(2\,\sin(\tfrac{4\pi}{7})\big)^{2n}\big)\big(\mathbb{X} - \big(2\,\sin(\tfrac{8\pi}{7})\big)^{2n}\big) = \\ = \mathbb{X}^3 - 2^n\,\mathcal{A}_n\big(-\tfrac{1}{2}\big)\,\mathbb{X}^2 + 2^{2n}\,\mathcal{B}_n\big(-\tfrac{1}{2}\big)\,\mathbb{X} - 7^n \quad (2.4)$$

and

$$\big(\mathbb{X} - \big(2\,\cos(\tfrac{2\pi}{7})\big)^{2n}\big)\big(\mathbb{X} - \big(2\,\cos(\tfrac{4\pi}{7})\big)^{2n}\big)\big(\mathbb{X} - \big(2\,\cos(\tfrac{8\pi}{7})\big)^{2n}\big) = \\ = \mathbb{X}^3 - 2^n\,\mathcal{A}_n\big(\tfrac{1}{2}\big)\,\mathbb{X}^2 + 2^{2n}\,\mathcal{B}_n\big(\tfrac{1}{2}\big)\,\mathbb{X} - 1. \quad (2.5)$$

Comparing formulas (2.1) and (2.5) two new identities are generated

$$\mathcal{A}_{2n} = 2^{2n}\,\mathcal{B}_n\big(\tfrac{1}{2}\big) \quad \text{and} \quad \mathcal{B}_{2n} = 2^n\,\mathcal{A}_n\big(\tfrac{1}{2}\big). \quad (2.6)$$

We also have the decomposition

$$\Big(\mathbb{X} - \big(\big(1 + \delta\,(\xi + \xi^6)\big)\,\big(1 + \delta\,(\xi^2 + \xi^5)\big)\big)^n\Big)\Big(\mathbb{X} - \big(\big(1 + \delta\,(\xi + \xi^6)\big)\,\big(1 + \delta\,(\xi^3 + \xi^4)\big)\big)^n\Big) \times \\ \times \Big(\mathbb{X} - \big(\big(1 + \delta\,(\xi^2 + \xi^5)\big)\,\big(1 + \delta\,(\xi^3 + \xi^4)\big)\big)^n\Big) = \\ = \Big(\mathbb{X} - \big(\big(1 + 2\,\delta\,\cos(\tfrac{2\pi}{7})\big)\,\big(1 + 2\,\delta\,\cos(\tfrac{4\pi}{7})\big)\big)^n\Big)\Big(\mathbb{X} - \big(\big(1 + 2\,\delta\,\cos(\tfrac{2\pi}{7})\big)\,\big(1 + 2\,\delta\,\cos(\tfrac{6\pi}{7})\big)\big)^n\Big) \times \\ \times \Big(\mathbb{X} - \big(\big(1 + 2\,\delta\,\cos(\tfrac{4\pi}{7})\big)\,\big(1 + 2\,\delta\,\cos(\tfrac{6\pi}{7})\big)\big)^n\Big) = \\ = \mathbb{X}^3 - \mathcal{B}_n(\delta)\,\mathbb{X}^2 + \big(1 - \delta - 2\,\delta^2 + \delta^3\big)\,\mathcal{A}_n(\delta)\,\mathbb{X} - \big(1 - \delta - 2\,\delta^2 + \delta^3\big)^{2n} = \\ := r_n(\mathbb{X};\delta). \quad (2.7)$$

Now let us set

$$\Xi_n := \Xi_n(\delta,\varepsilon,\eta) = 2^n\left(\delta\,\cos^n\left(\tfrac{2\pi}{7}\right)+\varepsilon\,\cos^n\left(\tfrac{4\pi}{7}\right)+\eta\,\cos^n\left(\tfrac{8\pi}{7}\right)\right), \tag{2.8}$$

$$\Upsilon_n := \Upsilon_n(\delta,\varepsilon,\eta) = 2^n\left(\varepsilon\,\cos^n\left(\tfrac{2\pi}{7}\right)+\eta\,\cos^n\left(\tfrac{4\pi}{7}\right)+\delta\,\cos^n\left(\tfrac{8\pi}{7}\right)\right), \tag{2.9}$$

$$\Theta_n := \Theta_n(\delta,\varepsilon,\eta) = 2^n\left(\eta\,\cos^n\left(\tfrac{2\pi}{7}\right)+\delta\,\cos^n\left(\tfrac{4\pi}{7}\right)+\varepsilon\,\cos^n\left(\tfrac{8\pi}{7}\right)\right) \tag{2.10}$$

for any $\delta,\varepsilon,\eta\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

**Lemma 2.1** *The following general decomposition formula holds*

$$\begin{aligned}(\mathbb{X}-\Xi_n)(\mathbb{X}-\Upsilon_n)(\mathbb{X}-\Theta_n) &= \\ &= \mathbb{X}^3-(\delta+\varepsilon+\eta)\,\mathcal{A}_n\,\mathbb{X}^2+\tfrac{1}{2}\,(-1)^n\left((\delta+\varepsilon)^2+(\delta+\eta)^2+(\varepsilon+\eta)^2\right)\mathcal{A}_n\,\mathbb{X}- \\ &\quad-\Big[\delta\,\varepsilon\,\eta\,(\mathcal{B}_{3n}+3)+\delta^3+\varepsilon^3+\eta^3+(-1)^n\left(\delta\,\varepsilon^2+\varepsilon\,\eta^2+\eta\,\delta^2\right)\mathcal{A}_n(-1)+ \\ &\qquad+2^{2n}\left(\delta^2\,\varepsilon+\varepsilon^2\,\eta+\eta^2\,\delta\right)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mathcal{A}_{2n-k}\left(\tfrac{1}{2}\right)\Big].\end{aligned} \tag{2.11}$$

Below an illustrative example connected with Lemma 2.1 will be presented.

**Examle 2.2** A. M. Yaglom and I. M. Yaglom [11] (see also [5, problem 230] and [6, problem 329]) considered the following polynomial:

$$w(x)=\binom{2m+1}{1}x^m-\binom{2m+1}{3}x^{m-1}+\binom{2m+1}{5}x^{m-2}-\ldots$$

and proved that it has the roots

$$x_k=\cot^2\left(\frac{k\,\pi}{2m+1}\right),\qquad k=1,2,\ldots,m.$$

In particular, for $m=3$ taking into account that

$$\cot^2\left(\tfrac{\pi}{7}\right)=\cot^2\left(\tfrac{8\pi}{7}\right),\qquad \cot^2\left(\tfrac{3\pi}{7}\right)=\cot^2\left(\tfrac{4\pi}{7}\right)$$

we have the decomposition (see also formula (6.14) below):

$$\left(x-\cot^2\left(\tfrac{2\pi}{7}\right)\right)\left(x-\cot^2\left(\tfrac{4\pi}{7}\right)\right)\left(x-\cot^2\left(\tfrac{8\pi}{7}\right)\right)=x^3-5\,x^2+3\,x-\tfrac{1}{7}$$

and as a corollary – the decomposition

$$\left(x+7\,\cot^2\left(\tfrac{2\pi}{7}\right)\right)\left(x+7\,\cot^2\left(\tfrac{4\pi}{7}\right)\right)\left(x+7\,\cot^2\left(\tfrac{8\pi}{7}\right)\right)=x^3+35\,x^2+147\,x+49. \tag{2.12}$$

According to (2.8) for $n=1$, let us try to find a linear combination

$$-7\,\cot^2\left(\tfrac{2\pi}{7}\right)=2\left(\delta\,\cos\left(\tfrac{2\pi}{7}\right)+\varepsilon\,\cos\left(\tfrac{4\pi}{7}\right)+\eta\,\cos\left(\tfrac{8\pi}{7}\right)\right)$$

or, the same,

$$-7\,\cos^2\left(\tfrac{2\pi}{7}\right) = 2\left(\delta\,\cos\left(\tfrac{2\pi}{7}\right) + \varepsilon\,\cos\left(\tfrac{4\pi}{7}\right) + \eta\,\cos\left(\tfrac{8\pi}{7}\right)\right) - \\ - \delta\,\cos^3\left(\tfrac{2\pi}{7}\right) - \varepsilon\,\cos\left(\tfrac{4\pi}{7}\right)\,\cos^2\left(\tfrac{2\pi}{7}\right) - \eta\,\cos\left(\tfrac{8\pi}{7}\right)\,\cos^2\left(\tfrac{2\pi}{7}\right).$$

Decreasing powers and taking into account the identity

$$\cos\left(\tfrac{2\pi}{7}\right) + \cos\left(\tfrac{4\pi}{7}\right) + \cos\left(\tfrac{8\pi}{7}\right) = -\frac{1}{2}$$

we find

$$\left(2\,\delta + \varepsilon - 3\,\eta\right)\cos\left(\tfrac{2\pi}{7}\right) + \left(\delta + 3\,\varepsilon - 3\,\eta + 7\right)\cos\left(\tfrac{4\pi}{7}\right) = -\frac{\delta}{2} + \frac{\varepsilon}{2} + \eta - 7.$$

Thus, for finding $\delta$, $\varepsilon$, $\eta$ we have the linear system (see Corollary 2.5 in [10]):

$$\begin{cases} 2\,\delta + \varepsilon - 3\,\eta = 0 \\ \delta + 3\,\varepsilon - 3\,\eta = -7 \\ \delta - \varepsilon - 2\,\eta = -14 \end{cases}$$

with the solution

$$\delta = 17, \qquad \varepsilon = 5, \qquad \eta = 13.$$

Hence,

$$-7\,\cot^2\left(\tfrac{2\pi}{7}\right) = 2\left(17\,\cos\left(\tfrac{2\pi}{7}\right) + 5\,\cos\left(\tfrac{4\pi}{7}\right) + 13\,\cos\left(\tfrac{8\pi}{7}\right)\right).$$

Analogously, we obtain

$$-7\,\cot^2\left(\tfrac{4\pi}{7}\right) = 2\left(13\,\cos\left(\tfrac{2\pi}{7}\right) + 17\,\cos\left(\tfrac{4\pi}{7}\right) + 5\,\cos\left(\tfrac{8\pi}{7}\right)\right),$$
$$-7\,\cot^2\left(\tfrac{8\pi}{7}\right) = 2\left(5\,\cos\left(\tfrac{2\pi}{7}\right) + 13\,\cos\left(\tfrac{4\pi}{7}\right) + 17\,\cos\left(\tfrac{8\pi}{7}\right)\right).$$

Thus the decomposition (2.12) corresponds to Lemma 2.1 with

$$\Xi_1(17,5,13), \quad \Upsilon_1(17,5,13), \quad \Theta_1(17,5,13).$$

**Remark 2.3** The identity (2.12) was found earlier by Shevelev [4].

**Remark 2.4** The formula (2.11), in some cases which are subject of our interest, especially when coefficients $\delta$, $\varepsilon$, $\eta$ are the corresponding values of trigonometric functions, becomes rather complicated. Thus, in the next two sections we attempt to designate the relevant coefficients of decomposition (2.11), including the recurrent coefficients, on the grounds of new sequences that are easier to analyze.

# 3 The first group of special cases of (2.11)

Let us set

$$a_n = 2^{2n+1}\Big[\sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n}\Big], \tag{3.1}$$

$$b_n = 2^{2n+1}\Big[\sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n}\Big], \tag{3.2}$$

$$c_n = 2^{2n+1}\Big[\sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n}\Big], \tag{3.3}$$

for $n = 0, 1, 2, \ldots$

**Lemma 3.1** *The following recurrence relations hold:*

$$\begin{cases} a_{n+1} = 2\,a_n + b_n, \\ b_{n+1} = a_n + 2\,b_n - c_n, \\ c_{n+1} = c_n - b_n, \end{cases} \tag{3.4}$$

*for $n = 0, 1, 2, \ldots$ and $a_0 = b_0 = c_0 = \sqrt{7}$. Moreover, elements of each sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ satisfy the recurrence equation*

$$x_{n+2} - 5\,x_{n+1} + 6\,x_n - x_{n-1} = 0, \qquad n = 0, 1, 2, \ldots \tag{3.5}$$

*(in view of decomposition (1.5) an appropriate characteristic polynomial is compatible with the definition of numbers $a_n$, $b_n$ and $c_n$).*

The first twelve values of numbers $a_n^* = a_n/\sqrt{7}$, $b_n^* = b_n/\sqrt{7}$ and $c_n^* = c_n/\sqrt{7}$ are presented in Table 1.

Now, let us set

$$\alpha_n = 2^{2n}\Big[\sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n-1} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n-1} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n-1}\Big], \tag{3.6}$$

$$\beta_n = 2^{2n}\Big[\sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n-1} + \sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n-1} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n-1}\Big], \tag{3.7}$$

$$\gamma_n = 2^{2n}\Big[\sin(\tfrac{8\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2n-1} + \sin(\tfrac{2\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2n-1} + \sin(\tfrac{4\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2n-1}\Big], \tag{3.8}$$

for $n = 1, 2, \ldots$

**Lemma 3.2** *We have*

$$\alpha_1 = 0, \qquad \beta_1 = -2\sqrt{7}, \qquad \textit{and} \qquad \gamma_1 = \sqrt{7}.$$

*The elements of sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the system of recurrence relations (3.4) and, selectively, recurrence relation (3.5).*

The first twelve values of numbers $\alpha_n^* = \alpha_n/\sqrt{7}$, $\beta_n^* = \beta_n/\sqrt{7}$ and $\gamma_n^* = \gamma_n/\sqrt{7}$ are presented in Table 1.

**Remark 3.3** There exists a simple relationships between numbers $\alpha_n$, $\beta_n$, $\gamma_n$, $n \in \mathbb{N}$, and $a_n$, $b_n$, $c_n$, $n \in \mathbb{N}$. We have

$$\alpha_n \equiv c_n, \qquad \beta_n \equiv -a_{n-1} - b_{n-1}, \qquad \gamma_n \equiv a_{n-1}.$$

These relations are easily derived from the definitions of respective numbers, for example

$$\begin{aligned}
\beta_n &= 2^{2(n-1)+1}\Big[2\,\sin(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^{2(n-1)} + \\
&\quad + 2\,\sin(\tfrac{8\pi}{7})\,\cos(\tfrac{2\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^{2(n-1)} + 2\,\sin(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^{2(n-1)}\Big] = \\
&= 2^{2(n-1)+1}\Big[-\big(\sin(\tfrac{2\pi}{7}) + \sin(\tfrac{4\pi}{7})\big)\big(\cos(\tfrac{8\pi}{7})\big)^{2(n-1)} + \\
&\quad + \big(-\sin(\tfrac{4\pi}{7}) - \sin(\tfrac{8\pi}{7})\big)\big(\cos(\tfrac{2\pi}{7})\big)^{2(n-1)} - \big(\sin(\tfrac{2\pi}{7}) + \sin(\tfrac{8\pi}{7})\big)\big(\cos(\tfrac{4\pi}{7})\big)^{2(n-1)}\Big] = \\
&= -a_{n-1} - b_{n-1}.
\end{aligned}$$

Let us also set

$$f_n = 2^{n+1}\left[\cos(\tfrac{2\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^n + \cos(\tfrac{4\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^n + \cos(\tfrac{8\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^n\right], \tag{3.9}$$

$$g_n = 2^{n+1}\left[\cos(\tfrac{8\pi}{7})\big(\cos(\tfrac{4\pi}{7})\big)^n + \cos(\tfrac{2\pi}{7})\big(\cos(\tfrac{8\pi}{7})\big)^n + \cos(\tfrac{4\pi}{7})\big(\cos(\tfrac{2\pi}{7})\big)^n\right], \tag{3.10}$$

$$h_n = 2^{n+1}\left[\big(\cos(\tfrac{2\pi}{7})\big)^{n+1} + \big(\cos(\tfrac{4\pi}{7})\big)^{n+1} + \big(\cos(\tfrac{8\pi}{7})\big)^{n+1}\right], \tag{3.11}$$

for $n = 0, 1, 2, \ldots$

**Lemma 3.4** *We have*

$$f_0 = g_0 = h_0 = -1, \qquad \textit{and} \qquad h_1 = 5$$

*and*

$$\begin{cases} f_{n+1} = f_n + g_n, & n \geq 0, \\ g_{n+1} = f_n + h_n, & n \geq 0, \\ h_{n+1} = g_n + 2\,h_{n-1}, & n \geq 1. \end{cases} \tag{3.12}$$

*The elements of sequences $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ satisfy the following recurrence relation (see formula (1.4)):*

$$x_{n+3} + x_{n+2} - 2\,x_{n+1} - x_n = 0, \qquad n = 0, 1, \ldots \tag{3.13}$$

*Proof:* By (3.12) we obtain

$$g_n = f_{n+1} - f_n, \tag{3.14}$$

$$h_n = g_{n+1} - f_n = f_{n+2} - f_{n+1} - f_n, \tag{3.15}$$

and, finally, the following identity

$$f_{n+3} - f_{n+2} - f_{n+1} = 2\left(f_{n+1} - f_n - f_{n-1}\right) + f_{n+1} - f_n,$$

i.e.,

$$f_{n+3} - f_{n+2} - 4\,f_{n+1} + 3\,f_n + 2\,f_{n-1} = 0. \tag{3.16}$$

But we also have the following decomposition of respective characteristic polynomial

$$x^4 - x^3 - 4\,x^2 + 3\,x + 2 = (x-2)(x^3 + x^2 - 2\,x - 1),$$

which implies the following form of (3.16):

$$f_{n+3} + f_{n+2} - 2\,f_{n+1} - f_n = 2\,\big(f_{n+2} + f_{n+1} - 2\,f_n - f_{n-1}\big). \tag{3.17}$$

Since

$$f_3 + f_2 - 2\,f_1 - f_0 = 0$$

so, we obtain the required identity (3.13) from (3.17), (3.14) and (3.15). □

The first twelve elements of the sequences $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ are presented in Table 1.

**Remark 3.5** We note, that

$$h_{n-1} = \mathcal{B}_n = \frac{1}{2}\Big(\big(\mathcal{A}_n\big)^2 - \mathcal{A}_{2n}\Big), \qquad n = 1, 2, \ldots \tag{3.18}$$

is an accelerator sequence for Catalan's constant (see [1] and A094648 [7]).

The next lemma contains a sequence of eight identities and simultanously six newly defined auxiliary sequences of real numbers $\{\widetilde{A}_n\}$, $\{\widetilde{B}_n\}$, $\{\widetilde{C}_n\}$, $\{F_n\}$, $\{G_n\}$ and $\{H_n\}$.

**Lemma 3.6** *The following identities hold*

$$\begin{aligned} 4\,\cos(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + 4\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n + \\ + 4\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n = \\ = f_n^2 + g_n^2 - h_n^2 + h_{2n+1} - 2\,h_{2n} - 4\,h_{2n-1} - f_{2n} := 2\,F_n, \end{aligned} \tag{3.19}$$

$$\begin{aligned} 2\,\cos(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + \\ + 2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n = \\ = F_n + h_n^2 - f_n^2 - h_{2n+1} + 2\,h_{2n} + 2\,h_{2n-1} := G_n, \end{aligned} \tag{3.20}$$

$$\begin{aligned} 2\,\cos(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n + \\ + 2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n = \\ = f_n^2 - 2\,(h_{2n} + h_{2n-1}) - F_n = h_n^2 - h_{2n+1} - G_n := H_n, \end{aligned} \tag{3.21}$$

$$g_n^2 - 2\,h_{2n-1} - f_{2n} = (-1)^n\big(\mathcal{A}_n + \mathcal{A}_{n+1} - 7\,A_n\big) = G_n + F_n, \tag{3.22}$$

$$f_n^2 - 2\,h_{2n} - 2\,h_{2n-1} = (-1)^n\big(\mathcal{A}_n - \mathcal{A}_{n-2}\big) = H_n + F_n, \tag{3.23}$$

$$2\,\sin(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{2n}+2\,\sin(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+$$
$$+\,2\,\sin(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}=$$
$$=a_{2n}-a_n\,h_{2n-1}+\tfrac{\sqrt{7}}{2}\,(h_{2n-1}^2-h_{4n-1}):=\widetilde{A}_n,\quad(3.24)$$

$$2\,\sin(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+2\,\sin(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{2n}+$$
$$+\,2\,\sin(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}=$$
$$=b_{2n}-b_n\,h_{2n-1}+\tfrac{\sqrt{7}}{2}\,(h_{2n-1}^2-h_{4n-1}):=\widetilde{B}_n,\quad(3.25)$$

$$2\,\sin(\tfrac{2\pi}{7})\,\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+2\,\sin(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{2n}+$$
$$+\,2\,\sin(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{2n}=$$
$$=c_{2n}-c_n\,h_{2n-1}+\tfrac{\sqrt{7}}{2}\,(h_{2n-1}^2-h_{4n-1}):=\widetilde{C}_n,\quad(3.26)$$

Now we are ready to present the final result of this section. All recurrent sequences defined in this section are applied below to the description of the coefficients of certain polynomials.

**Theorem 3.7** *The following decompositions of polynomials hold:*

$$\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)=$$
$$=\begin{cases}\mathbb{X}^3-a_k\,\mathbb{X}^2+7\,B_{2k}\,\mathbb{X}+\sqrt{7}, & \text{for } n=2\,k,\\ \mathbb{X}^3-\alpha_{k-1}\,\mathbb{X}^2-7\,B_{2k-1}\,\mathbb{X}+\sqrt{7}, & \text{for } n=2\,k-1,\end{cases}\quad(3.27)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)=$$
$$=\begin{cases}\mathbb{X}^3-b_k\,\mathbb{X}^2+7\,(C_{2k}-B_{2k})\,\mathbb{X}+\sqrt{7}, & \text{for } n=2\,k,\\ \mathbb{X}^3-\beta_{k-1}\,\mathbb{X}^2+7\,(B_{2k-1}-C_{2k-1})\,\mathbb{X}+\sqrt{7}, & \text{for } n=2\,k-1,\end{cases}\quad(3.28)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\begin{cases}\mathbb{X}^3-c_k\,\mathbb{X}^2-7\,C_{2k}\,\mathbb{X}+\sqrt{7}, & \text{for } n=2\,k,\\ \mathbb{X}^3-\gamma_{k-1}\,\mathbb{X}^2+7\,C_{2k-1}\,\mathbb{X}+\sqrt{7}, & \text{for } n=2\,k-1,\end{cases}\quad(3.29)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-f_n\,\mathbb{X}^2+(-1)^n\,(7\,A_n-3\,\mathcal{A}_n)\,\mathbb{X}-1,\quad(3.30)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\cos(\tfrac{2\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-g_n\,\mathbb{X}^2+(-1)^n\,(\mathcal{A}_n+\mathcal{A}_{n+1}-7\,A_n)\,\mathbb{X}-1,\quad(3.31)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-F_n\,\mathbb{X}^2+(f_n+h_n)\,\mathbb{X}-1, \quad (3.32)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-G_n\,\mathbb{X}^2+(f_n+g_n)\,\mathbb{X}-1, \quad (3.33)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n\big)\times$$
$$\times\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-H_n\,\mathbb{X}^2+(g_n+h_n)\,\mathbb{X}-1. \quad (3.34)$$

*Proof:* Ad (3.30) We have

$$4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})=2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{4\pi}{7})$$

hence, by (1.4) we obtain

$$\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{n+1}=\big(2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{4\pi}{7})\big)\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n=$$
$$=\Big[\big(2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{4\pi}{7})+2\,\cos(\tfrac{8\pi}{7})\big)-\big(2\,\cos(\tfrac{2\pi}{7})+2\,\cos(\tfrac{8\pi}{7})\big)+2\,\cos(\tfrac{2\pi}{7})\Big]\times$$
$$\times\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n=-\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n-4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n+$$
$$+\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{n-1}.$$

Treating, in a similar way, the following products:

$$\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^{n+1}\qquad\text{and}\qquad\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^{n+1},$$

and adding all three received decompositions, by (2.1), we obtain

$$(-1)^n\,\big(\mathcal{A}_{n+1}-\mathcal{A}_n-\mathcal{A}_{n-1}\big)=4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big(4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n+$$
$$+4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{4\pi}{7})\big)^n+4\,\cos(\tfrac{4\pi}{7})\,\cos(\tfrac{8\pi}{7})\big(4\,\cos(\tfrac{2\pi}{7})\,\cos(\tfrac{8\pi}{7})\big)^n.$$

But, by [10], we have

$$\mathcal{A}_{n+3}-2\,\mathcal{A}_{n+2}-\mathcal{A}_{n+1}+\mathcal{A}_n\equiv 0,$$

hence

$$\mathcal{A}_{n+1}-\mathcal{A}_n-\mathcal{A}_{n-1}=\mathcal{A}_n-\mathcal{A}_{n-2},$$

which implies decomposition (3.30). □

**Remark 3.8** It should also be noted that the following interesting identity holds:

$$4\,\mathcal{A}_n-\mathcal{A}_{n-2}=7\,A_n. \quad (3.35)$$

# 4 The second group of special cases of (2.11)

Let us set

$$u_n = 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{4.1}$$

$$v_n = 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{4.2}$$

$$w_n = 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{4.3}$$

$$x_n = 2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{4.4}$$

$$y_n = 2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n + 2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n + 2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n, \tag{4.5}$$

$$z_n = \big(2\,\sin(\tfrac{2\pi}{7})\big)^{n+1} + \big(2\,\sin(\tfrac{4\pi}{7})\big)^{n+1} + \big(2\,\sin(\tfrac{8\pi}{7})\big)^{n+1}, \tag{4.6}$$

for $n \in \mathbb{N}$ and $u_0 = v_0 = w_0 = -1$ and $z_0 = y_0 = z_0 = \sqrt{7}$, $z_1 = 7$ (see Table 3, and for sequence $\{-w_{2n+1}/\sqrt{7}\}$ see A115146 [7], and see A079309 [7] for sequence $\{z_{2n}/\sqrt{7}\}$).

Then, as may by verified without difficulty, the following recurrence relations hold

$$\begin{cases} u_{n+1} = x_n, \\ v_{n+1} = -y_n - z_n = x_n - \sqrt{7}\,z_{n-1}, \\ w_{n+1} = y_n - x_n, \\ x_{n+1} = u_n - w_n, \\ y_{n+1} = w_n - v_n, \\ z_{n+1} = 2\,z_{n-1} - v_n, \end{cases} \tag{4.7}$$

for every $n \in \mathbb{N}$. Hence, we easily obtain

$$\begin{cases} x_{n+2} = x_n - w_{n+1} = 2\,x_n - y_n, \\ y_{n+2} = y_n - v_{n+1} - x_n = 2\,y_n - x_n + z_n, \\ z_{n+2} = y_n + 3\,z_n, \end{cases} \tag{4.8}$$

for every $n \in \mathbb{N}$ and finally the recurrence relation (see also equation (1.2)):

$$z_{n+6} - 7\,z_{n+4} + 14\,z_{n+2} - 7\,z_n = 0, \tag{4.9}$$

which also satisfies the remaining sequences discussed in this section: $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$.

**Remark 4.1** The characteristic polynomial of equation (4.9) (after rescaling) has the form of (1.2) and was recognized by Johannes Kepler (1571–1630). The roots of this polynomial are equal to $|A_1A_2|^2$, $|A_1A_3|^2$ and $|A_1A_4|^2$, where $A_1A_2\ldots A_7$ is a regular convex heptagon inscribed in the unit circle [2].

**Theorem 4.2** *The following decompositions of polynomials hold:*

$$\big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - u_n\,\mathbb{X}^2 + \tfrac{1}{2}\big(u_n^2 - v_{2n} - 2\,z_{2n-1}\big)\,\mathbb{X} - (-\sqrt{7}\,)^n, \tag{4.10}$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-v_n\,\mathbb{X}^2+\tfrac{1}{2}\big(v_n^2-w_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^n, \quad (4.11)$$

$$\big(\mathbb{X}-2\,\cos(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\cos(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-w_n\,\mathbb{X}^2+\tfrac{1}{2}\big(w_n^2-u_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^n, \quad (4.12)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-x_n\,\mathbb{X}^2+\tfrac{1}{2}\big(x_n^2+w_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^{n+1}, \quad (4.13)$$

$$\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(2\,\sin(\tfrac{2\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(2\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(2\,\sin(\tfrac{8\pi}{7})\big)^n\big)=$$
$$=\mathbb{X}^3-y_n\,\mathbb{X}^2+\tfrac{1}{2}\big(y_n^2+u_{2n}-2\,z_{2n-1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^{n+1}, \quad (4.14)$$

$$\big(\mathbb{X}-\big(2\,\sin(\tfrac{2\pi}{7})\big)^{n+1}\big)\big(\mathbb{X}-\big(2\,\sin(\tfrac{4\pi}{7})\big)^{n+1}\big)\big(\mathbb{X}-\big(2\,\sin(\tfrac{8\pi}{7})\big)^{n+1}\big)=$$
$$=\mathbb{X}^3-z_n\,\mathbb{X}^2+\tfrac{1}{2}\big(z_n^2-z_{2n+1}\big)\,\mathbb{X}-(-\sqrt{7}\,)^{n+1}. \quad (4.15)$$

Additionally, primarily to describe the generating functions of the sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ (see Section 7) six new recurrence sequences were introduced, which, from a certain point of view, are conjugated with sequences $\{u_n\}$, …, $\{z_n\}$. Let us set

$$u_n^*=2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+$$
$$+2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n, \quad (4.16)$$
$$v_n^*=2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+$$
$$+2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.17)$$
$$w_n^*=2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+$$
$$+2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.18)$$
$$x_n^*=2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+$$
$$+2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.19)$$
$$y_n^*=2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n+$$
$$+2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n, \quad (4.20)$$
$$z_n^*=2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n+$$
$$+2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n, \quad (4.21)$$

for $n\in\mathbb{N}$ and $u_0^*=v_0^*=w_0^*=-1$ and $x_0^*=y_0^*=z_0^*=\sqrt{7}$ (see Table 4).

An easy computation shows that the following recurrence relations hold

$$\begin{cases} u_{n+1}^*=u_n^*+w_n^*-v_n^*-z_{n-1}^2+z_{2n-1},\\ v_{n+1}^*=z_{n-1}^2-z_{2n-1}-u_n^*,\\ w_{n+1}^*=u_n^*-w_n^*, \end{cases} \quad (4.22)$$

for every $n \geq 1$. Hence, we obtain

$$u^*_{n+1} + v^*_{n+1} = w^*_n - v^*_n, \tag{4.23}$$
$$u^*_n = w^*_{n+1} + w^*_n, \tag{4.24}$$
$$w^*_{n+2} + w^*_{n+1} - w^*_n = -v^*_{n+1} - v^*_n, \tag{4.25}$$
$$w^*_{n+3} - 3\, w^*_{n+1} - w^*_n = z_{2n+1} + z_{2n-1} - z^2_n - z^2_{n-1}. \tag{4.26}$$

for every $n \geq 1$.

Also the following recurrence relations hold

$$\begin{cases} x^*_{n+1} = 2\, y^*_n - z^*_n, \\ y^*_{n+1} = 2\, x^*_n + y^*_n - z^*_n, \\ z^*_{n+1} = -x^*_n - y^*_n - z^*_n, \end{cases} \tag{4.27}$$

for every $n \in \mathbb{N}$, which implies the identities

$$y^*_{n+1} = -x^*_{n+2} + 2\, x^*_{n+1} + 7\, x^*_n, \tag{4.28}$$
$$x^*_{n+2} + 7\, x^*_n + 7\, x^*_{n-1} = 0 \tag{4.29}$$

for every $n \geq 1$.

**Theorem 4.3** *The following decompositions of polynomials hold:*

$$\begin{aligned} \big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) \times \\ \times \big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - u^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(u_n + v_n\big)\,\mathbb{X} - 7^n, \end{aligned} \tag{4.30}$$

$$\begin{aligned} \big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) \times \\ \times \big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - v^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(v_n + w_n\big)\,\mathbb{X} - 7^n, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \big(\mathbb{X} - 2\,\cos(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\cos(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big) \times \\ \times \big(\mathbb{X} - 2\,\cos(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - w^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(u_n + w_n\big)\,\mathbb{X} - 7^n, \end{aligned} \tag{4.32}$$

$$\begin{aligned} \big(\mathbb{X} - 2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)\big(\mathbb{X} - 2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) \times \\ \times \big(\mathbb{X} - 2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big) = \\ = \mathbb{X}^3 - x^*_n\,\mathbb{X}^2 + (-\sqrt{7}\,)^n\,\big(w_n - v_n\big)\,\mathbb{X} - 7^n, \end{aligned} \tag{4.33}$$

$$\begin{aligned}\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)\times\\ \times\,\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)=\\ =\mathbb{X}^3-y_n^*\,\mathbb{X}^2+(-\sqrt{7}\,)^n\,\big(u_n-w_n\big)\,\mathbb{X}-7^n,\quad(4.34)\end{aligned}$$

$$\begin{aligned}\big(\mathbb{X}-2\,\sin(\tfrac{2\pi}{7})\big(4\,\sin(\tfrac{4\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\big(\mathbb{X}-2\,\sin(\tfrac{4\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{8\pi}{7})\big)^n\big)\times\\ \times\,\big(\mathbb{X}-2\,\sin(\tfrac{8\pi}{7})\big(4\,\sin(\tfrac{2\pi}{7})\,\sin(\tfrac{4\pi}{7})\big)^n\big)=\\ =\mathbb{X}^3-z_n^*\,\mathbb{X}^2+(-\sqrt{7}\,)^n\,\big(v_n-w_n\big)\,\mathbb{X}-7^n.\quad(4.35)\end{aligned}$$

**Lemma 4.4** *The following two groups of identities hold:*

$$\begin{aligned}u_n^2&=2\,z_{2n-1}+v_{2n}+v_n^*+u_n^*,\\ v_n^2&=2\,z_{2n-1}+w_{2n}+v_n^*+w_n^*,\\ w_n^2&=2\,z_{2n-1}+u_{2n}+u_n^*+w_n^*,\\ x_n^2&=2\,z_{2n-1}-w_{2n}+w_n^*-v_n^*,\\ y_n^2&=2\,z_{2n-1}-u_{2n}+u_n^*-w_n^*,\\ z_n^2&=2\,z_{2n-1}+2\,v_n^{'}-2\,u_n^*,\end{aligned}\qquad(4.36)$$

*and*

$$\begin{aligned}\big(u_n^*\big)^2&=z_{2n-1}^2-z_{4n-1}+v_n^*+2\,\big(-\sqrt{7}\,\big)^n\big(u_n+v_n\big),\\ \big(v_n^*\big)^2&=z_{2n-1}^2-z_{4n-1}+w_n^*+2\,\big(-\sqrt{7}\,\big)^n\big(v_n+w_n\big),\\ \big(w_n^*\big)^2&=z_{2n-1}^2-z_{4n-1}+u_n^*+2\,\big(-\sqrt{7}\,\big)^n\big(u_n+w_n\big),\\ \big(x_n^*\big)^2&=z_{2n-1}^2-z_{4n-1}-w_n^*+2\,\big(-\sqrt{7}\,\big)^n\big(w_n-v_n\big),\\ \big(y_n^*\big)^2&=z_{2n-1}^2-z_{4n-1}-u_n^*+2\,\big(-\sqrt{7}\,\big)^n\big(u_n-w_n\big),\\ \big(z_n^*\big)^2&=z_{2n-1}^2-z_{4n-1}-v_n^*+2\,\big(-\sqrt{7}\,\big)^n\big(v_n-u_n\big).\end{aligned}\qquad(4.37)$$

# 5 Some Ramanujan-type formulas

Let $p,q,r\in\mathbb{R}$. Shevelev [4] (see also [2]) proved that if $z_1$, $z_2$, $z_3$ are roots of the polynomial

$$w(z)=z^3+p\,z^2+q\,z+r$$

and $z_1$, $z_2$, $z_3$ are all reals and at least the following condition holds

$$p\,\sqrt[3]{r}+3\,\big(\sqrt[3]{r}\big)^2+q=0,\qquad(5.1)$$

then the following formula holds

$$\begin{aligned}\sqrt[3]{z_1}+\sqrt[3]{z_2}+\sqrt[3]{z_3}&=\sqrt[3]{-p-6\,\sqrt[3]{r}+\sqrt[3]{(p+6\,\sqrt[3]{r})^3-(p-3\,\sqrt[3]{r})^3}}=\\ &=\sqrt[3]{-p-6\,\sqrt[3]{r}+3\,\sqrt[3]{9\,r+3\,p\,(\sqrt[3]{r})^2+p^2\,\sqrt[3]{r}}}\qquad(5.2)\end{aligned}$$

(all radicals are determined to be real).

It was verified that only following polynomials (among all discussed in this paper) satisfy the condition (5.1); simultanously below the respective form of identity (5.2) is presented:

polynomial (1.4) and polynomial (2.1) for $n = 1$ (it is the classical Ramanujan formula):

$$\sqrt[3]{2\,\cos(\tfrac{2\pi}{7})} + \sqrt[3]{2\,\cos(\tfrac{4\pi}{7})} + \sqrt[3]{2\,\cos(\tfrac{8\pi}{7})} = \sqrt[3]{5 - 3\,\sqrt[3]{7}}; \tag{5.3}$$

polynomial (3.31) for $n = 4$:

$$\cos(\tfrac{2\pi}{7})\,\sqrt[3]{\sec(\tfrac{8\pi}{7})} + \cos(\tfrac{4\pi}{7})\,\sqrt[3]{\sec(\tfrac{2\pi}{7})} + \cos(\tfrac{8\pi}{7})\,\sqrt[3]{\sec(\tfrac{4\pi}{7})} = \frac{1}{2}\,\sqrt[3]{18\,(2 - \sqrt[3]{7})}; \tag{5.4}$$

polynomial (3.31) for $n = 3$:

$$2\,\cos(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})} + 2\,\cos(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})} + 2\,\cos(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})} = \\ = \sqrt[3]{-2 - 3\,\sqrt[3]{49}}; \tag{5.5}$$

polynomial (3.32) for $n = 2, 3$, respectively:

$$\sqrt[3]{\frac{\cos(\frac{2\pi}{7})}{1 + \cos(\frac{8\pi}{7})}} + \sqrt[3]{\frac{\cos(\frac{4\pi}{7})}{1 + \cos(\frac{2\pi}{7})}} + \sqrt[3]{\frac{\cos(\frac{8\pi}{7})}{1 + \cos(\frac{4\pi}{7})}} = \sqrt[3]{11 - 3\,\sqrt[3]{49}}, \tag{5.6}$$

$$\sec(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})} + \sec(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \sec(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})} = -2\,\sqrt[3]{9\,(1 + \sqrt[3]{7})}; \tag{5.7}$$

polynomial (3.30) for $n = 1$, polynomial (3.31) for $n = 1$ and polynomial (3.34) for $n = 2$:

$$\sqrt[3]{\sec(\tfrac{2\pi}{7})} + \sqrt[3]{\sec(\tfrac{4\pi}{7})} + \sqrt[3]{\sec(\tfrac{8\pi}{7})} = \sqrt[3]{8 - 6\,\sqrt[3]{7}}; \tag{5.8}$$

polynomial (4.11) for $n = 6$:

$$\sin^2(\tfrac{2\pi}{7})\,\sqrt[3]{2\cos(\tfrac{4\pi}{7})} + \sin^2(\tfrac{4\pi}{7})\,\sqrt[3]{2\cos(\tfrac{8\pi}{7})} + \sin^2(\tfrac{8\pi}{7})\,\sqrt[3]{2\cos(\tfrac{2\pi}{7})} = \\ = -\frac{1}{4}\,\sqrt[3]{63\,(1 + \sqrt[3]{7})}; \tag{5.9}$$

polynomial (4.13) for $n = 2$:

$$\sqrt[3]{\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{8\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{4\pi}{7})}{\sin(\frac{2\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{8\pi}{7})}{\sin(\frac{4\pi}{7})}} = \sqrt[3]{5 - 3\,\sqrt[3]{7}}; \tag{5.10}$$

polynomial (4.14) for $n = 2$:

$$\sqrt[3]{\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{4\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{4\pi}{7})}{\sin(\frac{8\pi}{7})}} + \sqrt[3]{\frac{\sin(\frac{8\pi}{7})}{\sin(\frac{2\pi}{7})}} = \sqrt[3]{4 - 3\,\sqrt[3]{7}}; \tag{5.11}$$

polynomial (4.14) for $n=5$:

$$\sin^2(\tfrac{2\pi}{7})\sqrt[3]{\frac{\sin(\frac{8\pi}{7})}{\sin(\frac{2\pi}{7})}}+\sin^2(\tfrac{4\pi}{7})\sqrt[3]{\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{4\pi}{7})}}+\sin^2(\tfrac{8\pi}{7})\sqrt[3]{\frac{\sin(\frac{4\pi}{7})}{\sin(\frac{8\pi}{7})}}=\frac{1}{4}\sqrt[3]{77-21\sqrt[3]{49}}; \quad (5.12)$$

polynomial (4.30) for $n=3$:

$$\sqrt[6]{7}\left(\csc(\tfrac{2\pi}{7})\sqrt[3]{2\cos(\tfrac{2\pi}{7})}+\csc(\tfrac{4\pi}{7})\sqrt[3]{2\cos(\tfrac{4\pi}{7})}+\csc(\tfrac{8\pi}{7})\sqrt[3]{2\cos(\tfrac{8\pi}{7})}\right)= \\ =2\sqrt[3]{2+3\sqrt[3]{49}}; \quad (5.13)$$

polynomial (4.32) for $n=3$:

$$\sqrt[6]{7}\left(\csc(\tfrac{4\pi}{7})\sqrt[3]{2\cos(\tfrac{2\pi}{7})}+\csc(\tfrac{8\pi}{7})\sqrt[3]{2\cos(\tfrac{4\pi}{7})}+\csc(\tfrac{2\pi}{7})\sqrt[3]{2\cos(\tfrac{8\pi}{7})}\right)= \\ =2\sqrt[3]{3\sqrt[3]{7}-5}; \quad (5.14)$$

polynomial (4.32) for $n=6$:

$$\csc^2(\tfrac{4\pi}{7})\sqrt[3]{2\cos(\tfrac{2\pi}{7})}+\csc^2(\tfrac{8\pi}{7})\sqrt[3]{2\cos(\tfrac{4\pi}{7})}+\csc^2(\tfrac{2\pi}{7})\sqrt[3]{2\cos(\tfrac{8\pi}{7})}= \\ =-4\sqrt[3]{\frac{2+3\sqrt[3]{49}}{7}}. \quad (5.15)$$

**Remark 5.1** Shevelev [4] presented the identities (5.10) and (5.11) in the following alternative form:

$$\sqrt[3]{2+\sec(\tfrac{2\pi}{7})}+\sqrt[3]{2+\sec(\tfrac{4\pi}{7})}+\sqrt[3]{2+\sec(\tfrac{8\pi}{7})}=\sqrt[3]{6\sqrt[3]{7}-10} \quad (5.16)$$

and

$$\sqrt[3]{1+2\cos(\tfrac{2\pi}{7})}+\sqrt[3]{1+2\cos(\tfrac{4\pi}{7})}+\sqrt[3]{1+2\cos(\tfrac{8\pi}{7})}= \\ =\sqrt[3]{\frac{2\cos(\frac{2\pi}{7})}{2\cos(\frac{2\pi}{7})+1}}+\sqrt[3]{\frac{2\cos(\frac{4\pi}{7})}{2\cos(\frac{4\pi}{7})+1}}+\sqrt[3]{\frac{2\cos(\frac{8\pi}{7})}{2\cos(\frac{8\pi}{7})+1}}=\sqrt[3]{3\sqrt[3]{7}-4}, \quad (5.17)$$

respectively.

**Remark 5.2** By (5.9), (5.3) and by applied the formula $2\sin^2\alpha=1-\cos(2\alpha)$ we obtain

$$2\cos\left(\tfrac{2\pi}{7}\right)\sqrt[3]{2\cos\left(\tfrac{2\pi}{7}\right)}+2\cos\left(\tfrac{4\pi}{7}\right)\sqrt[3]{2\cos\left(\tfrac{4\pi}{7}\right)}+2\cos\left(\tfrac{8\pi}{7}\right)\sqrt[3]{2\cos\left(\tfrac{8\pi}{7}\right)}= \\ =\sqrt[3]{63\left(1+\sqrt[3]{7}\right)}-2\sqrt[3]{5-3\sqrt[3]{7}}. \quad (5.18)$$

Moreover, from (5.3), (5.8) and the equality $8\cos(\frac{2\pi}{7})\cos(\frac{4\pi}{7})\cos(\frac{8\pi}{7})=1$ we get

$$\sqrt[3]{\left(2\cos\left(\tfrac{2\pi}{7}\right)\right)^2}+\sqrt[3]{\left(2\cos\left(\tfrac{4\pi}{7}\right)\right)^2}+\sqrt[3]{\left(2\cos\left(\tfrac{8\pi}{7}\right)\right)^2}= \\ =\sqrt[3]{\left(5-3\sqrt[3]{7}\right)^2}-2\sqrt[3]{4-3\sqrt[3]{7}}. \quad (5.19)$$

# 6 The sine-Fibonacci numbers of order 7

The quasi-Fibonacci numbers of order 7 introduced by Wituła et al. in [10] and further developed in Section 1 constitute the simplest, only one-parameter type of the so-called cosine-Fibonacci numbers of order 7. The name "cosine-Fibonacci numbers" is derived from the form of the decomposition of formulas: $(1+\delta\,(\xi+\xi^6))^n = (1+2\,\delta\,\cos(\frac{2\pi}{7}))^n$, ... In this Section we shall introduce and analyze a sine variety of these numbers, created in the course of decomposing formulas: $(1+\delta\,(\xi-\xi^6))^n = (1+2\,i\,\delta\,\sin(\frac{2\pi}{7}))^n$, ...

**Theorem 6.1** *Let $\xi, \delta \in \mathbb{C}$, $\xi^7 = 1$ and $\xi \neq 1$. Then, for every $n \in \mathbb{N}$, there exist polynomials $p_n, r_n, s_n, k_n, l_n \in \mathbb{Z}[\delta]$, called here the sine-Fibonacci numbers of order 7, so that*

$$\begin{aligned}(1+\delta\,(\xi\ -\xi^6))^n &= p_n(\delta) + r_n(\delta)\,(\xi\ -\xi^6) + s_n(\delta)\,(\xi^2-\xi^5) + \\ &\quad + k_n(\delta)\,(\xi\ +\xi^6) + l_n(\delta)\,(\xi^2+\xi^5), \qquad (6.1)\\ (1+\delta\,(\xi^2-\xi^5))^n &= p_n(\delta) + r_n(\delta)\,(\xi^2-\xi^5) + s_n(\delta)\,(\xi^4-\xi^3) + \\ &\quad + k_n(\delta)\,(\xi^2+\xi^5) + l_n(\delta)\,(\xi^4+\xi^3), \qquad (6.2)\\ (1+\delta\,(\xi^4-\xi^3))^n &= p_n(\delta) + r_n(\delta)\,(\xi^4-\xi^3) + s_n(\delta)\,(\xi\ -\xi^6) + \\ &\quad + k_n(\delta)\,(\xi^4+\xi^3) + l_n(\delta)\,(\xi\ +\xi^6). \qquad (6.3)\end{aligned}$$

*We have $p_1(\delta) = 1$, $r_1(\delta) = \delta$ and $s_1(\delta) = k_1(\delta) = l_1(\delta) = 0$ (see Table 7, where the initial values of these polynomial are presented). These polynomials are connected by recurrence relations*

$$\begin{cases} p_{n+1}(\delta) = p_n(\delta) - \delta\,s_n(\delta) - 2\,\delta\,r_n(\delta) - i\,\sqrt{7}\,\delta l_n(\delta), \\ r_{n+1}(\delta) = r_n(\delta) + \delta\,p_n(\delta), \\ s_{n+1}(\delta) = s_n(\delta) + \delta\,k_n(\delta) + \delta\,l_n(\delta), \\ k_{n+1}(\delta) = k_n(\delta) - 2\,\delta\,s_n(\delta), \\ l_{n+1}(\delta) = l_n(\delta) + \delta\,r_n(\delta) - \delta\,s_n(\delta), \end{cases} \qquad (6.4)$$

*for $n \in \mathbb{N}$. Hence, the following relationships can be deduced:*

$$\delta\,p_n(\delta) = r_{n+1}(\delta) - r_n(\delta), \qquad (6.5)$$

$$2\,\delta\,s_n(\delta) = k_{n+1}(\delta) - k_n(\delta), \qquad (6.6)$$

$$2\,\delta^2\,l_n(\delta) = -k_{n+2}(\delta) + 2\,k_{n+1}(\delta) - (1+2\,\delta^2)\,k_n(\delta), \qquad (6.7)$$

$$2\,\delta^3\,r_n(\delta) = -k_{n+3}(\delta) + 3\,k_{n+2}(\delta) - 3\,(1+\delta^2)\,k_{n+1}(\delta) + (1+3\,\delta^2)\,k_n(\delta) \qquad (6.8)$$

*and, at last, the main identity*

$$\begin{aligned}k_{n+5}(\delta) - 5\,k_{n+4} + (10+5\,\delta^2)\,k_{n+3}(\delta) + (i\,\sqrt{7}\,\delta^3 - 15\,\delta^2 - 10)\,k_{n+2}(\delta) + \\ + (7\,\delta^4 - i\,2\,\sqrt{7}\,\delta^3 + 15\,\delta^2 + 5)\,k_{n+1}(\delta) + \\ + (i\,2\,\sqrt{7}\,\delta^5 - 7\,\delta^4 + i\,\sqrt{7}\,\delta^3 - 5\,\delta^2 - 1)\,k_n(\delta) = 0. \qquad (6.9)\end{aligned}$$

*This identity satisfies, also by (6.5)–(6.8), the remaining polynomials: $p_n(\delta)$, $r_n(\delta)$, $s_n(\delta)$, and $l_n(\delta)$, $n \in \mathbb{N}$. The characteristic polynomial corresponding to identity (6.9) has the*

*following decomposition:*

$$\mathbb{X}^5 - 5\,\mathbb{X}^4 + (10 + 5\,\delta^2)\,\mathbb{X}^3 + (i\,\sqrt{7}\,\delta^3 - 15\,\delta^2 - 10)\,\mathbb{X}^2 + (7\,\delta^4 - i\,2\,\sqrt{7}\,\delta^3 + 15\,\delta^2 + 5)\,\mathbb{X} +$$
$$+ i\,2\,\sqrt{7}\,\delta^5 - 7\,\delta^4 + i\,\sqrt{7}\,\delta^3 - 5\,\delta^2 - 1 = (\mathbb{X} - 1 - \delta\,(\xi - \xi^6))(\mathbb{X} - 1 - \delta\,(\xi^2 - \xi^5)) \times$$
$$\times\,(\mathbb{X} - 1 - \delta\,(\xi^4 - \xi^3))(\mathbb{X} - 1 + \delta\,(\xi + \xi^2 + \xi^4))(\mathbb{X} - 1 + \delta\,(\xi^3 + \xi^5 + \xi^6)) =$$
$$= \big(\mathbb{X}^3 + (-3 - i\,\sqrt{7}\,\delta)\,\mathbb{X}^2 + (3 + i\,2\,\sqrt{7}\,\delta)\,\mathbb{X} - i\,\sqrt{7}\,\delta^3 - i\,\sqrt{7}\,\delta - 1\big) \times$$
$$\times\,\big(\mathbb{X}^2 + (-2 + i\,\sqrt{7}\,\delta)\,\mathbb{X} + 1 - 2\,\delta^2 - i\,\sqrt{7}\,\delta\big). \quad (6.10)$$

*Hence, we obtain, for example, the following explicit form of* $k_n(\delta)$*:*

$$k_n(\delta) = a\,(1 + \delta\,(\xi - \xi^6))^n + b\,(1 + \delta\,(\xi^2 - \xi^5))^n +$$
$$+ c\,(1 + \delta\,(\xi^4 - \xi^3))^n + d\,(1 - \delta\,(\xi + \xi^2 + \xi^4))^n + e\,(1 + \delta\,(\xi^3 + \xi^5 + \xi^6))^n, \quad (6.11)$$

*where*

$$a = -2\big(5 + (\xi^2 + \xi^5) + 7\,(\xi^3 + \xi^4)\big)^{-1} = \tfrac{2}{301}\big(46\,(\xi + \xi^6) + 5\,(\xi^3 + \xi^4) - 11\big),$$
$$b = \quad 2\big(2 + 7\,(\xi^2 + \xi^5) + 6\,(\xi^3 + \xi^4)\big)^{-1} = \tfrac{2}{187}\big(19\,(\xi + \xi^6) + 46\,(\xi^2 + \xi^5) - 5\big),$$
$$c = -2\big(4 + 6\,(\xi^2 + \xi^5) - (\xi^3 + \xi^4)\big)^{-1} = \tfrac{2}{301}\big(5\,(\xi^2 + \xi^5) + 46\,(\xi^3 + \xi^4) - 11\big),$$
$$d = -2\big(-5 + 2\,(\xi + \xi^2 + \xi^4)\big)^{-1} = \tfrac{2}{43}\big(5 - 2\,(\xi^3 + \xi^5 + \xi^6)\big),$$
$$e = \quad 2\big(7 + 2\,(\xi + \xi^2 + \xi^4)\big)^{-1} = \tfrac{2}{43}\big(7 + 2\,(\xi^3 + \xi^5 + \xi^6)\big).$$

**Corollary 6.2** *The following identity holds:*

$$\Omega_n(\delta) := (1 + \delta\,(\xi - \xi^6))^n + (1 + \delta\,(\xi^2 - \xi^5))^n + (1 + \delta\,(\xi^4 - \xi^3))^n =$$
$$= 3\,p_n(\delta) - k_n(\delta) - l_n(\delta) + i\,\sqrt{7}\big(r_n(\delta) + s_n(\delta)\big). \quad (6.12)$$

**Definition 6.3** *To simplify the notation, we shall write* $\Omega_n$ *instead of* $\Omega_n(1)$*.*

**Theorem 6.4** *The following decompositions of the polynomials hold:*

$$q_n(\mathbb{X};\delta) :=$$
$$= \big(\mathbb{X} - \big(1 + \delta\,(2\,i\,\sin(\tfrac{2\,\pi}{7}))^n\big)\big)\big(\mathbb{X} - \big(1 + \delta\,(2\,i\,\sin(\tfrac{4\,\pi}{7}))^n\big)\big)\big(\mathbb{X} - \big(1 + \delta\,(2\,i\,\sin(\tfrac{8\,\pi}{7}))^n\big)\big) =$$
$$= \mathbb{X}^3 - \Omega_n(\delta)\,\mathbb{X}^2 + \tfrac{1}{2}\big((\Omega_n(\delta))^2 - \Omega_{2n}(\delta)\big)\,\mathbb{X} - \big(1 + i\,\sqrt{7}\,\delta + i\,\sqrt{7}\,\delta^3\big)^n, \quad (6.13)$$

$$\big(x - \big(\cot(\tfrac{2\,\pi}{7})\big)^n\big)\big(x - \big(\cot(\tfrac{4\,\pi}{7})\big)^n\big)\big(x - \big(\cot(\tfrac{8\,\pi}{7})\big)^n\big) =$$
$$= x^3 - \big(\tfrac{3}{\sqrt{7}}\big)^n\,\mathcal{A}_n(\tfrac{2}{3})\,x^2 + \Omega_n(\tfrac{2\,i}{\sqrt{7}})\,x + \frac{(-1)^{n-1}}{\big(\sqrt{7}\big)^n}. \quad (6.14)$$

*Moreover, the following identity holds:*

$$\Omega_n^2(\delta) = \Omega_{2n}(\delta) + 2\,\big(3\,p_n^2(\delta) - 2\,l_n^2(\delta) - 2\,k_n^2(\delta) - 2\,p_n(\delta)\,k_n(\delta) - 2\,p_n(\delta)\,l_n(\delta) - 7\,r_n(\delta)\,s_n(\delta) +$$
$$+ 3\,k_n(\delta)\,l_n(\delta) + i\,2\,\sqrt{7}\,(p_n(\delta)\,r_n(\delta) + p_n(\delta)\,s_n(\delta) - r_n(\delta)\,k_n(\delta) - s_n(\delta)\,l_n(\delta)) +$$
$$+ i\,\sqrt{7}(r_n(\delta)\,l_n(\delta) - k_n(\delta)\,s_n(\delta))\big) =$$
$$= \Omega_{2n}(\delta) + 4\,p_n(\delta)\,\Omega_n(\delta) + 2\,\big(-3\,p_n^2(\delta) - 2\,l_n^2(\delta) - 2\,k_n^2(\delta) - 7\,r_n(\delta)\,s_n(\delta) + 3\,k_n(\delta)\,l_n(\delta) +$$
$$+ i\,\sqrt{7}(r_n(\delta)\,l_n(\delta) - 2\,r_n(\delta)\,k_n(\delta) - k_n(\delta)\,s_n(\delta) - 2\,s_n(\delta)\,l_n(\delta))\big). \quad (6.15)$$

# 7 Generating functions

The generating functions of almost all sequences discussed and defined in this paper are presented in this section.

By (1.9) and (2.2) we obtain (for $k \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \mathcal{A}_{kn}(\delta)\,\mathbb{X}^n = \left(1-\left(1+\delta\,(\xi+\xi^6)\right)^k\mathbb{X}\right)^{-1} + \left(1-\left(1+\delta\,(\xi^2+\xi^5)\right)^k\mathbb{X}\right)^{-1} +$$
$$+\left(1-\left(1+\delta\,(\xi^3+\xi^4)\right)^k\mathbb{X}\right)^{-1} = \frac{3-2\,\mathcal{A}_k(\delta)\,\mathbb{X}+\mathcal{B}_k(\delta)\,\mathbb{X}^2}{1-\mathcal{A}_k(\delta)\,\mathbb{X}+\mathcal{B}_k(\delta)\,\mathbb{X}^2-(1-\delta-2\,\delta^2+\delta^3)^k\,\mathbb{X}^3}, \quad (7.1)$$

and, in the sequel, for $k=1$ we get

$$\sum_{n=0}^{\infty} \mathcal{A}_{n}(\delta)\,\mathbb{X}^n = \left(1-\left(1+\delta\,(\xi+\xi^6)\right)\mathbb{X}\right)^{-1} + \left(1-\left(1+\delta\,(\xi^2+\xi^5)\right)\mathbb{X}\right)^{-1} +$$
$$+\left(1-\left(1+\delta\,(\xi^3+\xi^4)\right)\mathbb{X}\right)^{-1} = \frac{\mathbb{X}^2\,p_7'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\,p_7\big(1/\mathbb{X};\delta\big)} =$$
$$= \frac{3+2\,(\delta-3)\,\mathbb{X}+(3-2\,\delta-2\,\delta^2)\,\mathbb{X}^2}{1+(\delta-3)\,\mathbb{X}+(3-2\,\delta-2\,\delta^2)\,\mathbb{X}^2+(-1+\delta+2\,\delta^2-\delta^3)\,\mathbb{X}^3}, \quad (7.2)$$

where $p_7(\mathbb{X},\delta)=\mathbb{X}^3+(\delta-3)\mathbb{X}^2+(3-2\delta-2\delta^2)\mathbb{X}+(-1+\delta+2\delta^2-\delta^3)$ (see [10]).

By (1.10) and (2.7) we get (for $k \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \mathcal{B}_{kn}(\delta)\,\mathbb{X}^n = \frac{\mathbb{X}^2\,r_k'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\,r_k\big(1/\mathbb{X};\delta\big)} =$$
$$= \frac{3-2\,\mathcal{B}_k(\delta)\,\mathbb{X}+(1-\delta-2\,\delta^2+\delta^3)\,\mathcal{A}_k(\delta)\,\mathbb{X}^2}{1-\mathcal{B}_k(\delta)\,\mathbb{X}+(1-\delta-2\,\delta^2+\delta^3)\,\mathcal{A}_k(\delta)\,\mathbb{X}^2-(1-\delta-2\,\delta^2+\delta^3)^{2k}\,\mathbb{X}^3}, \quad (7.3)$$

where $r_n(\mathbb{X},\delta)$ is defined in (2.7). By (3.19) from [10] we obtain

$$\sum_{n=0}^{\infty} A_{n}(\delta)\,\mathbb{X}^n = \frac{2-\xi^3-\xi^4}{1-\left(1+\delta\,(\xi+\xi^6)\right)\mathbb{X}} + \frac{2-\xi-\xi^6}{1-\left(1+\delta\,(\xi^2+\xi^5)\right)\mathbb{X}} +$$
$$+\frac{2-\xi^2-\xi^5}{1-\left(1+\delta\,(\xi^3+\xi^4)\right)\mathbb{X}} = \frac{1+(-2+\delta)\,\mathbb{X}+(1-\delta)\,\mathbb{X}^2}{\mathbb{X}^3\,p_7\big(1/\mathbb{X};\delta\big)}; \quad (7.4)$$

by (3.18) and (3.17) from [10] we obtain respectively:

$$\sum_{n=0}^{\infty} B_{n}(\delta)\,\mathbb{X}^n = \frac{\xi+\xi^6-\xi^3-\xi^4}{1-\left(1+\delta\,(\xi+\xi^6)\right)\mathbb{X}} + \frac{\xi^2+\xi^5-\xi-\xi^6}{1-\left(1+\delta\,(\xi^2+\xi^5)\right)\mathbb{X}} +$$
$$+\frac{\xi^3+\xi^4-\xi^2-\xi^5}{1-\left(1+\delta\,(\xi^3+\xi^4)\right)\mathbb{X}} = \frac{\delta\,\mathbb{X}+(\delta^2-\delta)\,\mathbb{X}^2}{\mathbb{X}^3\,p_7\big(1/\mathbb{X};\delta\big)} \quad (7.5)$$

and

$$\sum_{n=0}^{\infty} C_n(\delta)\,\mathbb{X}^n = \sum_{n=0}^{\infty} \big(3A_n(\delta) - B_n(\delta) - \mathcal{A}_n(\delta)\big)\,\mathbb{X}^n = \frac{\delta^2\,\mathbb{X}^2}{\mathbb{X}^3\,p_7\big(1/\mathbb{X};\,\delta\big)}. \tag{7.6}$$

Equally, on the grounds of (3.1)–(3.3) and (3.24)–(3.26), we obtain the following formulas:

$$\sum_{n=0}^{\infty} a_{kn}\,\mathbb{X}^n = \frac{\sqrt{7} - (b_k + c_k)\,\mathbb{X} + \widetilde{A}_k\,\mathbb{X}^2}{1 - \mathcal{B}_{2k}\,\mathbb{X} + \mathcal{A}_{2k}\,\mathbb{X}^2 - \mathbb{X}^3}, \tag{7.7}$$

$$\sum_{n=0}^{\infty} b_{kn}\,\mathbb{X}^n = \frac{\sqrt{7} - (a_k + c_k)\,\mathbb{X} + \widetilde{B}_k\,\mathbb{X}^2}{1 - \mathcal{B}_{2k}\,\mathbb{X} + \mathcal{A}_{2k}\,\mathbb{X}^2 - \mathbb{X}^3}, \tag{7.8}$$

$$\sum_{n=0}^{\infty} c_{kn}\,\mathbb{X}^n = \frac{\sqrt{7} - (a_k + b_k)\,\mathbb{X} + \widetilde{C}_k\,\mathbb{X}^2}{1 - \mathcal{B}_{2k}\,\mathbb{X} + \mathcal{A}_{2k}\,\mathbb{X}^2 - \mathbb{X}^3}. \tag{7.9}$$

The special cases (for $k = 1$) can also be generated in the following selective ways: by (3.1) and (1.5) we obtain

$$\sum_{n=0}^{\infty} a_n\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\,\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{8\,\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{4\,\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{2\,\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{8\,\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{4\,\pi}{7}\right)\mathbb{X}} =$$

$$= \frac{\sqrt{7} - 2\,\sqrt{7}\,\mathbb{X} - \sqrt{7}\,\mathbb{X}^2}{1 - 5\,\mathbb{X} + 6\,\mathbb{X}^2 - \mathbb{X}^3}; \tag{7.10}$$

however, by (3.2) and (3.4) we obtain

$$\sum_{n=0}^{\infty} b_n\,\mathbb{X}^n = \sum_{n=0}^{\infty} \big(a_{n+1} + 2\,a_n\big)\,\mathbb{X}^n = \frac{1 + 2\,\mathbb{X}}{\mathbb{X}}\Big(\sum_{n=0}^{\infty} a_n\,\mathbb{X}^n\Big) - \frac{\sqrt{7}}{\mathbb{X}} =$$

$$= \frac{2\,\sin\left(\frac{4\,\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{8\,\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{8\,\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{2\,\pi}{7}\right)\mathbb{X}} + \frac{2\,\sin\left(\frac{2\,\pi}{7}\right)}{1 - 4\,\cos^2\left(\frac{4\,\pi}{7}\right)\mathbb{X}} =$$

$$= \frac{\sqrt{7} - 3\,\sqrt{7}\,\mathbb{X} + 3\,\sqrt{7}\,\mathbb{X}^2}{1 - 5\,\mathbb{X} + 6\,\mathbb{X}^2 - \mathbb{X}^3} \tag{7.11}$$

and

$$\sum_{n=0}^{\infty} c_n\,\mathbb{X}^n = \sum_{n=0}^{\infty} \big(a_n + 2\,b_n - b_{n+1}\big)\,\mathbb{X}^n = \frac{\sqrt{7} - 5\,\sqrt{7}\,\mathbb{X} + 4\,\sqrt{7}\,\mathbb{X}^2}{1 - 5\,\mathbb{X} + 6\,\mathbb{X}^2 - \mathbb{X}^3}. \tag{7.12}$$

By (6.12) and (6.13) we get

$$\sum_{n=0}^{\infty} \Omega_{kn}(\delta)\,\mathbb{X}^n = \big(1 - \big(1 + \delta\,(\xi - \xi^6)\big)^k\,\mathbb{X}\big)^{-1} + \big(1 - \big(1 + \delta\,(\xi^2 - \xi^5)\big)^k\,\mathbb{X}\big)^{-1} +$$

$$+ \big(1 - \big(1 + \delta\,(\xi^3 - \xi^4)\big)^k\,\mathbb{X}\big)^{-1} = \frac{\mathbb{X}^2\,q_k'\big(1/\mathbb{X};\,\delta\big)}{\mathbb{X}^3\,q_k\big(1/\mathbb{X};\,\delta\big)} =$$

$$= \frac{3 - 2\,\Omega_k(\delta)\,\mathbb{X} + \frac{1}{2}\,\big((\Omega_k(\delta))^2 - \Omega_{2k}(\delta)\big)\,\mathbb{X}^2}{1 - \Omega_k(\delta)\,\mathbb{X} + \frac{1}{2}\,\big((\Omega_k(\delta))^2 - \Omega_{2k}(\delta)\big)\,\mathbb{X}^2 - (1 + i\,\sqrt{7}\,\delta + i\,\sqrt{7}\,\delta^3)^k\,\mathbb{X}^3}, \tag{7.13}$$

and, the special case, for $k=1$ (by (6.12) and (6.10)):

$$\sum_{n=0}^{\infty}\Omega_n(\delta)\,\mathbb{X}^n=\big(1-\big(1+\delta\,(\xi-\xi^6)\big)\,\mathbb{X}\big)^{-1}+\big(1-\big(1+\delta\,(\xi^2-\xi^5)\big)\,\mathbb{X}\big)^{-1}+$$
$$+\big(1-\big(1+\delta\,(\xi^4-\xi^3)\big)\,\mathbb{X}\big)^{-1}=\frac{\mathbb{X}^2\,q_1'\big(1/\mathbb{X};\delta\big)}{\mathbb{X}^3\,q_1\big(1/\mathbb{X};\delta\big)}=$$
$$=\frac{3-2\,(3+i\,\sqrt{7}\,\delta)\,\mathbb{X}+(3+i\,2\,\sqrt{7}\,\delta)\,\mathbb{X}^2}{1-(3+i\,\sqrt{7}\,\delta)\,\mathbb{X}+(3+i\,2\,\sqrt{7}\,\delta)\,\mathbb{X}^2-(1+i\,\sqrt{7}\,\delta+i\,\sqrt{7}\,\delta^3)\,\mathbb{X}^3},\quad(7.14)$$

where $q_k(\mathbb{X};\delta)$ is defined in (6.13).

By (1.4), (3.9)–(3.11) and (3.19)–(3.21) the following formulas may be obtained:

$$\sum_{n=0}^{\infty}f_{kn}\,\mathbb{X}^n=\frac{2\,\cos\big(\frac{2\,\pi}{7}\big)}{1-\big(2\,\cos(\frac{4\,\pi}{7})\big)^k\,\mathbb{X}}+\frac{2\,\cos\big(\frac{4\,\pi}{7}\big)}{1-\big(2\,\cos(\frac{8\,\pi}{7})\big)^k\,\mathbb{X}}+\frac{2\,\cos\big(\frac{8\,\pi}{7}\big)}{1-\big(2\,\cos(\frac{2\,\pi}{7})\big)^k\,\mathbb{X}}=$$
$$=\frac{-1-(g_k+h_k)\,\mathbb{X}+F_k\,\mathbb{X}^2}{1-h_{k-1}\,\mathbb{X}+\frac{1}{2}\big(h_{k-1}^2-h_{2k-1}\big)\,\mathbb{X}^2-\mathbb{X}^3},\quad(7.15)$$

$$\sum_{n=0}^{\infty}g_{kn}\,\mathbb{X}^n=\frac{2\,\cos\big(\frac{2\,\pi}{7}\big)}{1-\big(2\,\cos(\frac{8\,\pi}{7})\big)^k\,\mathbb{X}}+\frac{2\,\cos\big(\frac{4\,\pi}{7}\big)}{1-\big(2\,\cos(\frac{2\,\pi}{7})\big)^k\,\mathbb{X}}+\frac{2\,\cos\big(\frac{8\,\pi}{7}\big)}{1-\big(2\,\cos(\frac{4\,\pi}{7})\big)^k\,\mathbb{X}}=$$
$$=\frac{-1-(f_k+h_k)\,\mathbb{X}+G_k\,\mathbb{X}^2}{1-h_{k-1}\,\mathbb{X}+\frac{1}{2}\big(h_{k-1}^2-h_{2k-1}\big)\,\mathbb{X}^2-\mathbb{X}^3},\quad(7.16)$$

$$\sum_{n=0}^{\infty}h_{kn-1}\,\mathbb{X}^n=\frac{1}{1-\big(2\,\cos(\frac{2\,\pi}{7})\big)^k\,\mathbb{X}}+\frac{1}{1-\big(2\,\cos(\frac{4\,\pi}{7})\big)^k\,\mathbb{X}}+\frac{1}{1-\big(2\,\cos(\frac{8\,\pi}{7})\big)^k\,\mathbb{X}}=$$
$$=\frac{3-2\,h_{k-1}\,\mathbb{X}+\frac{1}{2}\big(h_{k-1}^2-h_{2k-1}\big)\,\mathbb{X}^2}{1-h_{k-1}\,\mathbb{X}+\frac{1}{2}\big(h_{k-1}^2-h_{2k-1}\big)\,\mathbb{X}^2-\mathbb{X}^3}\overset{by\ (2.1)}{=}\frac{3-2\,\mathcal{B}_k\,\mathbb{X}+(-1)^k\,\mathcal{A}_k\,\mathbb{X}^2}{1-\mathcal{B}_k\,\mathbb{X}+(-1)^k\,\mathcal{A}_k\,\mathbb{X}^2-\mathbb{X}^3}.\quad(7.17)$$

($h_{-1}:=1$).

We note that the special case of the formulas (7.15) and (7.16) for $k=1$ can be treated in another way (polynomial (1.4) will be denoted here by $p_7(\mathbb{X})$):

$$\sum_{n=0}^{\infty}h_n\,\mathbb{X}^n=\frac{\xi+\xi^6}{1-(\xi+\xi^6)\,\mathbb{X}}+\frac{\xi^2+\xi^5}{1-(\xi^2+\xi^5)\,\mathbb{X}}+\frac{\xi^3+\xi^4}{1-(\xi^3+\xi^4)\,\mathbb{X}}=$$
$$=-\frac{\big(\mathbb{X}^3\,p_7\big(1/\mathbb{X}\big)\big)'}{\mathbb{X}^3\,p_7\big(1/\mathbb{X}\big)}=\frac{-1+4\,\mathbb{X}+3\,\mathbb{X}^2}{1+\mathbb{X}-2\,\mathbb{X}^2-\mathbb{X}^3},\quad(7.18)$$

hence by (3.12):

$$\sum_{n=0}^{\infty} g_n \mathbb{X}^n = -1 + \sum_{n=1}^{\infty} \big(h_{n+1} - 2\,h_{n-1}\big)\,\mathbb{X}^n =$$

$$= -1 + \mathbb{X}^{-1} \sum_{n=0}^{\infty} h_n \mathbb{X}^n - \mathbb{X}^{-1}\big(h_0 + h_1\,\mathbb{X}\big) - 2\,\mathbb{X} \sum_{n=0}^{\infty} h_n \mathbb{X}^n =$$

$$= -6 + \mathbb{X}^{-1} + \frac{1 - 2\,\mathbb{X}^2}{\mathbb{X}} \sum_{n=0}^{\infty} h_n \mathbb{X}^n = \frac{-1 - 3\,\mathbb{X} + 3\,\mathbb{X}^2}{1 + \mathbb{X} - 2\,\mathbb{X}^2 - \mathbb{X}^3} \quad (7.19)$$

and, finally, using (3.12) again we get

$$\sum_{n=0}^{\infty} f_n \mathbb{X}^n = \sum_{n=0}^{\infty} \big(g_{n+1} - h_n\big)\,\mathbb{X}^n =$$

$$= \mathbb{X}^{-1} \sum_{n=0}^{\infty} g_n \mathbb{X}^n + \mathbb{X}^{-1} - \sum_{n=0}^{\infty} h_n \mathbb{X}^n = -\frac{1 + 3\,\mathbb{X} + 4\,\mathbb{X}^2}{1 + \mathbb{X} - 2\,\mathbb{X}^2 - \mathbb{X}^3}. \quad (7.20)$$

By (1.1), (4.4)–(4.6), (4.15) and (4.19)–(4.21) we have

$$\sum_{n=0}^{\infty} x_{kn} \mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\sin(\frac{8\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\sin(\frac{2\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\sin(\frac{4\pi}{7})\big)^k\,\mathbb{X}} =$$

$$= \frac{\sqrt{7} - (y_k + z_k)\,\mathbb{X} + x_k^*\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\,\big)^k\,\mathbb{X}^3}, \quad (7.21)$$

$$\sum_{n=0}^{\infty} y_{kn} \mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\sin(\frac{4\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\sin(\frac{8\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\sin(\frac{2\pi}{7})\big)^k\,\mathbb{X}} =$$

$$= \frac{\sqrt{7} - (x_k + z_k)\,\mathbb{X} + y_k^*\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\,\big)^k\,\mathbb{X}^3}, \quad (7.22)$$

$$\sum_{n=0}^{\infty} z_{kn} \mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1 - \big(2\,\sin(\frac{2\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1 - \big(2\,\sin(\frac{4\pi}{7})\big)^k\,\mathbb{X}} + \frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1 - \big(2\,\sin(\frac{8\pi}{7})\big)^k\,\mathbb{X}} =$$

$$= \frac{\sqrt{7} - (x_k + y_k)\,\mathbb{X} + z_k^*\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\,\big)^k\,\mathbb{X}^3}, \quad (7.23)$$

and ($z_{-1} := 1$):

$$\sum_{n=0}^{\infty} z_{kn-1} \mathbb{X}^n = \Big(1 - \big(2\,\sin(\tfrac{2\pi}{7})\big)^k\,\mathbb{X}\Big)^{-1} + \Big(1 - \big(2\,\sin(\tfrac{4\pi}{7})\big)^k\,\mathbb{X}\Big)^{-1} +$$

$$+ \Big(1 - \big(2\,\sin(\tfrac{8\pi}{7})\big)^k\,\mathbb{X}\Big)^{-1} = \frac{3 - 2\,z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2}{1 - z_{k-1}\,\mathbb{X} + \frac{1}{2}\big(z_{k-1}^2 - z_{2k-1}\big)\,\mathbb{X}^2 - \big(-\sqrt{7}\,\big)^k\,\mathbb{X}^3}. \quad (7.24)$$

By (1.1), (1.4), (4.1)–(4.3), (4.15), (4.16)–(4.18), (4.30)–(4.32) and (4.6), respectively, we get

$$\sum_{n=0}^{\infty} u_{kn}\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{2\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{4\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{8\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} = \\ = \frac{-1-(v_k+w_k)\,\mathbb{X}+u_k^*\,\mathbb{X}^2}{1-z_{k-1}\,\mathbb{X}+\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}^2-\left(-\sqrt{7}\right)^k\,\mathbb{X}^3}, \quad (7.25)$$

$$\sum_{n=0}^{\infty} v_{kn}\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{4\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{2\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{8\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{4\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{2\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} = \\ = \frac{-1-(u_k+w_k)\,\mathbb{X}+v_k^*\,\mathbb{X}^2}{1-z_{k-1}\,\mathbb{X}+\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}^2-\left(-\sqrt{7}\right)^k\,\mathbb{X}^3}, \quad (7.26)$$

$$\sum_{n=0}^{\infty} w_{kn}\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{8\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{2\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{2\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{4\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\,\pi}{7}\right)}{1-\left(2\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} = \\ = \frac{-1-(u_k+v_k)\,\mathbb{X}+w_k^*\,\mathbb{X}^2}{1-z_{k-1}\,\mathbb{X}+\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}^2-\left(-\sqrt{7}\right)^k\,\mathbb{X}^3}, \quad (7.27)$$

$$\sum_{n=0}^{\infty} u_{kn}^*\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\,\pi}{7})\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\,\pi}{7})\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} + \\ + \frac{2\,\cos\left(\frac{8\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\,\pi}{7})\,\sin(\frac{4\,\pi}{7})\right)^k\,\mathbb{X}} = \frac{-1-(v_k^*+w_k^*)\,\mathbb{X}+\left(-\sqrt{7}\right)^k\,u_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k\,z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.28)$$

$$\sum_{n=0}^{\infty} v_{kn}^*\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\,\pi}{7})\,\sin(\frac{4\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\,\pi}{7})\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} + \\ + \frac{2\,\cos\left(\frac{8\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\,\pi}{7})\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} = \frac{-1-(u_k^*+w_k^*)\,\mathbb{X}+\left(-\sqrt{7}\right)^k\,v_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k\,z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.29)$$

$$\sum_{n=0}^{\infty} w_{kn}^*\,\mathbb{X}^n = \frac{2\,\cos\left(\frac{2\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\,\pi}{7})\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} + \frac{2\,\cos\left(\frac{4\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\,\pi}{7})\,\sin(\frac{4\,\pi}{7})\right)^k\,\mathbb{X}} + \\ + \frac{2\,\cos\left(\frac{8\,\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\,\pi}{7})\,\sin(\frac{8\,\pi}{7})\right)^k\,\mathbb{X}} = \frac{-1-(u_k^*+v_k^*)\,\mathbb{X}+\left(-\sqrt{7}\right)^k\,w_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z_{k-1}^2-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k\,z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}. \quad (7.30)$$

By (1.1), (4.19)–(4.21), (4.33)–(4.35) and (4.6), we obtain

$$\sum_{n=0}^{\infty} x^*_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} +$$
$$+\,\frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \frac{\sqrt{7}-(y^*_k+z^*_k)\,\mathbb{X}+\left(-\sqrt{7}\right)^k x_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z^2_{k-1}-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.31)$$

$$\sum_{n=0}^{\infty} y^*_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} +$$
$$+\,\frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} = \frac{\sqrt{7}-(x^*_k+z^*_k)\,\mathbb{X}+\left(-\sqrt{7}\right)^k y_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z^2_{k-1}-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}, \quad (7.32)$$

$$\sum_{n=0}^{\infty} z^*_{kn}\,\mathbb{X}^n = \frac{2\,\sin\left(\frac{2\pi}{7}\right)}{1-\left(4\,\sin(\frac{4\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} + \frac{2\,\sin\left(\frac{4\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{8\pi}{7})\right)^k\mathbb{X}} +$$
$$+\,\frac{2\,\sin\left(\frac{8\pi}{7}\right)}{1-\left(4\,\sin(\frac{2\pi}{7})\,\sin(\frac{4\pi}{7})\right)^k\mathbb{X}} = \frac{\sqrt{7}-(x^*_k+y^*_k)\,\mathbb{X}+\left(-\sqrt{7}\right)^k z_k\,\mathbb{X}^2}{1-\frac{1}{2}\left(z^2_{k-1}-z_{2k-1}\right)\mathbb{X}+\left(-\sqrt{7}\right)^k z_{k-1}\,\mathbb{X}^2-7^k\,\mathbb{X}^3}. \quad (7.33)$$

# 8 Jordan decomposition

For sequences $A_n(\delta)$, $B_n(\delta)$ and $C_n(\delta)$ we have the equality

$$\begin{bmatrix} A_{n+1}(\delta) \\ B_{n+1}(\delta) \\ C_{n+1}(\delta) \end{bmatrix} = \mathcal{W}(\delta) \begin{bmatrix} A_n(\delta) \\ B_n(\delta) \\ C_n(\delta) \end{bmatrix}, \qquad n \in \mathbb{N}, \quad (8.1)$$

where

$$\mathcal{W}(\delta) = \begin{bmatrix} 1 & 2\,\delta & -\delta \\ \delta & 1 & 0 \\ 0 & \delta & 1-\delta \end{bmatrix}. \quad (8.2)$$

Matrix $\mathcal{W}(\delta)$ is a diagonalized matrix, and the following decomposition can be obtained

$$\mathcal{W}(\delta) = A \cdot \begin{bmatrix} 1+\delta\,(\xi+\xi^6) & 0 & 0 \\ 0 & 1+\delta\,(\xi^2+\xi^5) & 0 \\ 0 & 0 & 1+\delta\,(\xi^3+\xi^4) \end{bmatrix} \cdot A^{-1}, \quad (8.3)$$

where

$$A = \begin{bmatrix} 1+(\xi+\xi^6)^{-1} & 1+(\xi^2+\xi^5)^{-1} & 1+(\xi^3+\xi^4)^{-1} \\ (\xi+\xi^6)^{-2}+(\xi+\xi^6)^{-1} & (\xi^2+\xi^5)^{-2}+(\xi^2+\xi^5)^{-1} & (\xi^3+\xi^4)^{-2}+(\xi^3+\xi^4)^{-1} \\ (\xi+\xi^6)^{-2} & (\xi^2+\xi^5)^{-2} & (\xi^3+\xi^4)^{-2} \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{7}\begin{bmatrix} (\xi\ +\xi^6)^2 & (\xi\ +\xi^6)^3 & (\xi\ +\xi^6)^4 \\ (\xi^2+\xi^5)^2 & (\xi^2+\xi^5)^3 & (\xi^2+\xi^5)^4 \\ (\xi^3+\xi^4)^2 & (\xi^3+\xi^4)^3 & (\xi^3+\xi^4)^4 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 & 5 \\ 1 & 1 & -2 \\ 2 & -1 & -2 \end{bmatrix}.$$

It should be noticed, that characteristic polynomial $w(\lambda)$ of matrix $\mathcal{W}(\delta)$ is equal to

$$w(\lambda) = -\delta^3\, p_7\Big(\frac{\lambda-1}{\delta}\Big).$$

# 9 Tables

Table 1:

| $n$ | $a_n/\sqrt{7}$ | $b_n/\sqrt{7}$ | $c_n/\sqrt{7}$ | $\alpha_n/\sqrt{7}$ | $\beta_n/\sqrt{7}$ | $\gamma_n/\sqrt{7}$ | $f_n$ | $g_n$ | $h_n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 | −2 | 1 | −1 | −1 | −1 |
| 1 | 3 | 2 | 0 | −2 | −5 | 3 | −2 | −2 | 5 |
| 2 | 8 | 7 | −2 | −9 | −15 | 8 | −4 | 3 | −4 |
| 3 | 23 | 24 | −9 | −33 | −47 | 23 | −1 | −8 | 13 |
| 4 | 70 | 80 | −33 | −113 | −150 | 70 | −9 | 12 | −16 |
| 5 | 220 | 263 | −113 | −376 | −483 | 220 | 3 | −25 | 38 |
| 6 | 703 | 859 | −376 | −1235 | −1562 | 703 | −22 | 41 | −57 |
| 7 | 2265 | 2797 | −1235 | −4032 | −5062 | 2265 | 19 | −79 | 117 |
| 8 | 7327 | 9094 | −4032 | −13126 | −16421 | 7327 | −60 | 136 | −193 |
| 9 | 23748 | 29547 | −13126 | −42673 | −53295 | 23748 | 76 | −253 | 370 |
| 10 | 77043 | 95968 | −42673 | −138641 | −173011 | 77043 | −177 | 446 | −639 |
| 11 | 250054 | 311652 | −138641 | −450293 | −561706 | 250054 | 269 | −816 | 1186 |

Table 2:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon_n$ | 1 | 0 | −2 | 2 | 2 | −6 | 2 | 10 | −14 | −6 | 34 | −22 |
| $\omega_n$ | 0 | 1 | −1 | −1 | 3 | −1 | −5 | 7 | 3 | −17 | 11 | 23 |
| $\Psi_n$ | 2 | −1 | −3 | 5 | 1 | −11 | 9 | 13 | −31 | 5 | 57 | −67 |

Table 3:

| $n$ | $u_n$ | $v_n$ | $w_n$ | $x_n$ | $y_n$ | $z_n$ |
|---|---|---|---|---|---|---|
| 0 | $-1$ | $-1$ | $-1$ | $\sqrt{7}$ | $\sqrt{7}$ | $\sqrt{7}$ |
| 1 | $\sqrt{7}$ | $-2\sqrt{7}$ | 0 | 0 | 0 | 7 |
| 2 | 0 | $-7$ | 0 | $\sqrt{7}$ | $2\sqrt{7}$ | $4\sqrt{7}$ |
| 3 | $\sqrt{7}$ | $-6\sqrt{7}$ | $\sqrt{7}$ | 0 | 7 | 21 |
| 4 | 0 | $-28$ | 7 | 0 | $7\sqrt{7}$ | $14\sqrt{7}$ |
| 5 | 0 | $-21\sqrt{7}$ | $7\sqrt{7}$ | $-7$ | 35 | 70 |
| 6 | $-7$ | $-105$ | 42 | $-7\sqrt{7}$ | $28\sqrt{7}$ | $49\sqrt{7}$ |
| 7 | $-7\sqrt{7}$ | $-77\sqrt{7}$ | $35\sqrt{7}$ | $-49$ | 147 | 245 |
| 8 | $-49$ | $-392$ | 196 | $-42\sqrt{7}$ | $112\sqrt{7}$ | $175\sqrt{7}$ |
| 9 | $-42\sqrt{7}$ | $-287\sqrt{7}$ | $154\sqrt{7}$ | $-245$ | 588 | 882 |
| 10 | $-245$ | $-1470$ | 833 | $-196\sqrt{7}$ | $441\sqrt{7}$ | $637\sqrt{7}$ |
| 11 | $-196\sqrt{7}$ | $-1078\sqrt{7}$ | $637\sqrt{7}$ | $-1078$ | 2303 | 3234 |

Table 4:

| $n$ | $u_n^*$ | $v_n^*$ | $w_n^*$ | $x_n^*/\sqrt{7}$ | $y_n^*/\sqrt{7}$ | $z_n^*/\sqrt{7}$ |
|---|---|---|---|---|---|---|
| 0 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 |
| 1 | $-7$ | 7 | 0 | 1 | 2 | $-3$ |
| 2 | $-14$ | 7 | $-7$ | 7 | 7 | 0 |
| 3 | $-56$ | 42 | $-7$ | 14 | 21 | $-14$ |
| 4 | $-147$ | 98 | $-49$ | 56 | 63 | $-21$ |
| 5 | $-490$ | 343 | $-98$ | 147 | 196 | $-98$ |
| 6 | $-1421$ | 980 | $-392$ | 490 | 588 | $-245$ |
| 7 | $-4459$ | 3087 | $-1029$ | 1421 | 1813 | $-833$ |
| 8 | $-13377$ | 9261 | $-3430$ | 4459 | 5488 | $-2401$ |
| 9 | $-41160$ | 28469 | $-9947$ | 13377 | 16807 | $-7546$ |
| 10 | $-124852$ | 86436 | $-31213$ | 41160 | 51107 | $-22638$ |
| 11 | $-381759$ | 264110 | $-93639$ | 124852 | 156065 | $-69629$ |

Table 5:

| $n$ | $F_n$ | $G_n$ | $H_n$ | $\widetilde{A}_n/\sqrt{7}$ | $\widetilde{B}_n/\sqrt{7}$ | $\widetilde{C}_n/\sqrt{7}$ |
|---|---|---|---|---|---|---|
| 0 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 |
| 1 | $-4$ | 3 | 3 | $-1$ | 3 | 4 |
| 2 | 5 | $-9$ | $-2$ | $-8$ | 15 | 19 |
| 3 | $-15$ | 20 | 6 | $-42$ | 76 | 95 |
| 4 | 31 | $-46$ | $-11$ | $-213$ | 384 | 479 |
| 5 | $-72$ | 103 | 26 | $-1076$ | 1939 | 2418 |
| 6 | 160 | $-232$ | $-57$ | $-5433$ | 9790 | 12208 |
| 7 | $-361$ | 521 | 129 | $-27431$ | 49429 | 61637 |
| 8 | 810 | $-1171$ | $-289$ | $-138497$ | 249563 | 311200 |
| 9 | $-1821$ | 2631 | 650 | $-699260$ | 1260023 | 1571223 |
| 10 | 4091 | $-5912$ | $-1460$ | $-3530506$ | 6361752 | 7932975 |
| 11 | $-9193$ | 13284 | 3281 | $-17825233$ | 32119960 | 40052935 |

Table 6:

| $n$ | $\Omega_n(\delta)$ |
|---|---|
| 0 | $3$ |
| 1 | $i\sqrt{7}\delta + 3$ |
| 2 | $-7\delta^2 + 2i\sqrt{7}\delta + 3$ |
| 3 | $-4i\sqrt{7}\delta^3 - 21\delta^2 + 3i\sqrt{7}\delta + 3$ |
| 4 | $21\delta^4 - 16i\sqrt{7}\delta^3 - 42\delta^2 + 4i\sqrt{7}\delta + 3$ |
| 5 | $14i\sqrt{7}\delta^5 + 105\delta^4 - 40i\sqrt{7}\delta^3 - 70\delta^2 + 5i\sqrt{7}\delta + 3$ |
| 6 | $-70\delta^6 + 84i\sqrt{7}\delta^5 + 315\delta^4 - 80i\sqrt{7}\delta^3 - 105\delta^2 + 6i\sqrt{7}\delta + 3$ |
| 7 | $-49i\sqrt{7}\delta^7 - 490\delta^6 + 294i\sqrt{7}\delta^5 + 735\delta^4 - 140i\sqrt{7}\delta^3 - 147\delta^2 + 7i\sqrt{7}\delta + 3$ |

Table 7:

| $n$ | $p_n(\delta)$ |
|---|---|
| 0 | $1$ |
| 1 | $1$ |
| 2 | $-2\delta^2+1$ |
| 3 | $-i\sqrt{7}\delta^3-6\delta^2+1$ |
| 4 | $3\delta^4-4i\sqrt{7}\delta^3-12\delta^2+1$ |
| 5 | $5i\sqrt{7}\delta^5+15\delta^4-10i\sqrt{7}\delta^3-20\delta^2+1$ |
| 6 | $-8\delta^6+30i\sqrt{7}\delta^5+45\delta^4-20i\sqrt{7}\delta^3-30\delta^2+1$ |
| 7 | $-17i\sqrt{7}\delta^7-56\delta^6+105i\sqrt{7}\delta^5+105\delta^4-35i\sqrt{7}\delta^3-42\delta^2+1$ |

| $n$ | $r_n(\delta)$ |
|---|---|
| 0 | $0$ |
| 1 | $\delta$ |
| 2 | $2\delta$ |
| 3 | $-2\delta^3+3\delta$ |
| 4 | $-i\sqrt{7}\delta^4-8\delta^3+4\delta$ |
| 5 | $3\delta^5-5i\sqrt{7}\delta^4-20\delta^3+5\delta$ |
| 6 | $5i\sqrt{7}\delta^6+18\delta^5-15i\sqrt{7}\delta^4-40\delta^3+6\delta$ |
| 7 | $-8\delta^7+35i\sqrt{7}\delta^6+63\delta^5-35i\sqrt{7}\delta^4-70\delta^3+7\delta$ |

| $n$ | $s_n(\delta)$ | $k_n(\delta)$ |
|---|---|---|
| 0 | $0$ | $0$ |
| 1 | $0$ | $0$ |
| 2 | $0$ | $0$ |
| 3 | $\delta^3$ | $0$ |
| 4 | $4\delta^3$ | $-2\delta^4$ |
| 5 | $-5\delta^5+10\delta^3$ | $-10\delta^4$ |
| 6 | $-i\sqrt{7}\delta^6-30\delta^5+20\delta^3$ | $10\delta^6-30\delta^4$ |
| 7 | $18\delta^7-7i\sqrt{7}\delta^6-105\delta^5+35\delta^3$ | $2i\sqrt{7}\delta^7+70\delta^6-70\delta^4$ |

| $n$ | $l_n(\delta)$ |
|---|---|
| 0 | $0$ |
| 1 | $0$ |
| 2 | $\delta^2$ |
| 3 | $3\delta^2$ |
| 4 | $-3\delta^4+6\delta^2$ |
| 5 | $-i\sqrt{7}\delta^5-15\delta^4+10\delta^2$ |
| 6 | $8\delta^6-6i\sqrt{7}\delta^5-45\delta^4+15\delta^2$ |
| 7 | $6i\sqrt{7}\delta^7+56\delta^6-21i\sqrt{7}\delta^5-105\delta^4+21\delta^2$ |

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