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Transformations Preserving the Hankel Transform

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Abstract

We classify all polynomial transformations of integer sequences which preserve the Hankel transform, thus generalizing examples due to Layman and Spivey & Steil. We also show that such transformations form a group under composition.

1 Introduction

Given a sequence of integers $\{a_i\} = a_0, a_1, a_2, \ldots$, the Hankel matrix of A is the infinite matrix whose (i, j) entry is a_{i+j} for $i \geq 0, j \geq 0$. The Hankel matrix of order n of A is the $n \times n$ matrix consisting of the first n rows and n columns of the Hankel matrix of A, and the Hankel sequence determined by A is the sequence of determinants of the Hankel matrices of order n. For example, the Hankel matrix of order 3 is given by

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}.$$

Let $R = \mathbb{Z}[a_0, a_1, a_2, \ldots]$. We will say that a sequence $\{b_i\} = \{b_0, b_1, b_2, \ldots\}$ of elements in R is an HTP sequence (Hankel Transform Preserving) if the Hankel transform of $\{b_i\}$ is formally equal to that of $\{a_i\}$; that is, if the following identity holds in R for all n:

$$\begin{vmatrix} b_0 & b_1 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_{n+1} \\ \vdots & & \ddots & \vdots \\ b_n & b_{n+1} & \cdots & b_{2n} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

An HTP sequence $\{b_i\}$ determines a ring homomorphism $b : R \to R$ by the equation $b(a_i) = b_i$. We will sometimes refer to the sequence $\{b_i\}$ as just b. It is easy to see that the composition of two ring homomorphisms associated to HTP sequences is the ring homomorphism associated to an HTP sequence, so that the set of all HTP sequences has a semigroup structure (with $b_i = a_i$ representing the identity). We will see (Theorem 3.6) that this is actually a group structure.

Example 1.1. Let $\sigma : R \to R$ be given by $\sigma(a_i) = (-1)^i a_i$. Then σ determines an HTP sequence. Indeed, the order n Hankel matrix of the sequence $\{(-1)^i a_i\}$ is given by conjugating the order n Hankel matrix A_n of $\{a_i\}$ by the diagonal matrix D_n with (i,i)-entry given by $(-1)^i$. The determinant of $D_n^{-1} A_n D_n$ is the same as that of A_n.

Example 1.2. For an integer k, the sequence defined by $b_n = \sum_{i=0}^n \binom{n}{i} k^{n-i} a_i$ is an HTP sequence. (Here, if $k = 0$, then we interpret 0^0 to be 1.) Spivey and Steil [5] called this the falling k-binomial transform, and they proved that this preserves the Hankel Transform. When $k = 1$, this gives the binomial transform, which Layman [3] originally proved preserves the Hankel transform.

Remark 1.3. In fact, Spivey and Steil allow k to be any real number. If k were not an integer, then b_n is not in $R = \mathbb{Z}[a_0, a_1, a_2, \ldots]$, so in our language, $\{b_n\}$ would not be an HTP sequence. Since we are interested in integer sequences, we have restricted to $\mathbb{Z}[a_0, a_1, a_2, \ldots]$.

Definition 1.4. *An HTP sequence b preserves a_0 through a_n if $b_i = a_i$ for $0 \leq i \leq n$.*

Suppose that $b(m)$ is a sequence of HTP sequences such that for each n, there is a number $M(n)$ such that $b(m)$ preserves a_0 through a_n for all $m > M(n)$. Then the *infinite composition*

$$\cdots \circ b(m) \circ b(m-1) \circ \cdots \circ b(0)$$

is itself a well-defined sequence. Indeed, the nth term in the sequence of this infinite composite is determined by $b(M(n)) \circ b(M(n)-1) \circ \cdots \circ b(0)$. It is easy to see that this infinite composition is itself an HTP sequence.

In order to find all HTP sequences, we first describe a special set of sequences, parametrized by R and the positive integers. In fact, for each positive integer n, and each $c \in R$, we will define an HTP sequence $b(n,c)$ which preserves a_0 through a_{2n}. Example 1.2 above arises when $n = 0$ and $c \in \mathbb{Z}$. Our main theorem is

Theorem 1.5. *If b is a given HTP sequence, then there is a sequence $c_0, c_1, c_2, \ldots$ in R and an $\epsilon \in \{0,1\}$ such that*

$$b = \cdots \circ b(n, c_n) \circ \cdots \circ b(1, c_1) \circ b(0, c_0) \circ \sigma^{\epsilon}.$$

Our goal in Section 2 is to define the sequences $b(n, c_n)$, which we do in Definition 2.5. In Section 3 we prove that the set of all HTP sequences forms a group (Theorem 3.6), and we prove our classification Theorem 1.5.

2 A collection of HTP sequences

Definition 2.1. *Let $T : R[x] \to R$ be the R-linear homomorphism defined by $T(x^k) = a_k$. For integers $i \geq j \geq 0$, define $T_{ij} : R[x] \to R$ to be the R-linear homomorphism given by*

$$T_{ij}(x^k) = (-1)^{i+j} \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_i \\ a_1 & a_2 & a_3 & \cdots & a_{i+1} \\ \vdots & & & \ddots & \vdots \\ a_{j-1} & a_j & a_{j+1} & \cdots & a_{j-1+i} \\ a_{j+1} & a_{j+2} & a_{j+3} & \cdots & a_{j+1+i} \\ \vdots & & & \ddots & \vdots \\ a_i & a_{i+1} & a_{i+2} & \cdots & a_{2i} \\ a_k & a_{k+1} & a_{k+2} & \cdots & a_{k+i} \end{vmatrix}.$$

Remark 2.2. If $0 \leq k \leq i$ and $k \neq j$, then $T_{ij}(x^k) = 0$, since two rows in the matrix defining $T_{ij}(x^k)$ are equal. Also, keeping track of signs, one sees that $T_{ij}(x^j) = T_{ii}(x^i)$. Finally, we note that $T_{00} = T$.

Definition 2.3. *For each $c \in R$ and each integer $i \geq 1$, we define a sequence of polynomials $f_{m,i,c}(x) \in R[x]$ recursively by*

$$f_{0,i,c} = 1, \qquad f_{m+1,i,c} = f_{m,i,c} \cdot (x + cT_{i-1,i-1}(x^{i-1})) - c\left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{m,i,c}) \cdot x^j\right).$$

Also, we define $f_{m,0,c} \in R[x]$ by $f_{m,0,c} = (x+c)^m$, or (equivalently) recursively by

$$f_{0,0,c} = 1, \qquad f_{m+1,0,c} = f_{m,0,c} \cdot (x + c).$$

We will show in Lemma 2.9 that for each $i \geq 0$, $m \geq 0$, and $c \in R$, $f_{m,i,c}(x)$ is a degree m polynomial in x with leading coefficient 1.

Definition 2.4. *Fixing a choice of i and c, define $U_{k,m}$ for each k, m by*

$$f_{m,i,c}(x) = \sum U_{k,m} x^k.$$

Let U be the infinite matrix whose (k,m) entry is given by $U_{k,m}$.

Thus U is upper triangular, with diagonal entries all 1. Let A be the infinite Hankel matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We will show in Lemma 2.14 that the product $U^t A U$ is a Hankel matrix.

Definition 2.5. *Let $b(i,c)$ denote the sequence whose Hankel matrix is the matrix $U^t AU$, where U is defined in terms of i and c as above.*

We show in Corollary 2.15 that $b(i,c)$ is an HTP sequence preserving a_0 through a_{2i}.

Example 2.6. If $i=1$ and $c=1$, then the recurrence relation of Definition 2.3 becomes

$$f_{m+1,1,1} = f_{m,1,1}\cdot(x+a_0) - T(f_{m,1,1}),$$

$$f_{0,1,1} = 1.$$

Thus,

$$f_{1,1,1} = (x+a_0) - T(1) = x + a_0 - a_0 = x,$$
$$f_{2,1,1} = x(x+a_0) - T(x) = x^2 + a_0x - a_1,$$
$$f_{3,1,1} = (x^2+a_0x-a_1)(x+a_0) - T(x^2+a_0x-a_1) = x^3 + 2a_0x^2 + (a_0^2 - a_1)x + -a_0a_1 - a_2.$$

Therefore, the upper left 4×4 submatrix of U is

$$\begin{pmatrix} 1 & 0 & -a_1 & -a_0a_1 - a_2 \\ 0 & 1 & a_0 & a_0^2 - a_1 \\ 0 & 0 & 1 & 2a_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we compute the upper left 4×4 submatrix of U^tAU to be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 - a_1^2 + a_0a_2 \\ a_1 & a_2 & a_3 - a_1^2 + a_0a_2 & x \\ a_2 & a_3 - a_1^2 + a_0a_2 & x & y \\ a_3 - a_1^2 + a_0a_2 & x & y & z \end{pmatrix}$$

where

$$x = a_4 - a_0a_1^2 + 2a_0a_3 - 2a_1a_2 + a_0^2a_2$$
$$y = a_5 - 4a_0a_1a_2 + 3a_0a_4 - 2a_1a_3 - a_0^2a_1^2 + 3a_0a_3 - a_2^2 + a_0^3a_2 + a_1^3$$
$$z = a_6 + a_0^4a_2 + 3a_1^2a_2 - a_0^3a_1^2 + 4a_0^3a_3 - 2a_2a_3 - 2a_1a_4 - 6a_0^2a_1a_2$$
$$+6a_0^2a_4 + 2a_0a_1^3 - 3a_0a_2^2 - 6a_0a_1a_3 + 4a_0a_5$$

Thus, the first seven terms of $b(1,1)$ are $a_0, a_1, a_2, a_3 - a_1^2 + a_0a_2, x, y, z$.

Example 2.7. If $i=2$ and $c=1$, then the recurrence relation of Definition 2.3 becomes

$$f_{m+1,2,1} = f_{m,2,1}\cdot(x + T_{1,1}(x)) - T_{1,1}(f_{m,2,1})x - T_{1,0}(f_{m,2,1}),$$

$$f_{0,1,1} = 1.$$

Thus,

$$f_{1,2,1} = (x + a_0a_2 - a_1^2) - 0x - (a_0a_2 - a_1^2) = x,$$
$$f_{2,2,1} = x(x + a_0a_2 - a_1^2) - (a_0a_2 - a_1^2)x + 0 = x^2,$$

$$\begin{aligned} f_{3,2,1} &= x^2(x + a_0a_2 - a_1^2) - (a_0a_3 - a_1a_2)x + a_1a_3 - a_2^2 \\ &= x^3 + (a_0a_2 - a_1^2)x^2 - (a_0a_3 - a_1a_2)x + a_1a_3 - a_2^2. \end{aligned}$$

Therefore, the upper left 4×4 submatrix of U is

$$\begin{pmatrix} 1 & 0 & 0 & a_1a_3 - a_2^2 \\ 0 & 1 & 0 & a_1a_2 - a_0a_3 \\ 0 & 0 & 1 & a_0a_2 - a_1^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we compute the upper left 4×4 submatrix of U^tAU to be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & x \\ a_3 & a_4 & x & y \end{pmatrix}$$

where

$$\begin{aligned} x &= a_5 - a_2^3 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 + a_0a_2a_4, \\ y &= a_6 - 2a_1^3a_2a_3 - 2a_2^2a_3 + a_1^4a_4 - a_0^2a_2a_3^2 + a_0^2a_2^2a_4 + 2a_0a_1a_2^2a_3 + 2a_1a_3^2 + 2a_1a_2a_4 \\ &\quad +a_1^2a_2^3 + a_0a_1^2a_3^2 - 2a_0a_1^2a_2a_4 - 2a_1^2a_5 - a_0a_2^4 - 2a_0a_3a_4 + 2a_0a_2a_5. \end{aligned}$$

Thus the first seven terms of $b(2,1)$ are $a_0, a_1, a_2, a_3, a_4, x, y$.

Lemma 2.8. *Suppose f and g are two polynomials in $R[x]$, and $i \geq 0$ is any integer. Then*

$$\sum_{j=0}^{i-1} T(f \cdot x^j) \cdot T_{i-1,j}(g) = \sum_{j=0}^{i-1} T(g \cdot x^j) \cdot T_{i-1,j}(f).$$

Proof. Since both sides of the equation are R-linear in both f and g, it suffices to consider the case when $f = x^m$ and $g = x^n$, so we need to show that

$$\sum_{j=0}^{i-1} T(x^{m+j}) \cdot T_{i-1,j}(x^n) = \sum_{j=0}^{i-1} T(x^{n+j}) \cdot T_{i-1,j}(x^m).$$

Consider the matrix below:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{i-1} & a_m \\ a_1 & a_2 & a_3 & \cdots & a_i & a_{m+1} \\ a_2 & a_3 & a_4 & \cdots & a_{i+1} & a_{m+2} \\ \vdots & & & \ddots & & \vdots \\ a_{i-1} & a_i & a_{i+1} & \cdots & a_{2i-2} & a_{m+i-1} \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+i-1} & 0 \end{pmatrix}$$

If we compute the determinant by expanding along the rightmost column, we get

$$\sum_{j=0}^{i-1} (-1)^{i+j} a_{m+j} \cdot T_{i-1,j}(x^n) = -\sum_{j=0}^{i-1} T(x^{m+j}) \cdot T_{i-1,j}(x^n).$$

If we compute the determinant by expanding along the bottom row, we get

$$\sum_{j=0}^{i-1}(-1)^{i+j}a_{n+j}\cdot T_{i-1,j}(x^m) = -\sum_{j=0}^{i-1}T(x^{n+j})\cdot T_{i-1,j}(x^m).$$

□

Lemma 2.9. *For each $i \geq 0$, $m \geq 0$, and $c \in R$, $f_{m,i,c}(x)$ is a degree m polynomial in x with leading coefficient* 1.

Proof. When $i = 0$ or $m = 0$, the claim is clear. We assume $i \geq 1$. Assume by induction on m that $f_{m,i,c}(x)$ has degree m and leading coefficient 1. It then suffices to show by the recursive definition of $f_{m+1,i,c}(x)$ that $T_{i-1,j}(f_{m,i,c}) = 0$ whenever $i > j > m$. This follows at once from Remark 2.2 and the inductive hypothesis. □

Lemma 2.10. *$f_{m,i,c} = x^m$ whenever $m \leq i$.*

Proof. Using the recursive definition, we see that it is enough to show that

$$x^m T_{i-1,i-1}(x^{i-1}) = \sum_{j=0}^{i-1} T_{i-1,j}(x^m)x^j$$

whenever $m \leq i-1$. This follows from Remark 2.2, since all terms in the right hand sum are 0, except for the term when $j = m$, and this is $T_{i-1,m}(x^m)x^m = T_{i-1,i-1}(x^{i-1})x^m$. □

Lemma 2.11. *For $m, n \geq 0$,*

$$T(f_{m+1,i,c}\cdot f_{n,i,c}) = T(f_{m,i,c}\cdot f_{n+1,i,c}).$$

Proof. First,

$$f_{m+1,0,c}\cdot f_{n,0,c} = (x+c)^{m+1}(x+c)^n = (x+c)^m(x+c)^{n+1} = f_{m,0,c}\cdot f_{n+1,0,c}.$$

We now assume $i \geq 1$. We then have

$$\begin{aligned}
&f_{m+1,i,c}\cdot f_{n,i,c} - f_{m,i,c}\cdot f_{n+1,i,c} = \\
&\left(f_{m,i,c}(x + cT_{i-1,i-1}(x^{i-1})) - c\left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{m,i,c})\cdot x^j\right)\right)\cdot f_{n,i,c} - \\
&f_{m,i,c}\cdot\left(f_{n,i,c}(x + cT_{i-1,i-1}(x^{i-1})) - c\left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{n,i,c})\cdot x^j\right)\right) \\
&= c\left(f_{m,i,c}\left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{n,i,c})\cdot x^j\right) - \left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{m,i,c})\cdot x^j\right)f_{n,i,c}\right) \\
&= c\sum_{j=0}^{i-1}\left(f_{m,i,c}\cdot x^j\cdot T_{i-1,j}(f_{n,i,c}) - f_{n,i,c}\cdot x^j\cdot T_{i-1,j}(f_{m,i,c})\right).
\end{aligned}$$

By Lemma 2.8, this term is in the kernel of T. □

Lemma 2.12. *$T(f_{m,i,c}(x)) = a_m$ whenever $0 \le m \le 2i$.*

Proof. Suppose $m = 2i$. Since $f_{0,i,c} = 1$, and by applying Lemma 2.11 i times,

$$T(f_{2i,i,c}) = T(f_{2i,i,c} \cdot f_{0,i,c}) = T(f_{i,i,c} \cdot f_{i,i,c}).$$

By Lemma 2.10, $f_{i,i,c} = x^i$, so $(f_{i,i,c})^2 = x^{2i}$, and $T(x^{2i}) = a_{2i}$. The other cases follow similarly. $\square$

Lemma 2.13.

$$T(f_{2i+1,i,c}(x)) = a_{2i+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \vdots & & \ddots & \vdots \\ a_i & a_{i+1} & \cdots & a_{2i} \end{vmatrix}.$$

Proof. If we expand the determinant on the right hand side of the equation along the last column, we get

$$\begin{aligned} & a_{2i+1} + ca_{2i}T_{i-1,i-1}(x^{i-1}) - c\sum_{j=0}^{i-1} T_{i-1,j}(x^i)a_{i+j} \\ &= T\left(x^{2i+1} + cx^{2i}T_{i-1,i-1}(x^{i-1}) - c\sum_{j=0}^{i-1} T_{i-1,j}(x^i)x^{i+j}\right) \\ &= T\left(x^i\left(x^i\left(x + cT_{i-1,i-1}(x^{i-1})\right) - c\sum_{j=0}^{i-1} T_{i-1,j}(f_{i,i,c})x^j\right)\right) \\ &= T\left(x^i\left(f_{i,i,c} \cdot \left(x + cT_{i-1,i-1}(x^{i-1})\right) - c\sum_{j=0}^{i-1} T_{i-1,j}(f_{i,i,c}) \cdot x^j\right)\right) \\ &= T(x^i \cdot f_{i+1,i,c}) = T(f_{i,i,c} \cdot f_{i+1,i,c}) = T(f_{2i+1,i,c}). \end{aligned}$$

Here, we have used Lemma 2.10 to identify x^i with $f_{i,i,c}$ and Lemma 2.11 for the last equality. $\square$

Now recall that A is the infinite Hankel matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Lemma 2.14. *The product U^tAU is a Hankel matrix.*

Proof. We have $f_{r,i,c} \cdot f_{u,i,c} = \sum_{s,t} U_{s,r}U_{t,u}x^{s+t}$, so that

$$T(f_{r,i,c} \cdot f_{u,i,c}) = \sum_{s,t} U_{s,r}U_{t,u}a_{s+t} = \sum_{s,t} U_{s,r}a_{s+t}U_{t,u}.$$

This is precisely the (r, u) entry of U^tAU. Now the result follows from Lemma 2.11. $\square$

Recall from Definition 2.5 that $b(i,c)$ denotes the sequence whose Hankel matrix is U^tAU.

Corollary 2.15. *We have $b(i,c)_n = T(f_{n,i,c})$, and $b(i,c)$ is an HTP sequence preserving a_0 through a_{2i}. Moreover*

$$b(i,c)_{2i+1} = a_{2i+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \vdots & & \ddots & \vdots \\ a_i & a_{i+1} & \cdots & a_{2i} \end{vmatrix}.$$

Proof. Since U^t and U are each triangular with 1s on the diagonal, it follows that the Hankel matrices of finite order associated to U^tAU have the same determinants as those of A. Thus the sequence of entries on the top row of U^tAU represents a transformation of A which preserves the Hankel transform. This is the same as the sequence of entries in the top row of AU since U^t preserves the top row. But it is easy to see that this sequence is given by $T(f_{n,i,c})$. The first $2i$ terms of the sequence are given by Lemma 2.12, while the $2i+1$ term is given by Lemma 2.13. □

3 Classifying all HTP sequences

Lemma 3.1. *For each integer $n \geq 0$, the determinant*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

is irreducible in R.

Proof. The statement is obvious if $n = 0$, so suppose $n \geq 1$. We make R into a graded ring by assigning $\deg(a_i) = 2n+1-i$. Then the determinant is a homogeneous polynomial of degree $(n+1)^2$. As a polynomial in a_{2n} with coefficients in $\mathbb{Z}[a_0, a_1, \ldots, a_{2n-1}]$, the determinant can be written

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} \cdot a_{2n} + \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

The coefficient of a_{2n} has degree $(n+1)^2 - 1$, while the constant coefficient has degree $(n+1)^2$. But no element in $\mathbb{Z}[a_0, a_1, \ldots, a_{2n-1}]$ has degree 1. Therefore, the coefficient of a_{2n} does not divide the constant coefficient. By induction, the coefficient of a_{2n} is irreducible. The result follows from this. □

Lemma 3.2. *Suppose that b is an HTP sequence which preserves a_0 through a_{2n-1}. Then $b_{2n} = a_{2n}$.*

Proof. In order to have

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & b_{2n} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix},$$

we must have

$$(b_{2n} - a_{2n}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} = 0.$$

Thus, $b_{2n} - a_{2n}$ must be 0. □

Lemma 3.3. *Suppose b is an HTP sequence. Then either $b_1 - a_1$ or $b_1 + a_1$ is divisible by a_0 in R.*

Proof. Clearly, $b_0 = a_0$, since the 0th terms in the Hankel transforms must coincide. Now, in order to have

$$\begin{vmatrix} a_0 & b_1 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}$$

we must have $b_2 a_0 - b_1^2 = a_2 a_0 - a_1^2$, so $a_0(b_2 - a_2) = b_1^2 - a_1^2 = (b_1 - a_1)(b_1 + a_1)$. Therefore, a_0 divides either $b_1 - a_1$ or $b_1 + a_1$. □

Lemma 3.4. *Fix an integer $n \geq 1$. Suppose b is an HTP sequence preserving a_0 through a_{2n}. Then $b_{2n+1} - a_{2n+1}$ is divisible by*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Proof. We have

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & b_{2n+2} \end{vmatrix}$$

$$= b_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & 0 \end{vmatrix}$$

The determinant on the right above can be written

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & b_{2n+1} \\ 0 & 0 & \cdots & b_{2n+1} & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ 0 & 0 & \cdots & b_{2n+1} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix}.$$

(In each of the four above matrices, all rows except the last two are the same.) Now, we can write this as

$$= -b_{2n+1}^2 \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} - b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} & 0 \end{vmatrix}$$

$$-b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix}.$$

The second and third terms in the above sum are equal, since the determinant of a matrix is equal to the determinant of its transpose.

Now suppose

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & b_{2n+2} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & a_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n+1} & a_{2n+2} \end{vmatrix}.$$

Expanding these determinants as above and setting them equal, we get

$$b_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} - b_{2n+1}^2 \begin{vmatrix} a_0 & \cdots & a_{n-1} \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-2} \end{vmatrix} - 2b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix} =$$

$$a_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} - a_{2n+1}^2 \begin{vmatrix} a_0 & \cdots & a_{n-1} \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-2} \end{vmatrix} - 2a_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

Thus, we must have

$$(b_{2n+2} - a_{2n+2}) \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

$$= (b_{2n+1}^2 - a_{2n+1}^2) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} + 2(b_{2n+1} - a_{2n+1}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

By Lemma 3.1, either the claim is true or

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

must divide

$$(b_{2n+1} + a_{2n+1}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} + 2 \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

But in the quotient by the ideal I generated by $a_0, a_1, \ldots, a_{n-1}, a_{n+2}, a_{n+3}, \ldots, a_{2n+1}$, the first determinant becomes a_n^{n+1} (up to a sign), while the sum above becomes $2a_n^n * a_{n+1}$ (again up to a sign). Since a_n does not divide a_{n+1} in R/I, the claim must be true.

$\square$

We now turn to showing that the set of HTP sequences forms a group under composition. For this, we first need the following Lemma.

Lemma 3.5. *Suppose that b and b' are two HTP sequences which each preserve a_0 through a_{2n}, $n \geq 1$. Also, suppose that $b_{2n+1} + b'_{2n+1} = 2a_{2n+1}$. Then $b \circ b'$ preserves a_0 through a_{2n+2}.*

Proof. Let $d = b \circ b'$. Clearly d preserves a_0 through a_{2n}. By Lemma 3.4, there is a c such that

$$b_{2n+1} = a_{2n+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Since $b_{2n+1} + b'_{2n+1} = 2a_{2n+1}$, we must have

$$b'_{2n+1} = a_{2n+1} - c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Since both b and b' preserve a_0 through a_{2n}, it follows that $d_{2n+1} = a_{2n+1}$. Now $d_{2n+2} = a_{2n+2}$ by Lemma 3.2. □

Theorem 3.6. *The set of HTP sequences forms a group under composition.*

Proof. It suffices to show that any HTP sequence b has a left inverse. By Lemma 3.3, either $b_1 - a_1$ is divisible by a_0 or $b_1 + a_1$ is divisible by a_0. We will assume first that $b_1 - a_1$ is divisible by a_0. Now let n be the smallest number such that $b_{2n+1} \neq a_{2n+1}$ (so $n \geq 0$, since $b_0 = a_0$.) By Lemma 3.2, we know that $b_{2n} = a_{2n}$. Choose c_n as in Lemma 3.4 (or 3.3) so that

$$b_{2n+1} = a_{2n+1} - c_n \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Then by Lemma 3.5 and Corollary 2.15, the composition $b(n, c_n) \circ b$ preserves a_0 through a_{2n+2}.

Now, we inductively define a sequence $c_0, c_1, c_2, \ldots$ so that the composition

$$b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0) \circ b$$

preserves a_0 through a_{2n+2}. Thus

$$(\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0)) \circ b = \mathrm{id}.$$

We assumed above that $b_1 - a_1$ is divisible by a_0. If $b_1 + a_1$ is divisible by a_0, then we can reduce to the former case by replacing b by $b \circ \sigma$ (see Example 1.1). Then, as above, we can find a sequence $c_0, c_1, c_2, \ldots$ so that

$$\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0) \circ b \circ \sigma = \mathrm{id}.$$

Composing and precomposing both sides with σ (which satisfies $\sigma^2 = \mathrm{id}$), we get

$$\sigma \circ (\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0)) \circ b = \mathrm{id}.$$

□

Lemma 3.7. *The inverse $b(n,c)^{-1}$ of $b(n,c)$ preserves a_0 through a_{2n} and satisfies*

$$b(n,c)^{-1}_{2n+1} = a_{2n+1} - c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Proof. Since $b(n,c)$ preserves a_0 through a_{2n} by Corollary 2.15, it is clear that $b(n,c)^{-1}$ also preserves a_0 through a_{2n}. For the second part, $b(n,-c) = b(n,-c) \circ b(n,c) \circ b(n,c)^{-1}$. Since $b(n,-c) \circ b(n,c)$ preserves a_0 through a_{2n+2} by Lemma 3.5 and Corollary 2.15, it follows that $b(n,c)^{-1}$ must have the same $2n+1$ term as $b(n,-c)$. □

Remark 3.8. We do not know whether or not $b(n,c)^{-1} = b(n,-c)$ in general.

We now prove our main theorem.

Proof. As in Theorem 3.6, we can inductively define $c_0, c_1, \ldots$ such that either

$$b' := b \circ b(0,c_0)^{-1} \circ b(1,c_1)^{-1} \circ \cdots \circ b(n,c_n)^{-1}$$

or

$$b' := b \circ \sigma \circ b(0,c_0)^{-1} \circ b(1,c_1)^{-1} \circ \cdots \circ b(n,c_n)^{-1}$$

preserves a_0 through a_{2n+2}. Here, we use Lemma 3.7 together with Lemma 3.5 to complete the inductive step. Now either b or $b \circ \sigma$ can be written as

$$b' \circ b(n,c_n) \circ \cdots \circ b(1,c_1) \circ b(0,c_0).$$

It follows that

$$\cdots \circ b(n,c_n) \circ \cdots \circ b(1,c_1) \circ b(0,c_0)$$

agrees on all terms with b or $b \circ \sigma$. In the latter case, we multiply both sides by σ. □

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