Journal of Integer Sequences, Vol. 10 (2007),
Article 07.10.5

Inversion of Generating Functions using Determinants

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Abstract

In this note, we prove that the coefficients of an ordinary generating function can be deduced from some determinant formula about the coefficients in the reciprocal of the ordinary generating function. We use this result to obtain determinant identities for some well-known numbers.

1 Introduction

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. We define a sequence of determinants $\{D_n(a_k)\}_{n=0}^{\infty}$ related to the given sequence $\{a_n\}_{n=0}^{\infty}$: $D_0(a_k) = 1$, $D_1(a_k) = a_1$, $D_2(a_k) = \begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix} = a_1^2 - a_0 a_2$, and in general for $n \geq 1$,

$$D_n(a_k) = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}. \tag{1}$$

In 2005 Van Malderen [6] found the two identities

$$B_{2n} = \frac{(-1)^n(2n)!}{2}\left\{\sum_{k=0}^{n}\frac{(-1)^k D_{n-k}(1/(2k+1)!)}{(2k)!} + D_n\left(\frac{1}{(2k+1)!}\right)\right\} \tag{2}$$

and

$$B_{2n} = \frac{(-1)^{n+1}(2n)!D_n(1/(2k+1)!)}{2(2^{2n-1}-1)}. \tag{3}$$

where the Bernoulli numbers B_n are defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n t^n}{n!},$$

Motivated by the above work we obtained some interesting results. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be the ordinary generating functions of $\{a_n\}$ and $\{b_n\}$, respectively. If $A(z)B(z) = 1$, then

$$\sum_{k=0}^{n} a_k b_{n-k} = \delta_{0,n},$$

where $\delta_{0,n}$ is the Kronecker delta. These equations form a system of linear equations in $n+1$ unknowns $b_0, b_1, \ldots, b_n$. It should be mentioned that $a_0 \neq 0$. By Cramer's Rule, we can solve $b_n = (-1)^n D_n(a_k)/a_0^{n+1}$. We summarize the above result as the following theorem.

Theorem 1.1. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be the ordinary generating functions of $\{a_n\}$ and $\{b_n\}$, respectively. If $A(z)B(z) = 1$, then $a_0 \neq 0$ and b_n can be expressed as $(-1)^n D_n(a_k)/a_0^{n+1}$.* □

In fact, the above result could be generalized for $A_m(z) = \sum_{n=m}^{\infty} a_{n-m} z^n$ with $m \in \mathbb{Z}$ and $a_0 \neq 0$. For this case we can write $A_m(z) = z^m A(z)$. Applying Theorem 1.1 then yields $A_m(z)B_m(z) = 1$ where $B_m(z) = z^{-m}B(z)$ in which $B(z)$ is determined from $A(z)B(z) = 1$ using Theorem 1.1.

There are many interesting applications. For example, Euler [2] gave a generating function for the partition numbers $p(n)$ using the q-series

$$(q)_\infty = \prod_{m=1}^{\infty}(1-q^m) = \sum_{n=0}^{\infty} a_n q^n,$$

where

$$a_n = \begin{cases} 1, & \text{if } n = k(3k\pm 1)/2 \text{ and } k \text{ is even,} \\ -1, & \text{if } n = k(3k\pm 1)/2 \text{ and } k \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Here the exponents of nonzero terms are generalized pentagonal numbers. Then the partition numbers $p(n)$ are given by the generating function

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n.$$

Applying Theorem 1.1,

$$p(n) = (-1)^n D_n(a_k) \quad \text{and} \quad a_n = (-1)^n D_n(p(n)). \tag{4}$$

As application we use Theorem 1.1 to obtain determinant identities for well-known numbers by specializing to specific sequences $\{a_k\}$. We also give an elegant proof of Van Malderen's recent results on the formula for even-indexed Bernoulli numbers.

2 Some Properties of $D_n(a_k)$

We expand the determinant $D_n(a_k)$ according to the first row repeatedly. Then

$$D_n(a_k) = \sum_{k=1}^{n} (-a_0)^{k-1} a_k D_{n-k}(a_r).$$

This gives a recursive formula for $D_n(a_k)$.

Proposition 2.1. *(The Recursive Formula for $D_n(a_k)$) For each positive integer n, we have*

$$\sum_{k=0}^{n} (-a_0)^k a_k D_{n-k}(a_r) = 0. \tag{5}$$

□

The sequence of a lower triangular Toeplitz matrix $\{L_n(a_k)\}_{n=0}^{\infty}$ related to the given sequence $\{a_n\}_{n=0}^{\infty}$ can be defined as

$$L_n(a_k) = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}. \tag{6}$$

The determinant of $L_n(a_k)$ is a_0^{n+1}. The inverse matrix of $L_n(a_k)$ exists if $a_0 \neq 0$. Using the recursive formula for $D_n(a_k)$ we have the following corollary.

Corollary 2.2. *Let $a_0 \neq 0$. Then for each integer $n \geq 0$, the inverse matrix of $L_n(a_k)$ is $L_n((-1)^k D_k(a_r)/a_0^{k+1})$.* □

Theorem 2.3. *Let $D(a_k, z) = \sum_{n=0}^{\infty} D_n(a_k) z^n$, $A(a_k, z) = \sum_{n=0}^{\infty} a_n z^n$ be the ordinary generating functions of $\{D_n(a_k)\}$ and $\{a_n\}$, respectively. Then we have*

$$D(a_k, z) \cdot A(a_k, -a_0 z) = a_0. \tag{7}$$

Proof. Since

$$
\begin{aligned}
D(a_k, z)A(a_k, -a_0 z) &= \sum_{n=0}^{\infty} D_n(a_k) z^n \cdot \sum_{n=0}^{\infty} a_n(-a_0 z)^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-a_0)^k a_k D_{n-k}(a_r) z^n.
\end{aligned}
$$

For any positive integer n, the recursive formula for $D_n(a_k)$ implies that the coefficients of z^n on the right-hand side are all zeros. Hence all that remains of the above equation is the constant term. So

$$D(a_k, z)A(a_k, -a_0 z) = (-a_0)^0 a_0 D_0(a_k) = a_0.$$

□

If $A(z)B(z) = 1$, it implies that $a_0 b_0 = 1$. Thus the number a_0 must be nonzero. Set $t = -a_0 z$ in Eq. (7), $a_0 = D(a_k, -t/a_0)A(a_k, t)$. Therefore b_n can be expressed as $(-1)^n D_n(a_k)/a_0^{n+1}$. This gives another proof of Theorem 1.1.

Here we give a basic property of the determinant $D_n(a_k)$.

Proposition 2.4. *Let u be a complex number and n be a nonnegative integer. Then*

$$D_n(u^k a_k) = u^n D_n(a_k). \tag{8}$$

Proof. Using the recursive formula for $D_n(a_k)$ and the strong mathematical induction on the integer n, we easily get the assertion. □

3 Applications

Theorem 3.1. *Let n be a nonnegative integer. Then the ordinary generating function of $D_n(1/(2k+1)!)$ is*

$$D\left(\frac{1}{(2k+1)!}, z^2\right) = \frac{z}{\sin z}. \tag{9}$$

Moreover,

$$(2n)! D_n(1/(2k+1)!) = (-1)^{n+1}(2^{2n} - 2)B_{2n}. \tag{10}$$

Proof. The ordinary generating function of $1/(2n+1)!$ is related to the power series expansion of $\sin z$.

$$A\left(\frac{1}{(2k+1)!}, -z^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = \frac{\sin z}{z}.$$

In view of Theorem 1.1 the ordinary generating function of $D_n(1/(2k+1)!)$ becomes

$$D\left(\frac{1}{(2k+1)!}, z^2\right) = \frac{z}{\sin z}.$$

We express the function $z/\sin z$ in the form

$$\frac{z}{\sin z} = \frac{2iz}{e^{iz} - e^{-iz}} = \frac{2ize^{iz}}{e^{2iz} - 1}.$$

Therefore,

$$D\left(\frac{1}{(2k+1)!}, -z^2\right) = \frac{2ze^z}{e^{2z} - 1}.$$

Since the Bernoulli polynomial $B_n(z)$ is defined by

$$\frac{te^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)t^n}{n!},$$

this implies that

$$(2n)!D_n(1/(2k+1)!) = (-1)^n 2^{2n} B_{2n}(1/2).$$

Our assertion follows from Eq. 23.1.21 of [1], for $n \geq 0$,

$$B_n(1/2) = (2^{1-n} - 1)B_n.$$

$\square$

The Eq. (10) is just Eq. (3) obtained by Van Malderen [6]. We can rewrite the well-known recursive relation among Bernoulli numbers $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$ as

$$\sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} \cdot \frac{1}{(2n+1-2k)!} = \frac{1}{2(2n)!} - \frac{1}{(2n+1)!}. \tag{11}$$

This formula can be represented in the matrix form

$$L_n\left(\frac{1}{(2k+1)!}\right) \cdot \overrightarrow{\left(\frac{B_{2k}}{(2k)!}\right)}_{n+1} = \overrightarrow{\left(\frac{1}{2(2k)!} - \frac{1}{(2k+1)!}\right)}_{n+1}, \tag{12}$$

where $\overrightarrow{(v_k)}_n = (v_1, v_2, \ldots, v_n)^t$ be a $n \times 1$ vector. By Corollary 2.2 the inverse matrix of $L_n(1/(2k+1)!)$ is $L_n((-1)^k D_k(1/(2r+1)!))$. We can obtain Eq. (2) by multiplying this inverse matrix.

The Genocchi numbers G_n and the tangent numbers T_n are defined by [4]

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} \frac{G_n t^n}{n!} \quad \text{and} \quad 1 + \tanh t = \sum_{n=0}^{\infty} \frac{T_n t^n}{n!}.$$

They can be expressed in terms of the Bernoulli numbers as

$$G_{2n} = (2^{2n+1} - 2)B_{2n} \qquad \text{and} \qquad T_{2n-1} = \frac{2^{2n}(2^{2n} - 1)}{2n} B_{2n}.$$

Hence we can express them as $D_n(1/(2k+1)!)$ multiplied a suitable constant.

Corollary 3.2. *Let n, m be integers with $n \geq 1$, $m \geq 0$, we have*

$$(2^{2n}-1)(2n-1)!D_n(2^{2k}/(2k+1)!) = (-1)^{n+1}(2^{2n}-2)T_{2n-1} \tag{13}$$

and

$$(2^{2m}-1)(2m)!D_m(1/(2k+1)!) = (-1)^{m+1}(2^{2m-1}-1)G_{2m}. \tag{14}$$

The Bernoulli numbers of order m, $B_n^{(m)}$, are defined by

$$B(m,t) = \left(\frac{t}{e^t-1}\right)^m = \sum_{n=0}^{\infty} \frac{B_n^{(m)} t^n}{n!}.$$

The Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$ are defined by [5]

$$\frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!}.$$

Theorem 3.3. *Let n, m be nonnegative integers and $S_n^{(m)} = \left\{ {m+n \atop m} \right\} / \binom{m+n}{m}$. Then the ordinary generating function of $D_n(S_k^{(m)}/k!)$ is*

$$D(S_k^{(m)}/k!, z) = B(m, -z). \tag{15}$$

Moreover,

$$n!D_n(S_k^{(m)}/k!) = (-1)^n B_n^{(m)} \quad \textit{and} \quad n!D_n(B_k^{(m)}/k!) = (-1)^n S_n^{(m)}. \tag{16}$$

Proof. The ordinary generating function of the Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$ can be rewritten as

$$\frac{(e^z-1)^m}{m!} = \sum_{n=0}^{\infty} \left\{ {m+n \atop m} \right\} \frac{z^{m+n}}{(m+n)!}.$$

Therefore, the ordinary generating function of $S_n^{(m)}/n!$ is

$$A\left(\frac{S_k^{(m)}}{k!}, z\right) = \frac{(e^z-1)^m}{z^m}.$$

Applying Theorem 2.3 we have

$$D\left(\frac{S_k^{(m)}}{k!}, z\right) = \left(\frac{-z}{e^{-z}-1}\right)^m = B(m, -z).$$

Comparing the coefficients of z in the both sides of the above equation, we get Eq. (16). $\square$

The case of $m = 1$ in the above theorem gives a well-known formula for Bernoulli numbers (see Eq. (4) of [3]):

$$D_n\left(\frac{1}{(k+1)!}\right) = \frac{(-1)^n B_n}{n!}. \tag{17}$$

Using Proposition 2.4 we get an interesting identity from the above formula:

$$D_n\left(\frac{(-1)^k}{(k+1)!}\right) = \frac{B_n}{n!}. \tag{18}$$

The Harmonic numbers H_n are defined by

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},$$

for $n \geq 1$. Since [7]

$$H_n = \frac{(-1)^{n-1} B_{n-1}^{(n+1)}}{(n-1)!},$$

we have the following corollary.

Corollary 3.4. *For each positive integer n, we have*

$$H_n = D_{n-1}(S_k^{(n+1)}/k!). \tag{19}$$

The Euler numbers of order m, $E_n^{(m)}$, are defined by

$$E(m,t) = (\operatorname{sech} t)^m = \sum_{n=0}^{\infty} \frac{E_n^{(m)} t^n}{n!}.$$

Since $\operatorname{sech} t$ is an even function, $E_{2n+1}^{(m)} = 0$ for $n \geq 0$. This gives that

$$E(m,t) = \sum_{n=0}^{\infty} \frac{E_{2n}^{(m)} t^{2n}}{(2n)!}.$$

Theorem 3.5. *Let n, m be nonnegative integers and*

$$C_n^{(m)} = \sum_{\substack{p_i \geq 0 \\ p_1+\cdots+p_m=n}} \frac{(2n)!}{(2p_1)!\cdots(2p_m)!} = \sum_{\substack{p_i \geq 0 \\ p_1+\cdots+p_m=n}} \binom{2n}{2p_1,\ldots,2p_m}. \tag{20}$$

Then the ordinary generating function of $D_n(C_k^{(m)}/(2k)!)$ is

$$D(C_k^{(m)}/(2k)!, z^2) = E(m, iz). \tag{21}$$

Moreover,

$$(2n)! D_n(C_k^{(m)}/(2k)!) = (-1)^n E_{2n}^{(m)} \quad \textit{and} \quad (2n)! D_n(E_{2k}^{(m)}/(2k)!) = (-1)^n C_n^{(m)}. \tag{22}$$

Proof. The ordinary generating function of $C_n^{(m)}/(2n)!$ is related to the power series expansion of $(\cosh z)^m$.

$$A\left(\frac{C_k^{(m)}}{(2k)!}, z^2\right) = \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\right)^m = (\cosh z)^m.$$

The remaining proof is similar to the proofs of Theorems 3.1 and 3.3, therefore we omit it. $\square$

The Bernoulli numbers of the second kind of order m, $b_n^{(m)}$, are defined by

$$b(m,t) = \left(\frac{t}{\ln(1+t)}\right)^m = \sum_{n=0}^{\infty} \frac{b_n^{(m)} t^n}{n!}.$$

The Stirling numbers of the first kind $\left[{n \atop m}\right]$ are defined by [5]

$$\frac{(\ln(1+t))^m}{m!} = \sum_{n=m}^{\infty} \left[{n \atop m}\right] \frac{t^n}{n!}.$$

Theorem 3.6. *Let n, m be nonnegative integers and $s_n^{(m)} = \left[{m+n \atop m}\right] / \binom{m+n}{m}$. Then the ordinary generating function of $D_n(s_k^{(m)}/k!)$ is*

$$D(s_k^{(m)}/k!, z) = b(m, -z). \tag{23}$$

Moreover,

$$n! D_n(s_k^{(m)}/k!) = (-1)^n b_n^{(m)} \quad \text{and} \quad n! D_n(b_k^{(m)}/k!) = (-1)^n s_n^{(m)}. \tag{24}$$

Proof. We indicate that the definition of the Stirling numbers of the first kind $\left[{n \atop m}\right]$ can be rewritten as

$$\frac{(\ln(1+t))^m}{m!} = \sum_{n=0}^{\infty} \left[{n+m \atop m}\right] \frac{t^{m+n}}{(n+m)!}.$$

The results follow. $\square$

The case of $m = 1$ in the above theorem gives a new formula for the Bernoulli numbers of second kind:

$$D_n\left(\frac{1}{k+1}\right) = \frac{b_n}{n!}. \tag{25}$$

In the same spirit we can apply Theorem 1.1 to many other areas. For example, in the theory of symmetric functions the Jacobi-Trudi Determinants (page 154 of [8]) say that $s_\lambda = \det(h_{\lambda_i - i + j})$ and $s_{\lambda'} = \det(e_{\lambda_i - i + j})$ where s_λ is the Schur function associated with the partition λ, λ_i is the ith part of λ, λ' is the conjugate of λ, and h_n is the nth complete symmetric function, e_n is the nth elementary symmetric function.

The generating functions of $e_n(\mathbf{x})$ and $h_n(\mathbf{x})$ are

$$E(t) = \sum_{n=0}^{\infty} e_n(\mathbf{x}) t^n = \prod_{i=1}^{\infty} (1 + x_i t), \qquad H(t) = \sum_{n=0}^{\infty} h_n(\mathbf{x}) t^n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t},$$

respectively. Since $E(-t)H(t) = 1$, our Theorem 1.1 gives

$$e_n = D_n(h_k) \qquad \text{and} \qquad h_n = D_n(e_k). \tag{26}$$

These formulae are the same with the results specialized to $\lambda = 1^n$ in the Jacobi-Trudi Determinants.

Acknowledgements

The author would like to thank the referee for some useful comments and suggestions. This research was supported by the National Science Foundation of Taiwan, Republic of China, grant under NSC 96-2115-M-133-002.

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2000 *Mathematics Subject Classification*: Primary 11C20, Secondary 11B68.
Keywords: Generating function inversion, determinant formulas, Toeplitz matrix, Bernoulli numbers of order m, Stirling numbers.

(Concerned with sequences A000041, A000182, A001318, A001469, A008275, and A008277.)

Received October 10 2007; revised version received December 31 2007. Published in *Journal of Integer Sequences*, December 31 2007.

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