Journal of Integer Sequences, Vol. 10 (2007),
Article 07.9.8

On a One-Parameter Family of Riordan Arrays and the Weight Distribution of MDS Codes

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Abstract

We use the formalism of the Riordan group to study a one-parameter family of lower-triangular matrices related to the weight distribution of maximum distance separable codes. We obtain factorization results for these matrices. We then derive alternative expressions for the weight distribution of MDS codes. We define related weight ratios and show that they satisfy a certain linear recurrence.

1 Introduction

In this note, we report on a one-parameter family of transformation matrices which can be related to the weight distribution of maximum distance separable (MDS) codes. Regarded as transformations on integer sequences, they are easy to describe both by formula (in relation to the general term of a sequence) and in terms of their action on the ordinary generating function of a sequence. To achieve this, we use the language of the Riordan group of infinite lower-triangular integer matrices. They are also linked to several other known transformations, most notably the binomial transformation.

2 Error-correcting codes

Maximum distance separable codes are a special case of error-correcting code. By *error-correcting code*, we shall mean a *linear* code over $F_q = GF(q)$, that is, a vector subspace C of F_q^n for some $n > 0$. If C is a k-dimensional vector subspace of F_q^n, then the code is described as a *q-ary* $[n, k]$-code. The elements of C are called the codewords of the code. The weight $w(c)$ of a codeword c is the number of non-zero elements in the vector representation of c. An $[n, k]$ code with minimum weight d is called an $[n, k, d]$ code. A code is called a maximum separable code if the minimum weight of a non-zero codeword in the code is $n - k + 1$, that is, $d = n - k + 1$. The Reed-Solomon family of linear codes is a well-known family of MDS codes.

An important characteristic of a code is its *weight distribution*. This is defined to be the set of coefficients $A_0, A_1, \ldots, A_n$ where A_i is the number of codewords of weight i in C. The weight distribution of a code plays a significant role in calculating probabilities of error. Except for trivial or 'small' codes, the determination of the weight distribution is normally not easy. The MacWilliams identity for general linear codes is often used to simplify this task. The special case of MDS codes proves to be tractable. Using the MacWilliams identity [2] or otherwise [3, 9], we obtain the following equivalent results.

Proposition 1. *The number of codewords of weight i, where $n - k + 1 \le i \le n$, in a q-ary $[n, k]$ MDS code is given by*

$$\begin{aligned} A_i &= \binom{n}{i}(q-1)\sum_{j=0}^{i-d_{min}}(-1)^j\binom{i-1}{j}q^{i-d_{min}-j} && (1)\\ &= \binom{n}{i}\sum_{j=0}(-1)^j\binom{i}{j}(q^{i-d_{min}+1-j}-1) && (2)\\ &= \binom{n}{i}\sum_{j=d_{min}}^{i}(-1)^{i-j}\binom{i}{j}(q^{j-d_{min}+1}-1) && (3)\end{aligned}$$

where $d_{min} = n - k + 1$.

We note that the last expression can be written as

$$A_i = \binom{n}{i}\sum_{j=0}^{i-d_{min}}(-1)^{i-d_{min}-j}\binom{i}{j+d_{min}}(q^{j+1}-1) \tag{4}$$

by a simple change of variable.

We have $A_0 = 1$, and $A_i = 0$ for $1 \le i \le n-k$. The term $\binom{n}{i}$ is a scaling term, which also ensures that $A_i = 0$ for $i > n$. In the sequel, we shall study a one-parameter family of Riordan arrays associated to the equivalent summation expressions above.

3 Transformations on integer sequences and the Riordan Group

We shall introduce transformations that operate on integer sequences. An example of such a transformation that is widely used in the study of such sequences is the so-called *Binomial transform* [10], which associates to the sequence with general term a_n the sequence with general term b_n where

$$b_n = \sum_{j=0}^{n} \binom{n}{j} a_j. \tag{5}$$

If we consider the sequence with general term a_n to be the column vector $\mathbf{a} = (a_0, a_1, \ldots)'$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector by the lower-triangle matrix $\mathbf{B}$ whose (i, j)-th element is equal to $\binom{i}{j}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This transformation is invertible, with

$$a_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} b_j. \tag{6}$$

We note that $\mathbf{B}$ corresponds to Pascal's triangle. Its row sums are 2^n, while its diagonal sums are the Fibonacci numbers $F(n + 1)$. If $\mathbf{B}^k$ denotes the $k-$th power of $\mathbf{B}$, then the $n-$th term of $\mathbf{B}^k\mathbf{a}$ where $\mathbf{a} = \{a_j\}$ is given by $\sum_{j=0}^{n} k^{n-j}\binom{n}{j} a_j$.

If $\mathcal{A}(x)$ is the ordinary generating function of the sequence a_n, then the generating function of the transformed sequence b_n is $\frac{1}{1-x}\mathcal{A}(\frac{x}{1-x})$. The binomial transform is an element of the Riordan group, which can be defined as follows.

The *Riordan group* [1, 4, 8], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \ldots$ and $f(x) = f_1x + f_2x^2 + \ldots$ where $f_1 \neq 0$ [8]. We sometimes write $f(x) = xh(x)$ where $h(0) \neq 0$. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f). \tag{7}$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

To each Riordan array as defined above is associated an integer sequence $A = \{a_i\}$ with $a_0 \neq 0$ such that every element $d_{n+1,k+1}$ of the array (not lying in column 0 or row 0) can be expressed as a linear combination with coefficients in A of the elements in the preceding row, starting from the preceding column:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$$

$A = \{a_i\}$ is called the A-sequence of the array, and may be calculated according to

$$A(x) = [h(t)|t = xh(t)^{-1}].$$

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}\ (x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$.

Example 2. *The binomial matrix* $\mathbf{B}$ *is the element* $(\frac{1}{1-x}, \frac{x}{1-x})$ *of the Riordan group. It has general element* $\binom{n}{k}$. *More generally,* $\mathbf{B}^m$ *is the element* $(\frac{1}{1-mx}, \frac{x}{1-mx})$ *of the Riordan group, with general term* $\binom{n}{k}m^{n-k}$. *It is easy to show that the inverse* $\mathbf{B}^{-m}$ *of* $\mathbf{B}^m$ *is given by* $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The row sums of the matrix (g, f) have generating function $g(x)/(1 - f(x))$ while the diagonal sums of (g, f) have generating function $g(x)/(1 - xf(x))$.

Many interesting examples of Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences, [5, 6]. Sequences are frequently referred to by their OEIS number. For instance, the matrix $\mathbf{B}$ is A007318.

4 Introducing the one-parameter family of 'MDS' transforms

In this section, we shall frequently use n and k to address elements of infinite arrays. Thus the n,k-th element of an infinite array T refers to the element in the n-th row and the k-th column. Row and column indices will start at 0. This customary use of n, k, should not cause any confusion with the use of n, k above to describe $[n,k]$ codes.

We define $\mathbf{T}_m$ to be the transformation represented by the matrix

$$\mathbf{T}_m = \left(\frac{1+x}{1-mx}, \frac{x}{1+x}\right)$$

where $m \in \mathbf{N}$. For instance, we have

$$\mathbf{T}_1 = \left(\frac{1+x}{1-x}, \frac{x}{1+x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 1 & 0 & 0 & \ldots \\ 2 & 1 & 1 & -1 & 1 & 0 & \ldots \\ 2 & 1 & 0 & 2 & -2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This triangle is A113310, which has row sums $1, 3, 4, 4, 4, \ldots$ A113311 with generating function $\frac{(1+x)^2}{1-x}$. In general, the row sums of $\mathbf{T}_m$ have generating function $\frac{(1+x)^2}{1-mx}$. Note also that

$$\mathbf{T}_0 = \left(1+x, \frac{x}{1+x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -2 & 1 & 0 & \ldots \\ 0 & 0 & -1 & 3 & -3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with general term $\binom{n-2}{n-k}(-1)^{n-k}$. We can calculate the A-sequence for $\mathbf{T}_m$ as follows.

$$\begin{aligned} A(x) &= \left[\frac{1}{1+t} | t = xh(t)^{-1}\right] \\ &= \left[\frac{1}{1+t} | t = x(1+t)\right] \\ &= \left[\frac{1}{1+t} | t = \frac{x}{1-x}\right] \\ &= \frac{1}{1+\frac{x}{1-x}} = 1-x. \end{aligned}$$

Thus every element in $\mathbf{T}_m$ is given by the difference of the two elements above it, i.e.:

$$\mathbf{T}_m(n+1, k+1) = \mathbf{T}_m(n, k) - \mathbf{T}_m(n, k+1).$$

Proposition 3. *For each m, $\mathbf{T}_m$ is invertible with*

$$\mathbf{T}_m^{-1} = \left(1-(m+1)x, \frac{x}{1-x}\right).$$

Proof. Let $\mathbf{T}_m^{-1} = (g^*, \bar{f})$. This exists since $\mathbf{T}_m$ is an element of the Riordan group. Then

$$(g^*, \bar{f})\left(\frac{1+x}{1-mx}, \frac{x}{1+x}\right) = (1, x).$$

Hence

$$\frac{\bar{f}}{1+\bar{f}} = x \Rightarrow \bar{f} = \frac{x}{1-x}$$

and

$$g^* = \frac{1}{g \circ \bar{f}} \Rightarrow g^* = \frac{1-m\bar{f}}{1+\bar{f}} = 1-(m+1)x.$$

□

Corollary 4. *The general term of $\mathbf{T}_m^{-1}$ is given by*

$$\mathbf{T}_m^{-1}(n,k) = \binom{n-1}{n-k} - (m+1)\binom{n-2}{n-k-1}.$$

Proof. We have

$$\begin{aligned}
\mathbf{T}_m^{-1}(n,k) &= [x^n](1-(m+1)x)\left(\frac{x}{1-x}\right)^k\\
&= [x^{n-k}]\frac{1}{(1-x)^k}-(m+1)[x^{n-k-1}]\frac{1}{(1-x)^k}\\
&= \binom{-k}{n-k}(-1)^{n-k}-(m+1)\binom{-k}{n-k-1}(-1)^{n-k-1}\\
&= \binom{n-1}{n-k}-(m+1)\binom{n-2}{n-k-1}.
\end{aligned}$$

□

Our main goal in this section is to find expressions for the general term $\mathbf{T}_m(n,k)$ of $\mathbf{T}_m$. To this end, we exhibit certain useful factorizations of $\mathbf{T}_m$.

Proposition 5. *We have the following factorizations of the Riordan array* $\mathbf{T}_m$*:*

$$\begin{aligned}
\mathbf{T}_m &= \left(\frac{1+x}{1-mx},\frac{x}{1+x}\right)\\
&= (1+x,x)*\left(\frac{1}{1-mx},\frac{x}{1+x}\right)\\
&= \left(1,\frac{1}{1+x}\right)*\left(\frac{x}{1-(m+1)x},x\right)\\
&= \left(\frac{1}{1-mx},x\right)*\left(1+x,\frac{x}{1+x}\right)\\
&= \left(\frac{1}{1+x},\frac{x}{1+x}\right)*\left(\frac{1}{1-x}\cdot\frac{1}{1-(m+1)x},x\right).
\end{aligned}$$

Proof. Each of the assertions is a simple consequence of the product rule (equation (7)) for Riordan arrays. For instance,

$$\begin{aligned}
\left(1,\frac{x}{1+x}\right)*\left(\frac{1}{1-(m+1)x},x\right) &= \left(1\cdot\frac{1}{1-(m+1)\frac{x}{1+x}},\frac{x}{1+x}\right)\\
&= \left(\frac{1+x}{1+x-(m+1)x},\frac{x}{1+x}\right)\\
&= \left(\frac{1+x}{1-mx},\frac{x}{1+x}\right)=\mathbf{T}_m.
\end{aligned}$$

The other assertions follow in a similar manner. □

The last assertion, which can be written

$$\mathbf{T}_m=\mathbf{B}^{-1}*\left(\frac{1}{1-x}\cdot\frac{1}{1-(m+1)x},x\right),$$

is a consequence of the fact that the product $\mathbf{BT}_m$ takes on a simple form. We have

$$\begin{aligned}\mathbf{BT}_m &= \left(\frac{1}{1-x}, \frac{x}{1-x}\right) * \left(\frac{1+x}{1-mx}, \frac{x}{1+x}\right)\\ &= \left(\frac{1}{1-x} \cdot \frac{1+\frac{x}{1-x}}{1-m\frac{x}{1-x}}, \frac{\frac{x}{1-x}}{1+\frac{x}{1-x}}\right)\\ &= \left(\frac{1}{1-x} \cdot \frac{1}{1-(m+1)x}, x\right).\end{aligned}$$

We can interpret this as the sequence array for the partial sums of the sequence $(m+1)^n$, that is, the sequence array of $\frac{(m+1)^{n+1}-1}{(m+1)-1}$. Thus $\mathbf{T}_m$ is obtained by applying $\mathbf{B}^{-1}$ to this sequence array. We note that the inverse matrix $(\mathbf{BT}_m)^{-1}$ takes the special form

$$((1-x)(1-(m+1)x), x) = (1-(m+2)x+(m+1)x^2, x).$$

Thus this matrix is tri-diagonal, of the form

$$(\mathbf{BT}_m)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -(m+2) & 1 & 0 & 0 & 0 & 0 & \dots \\ m+1 & -(m+2) & 1 & 0 & 0 & 0 & \dots \\ 0 & m+1 & -(m+2) & 1 & 0 & 0 & \dots \\ 0 & 0 & m+1 & -(m+2) & 1 & 0 & \dots \\ 0 & 0 & 0 & m+1 & -(m+2) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Corollary 6. *The general term of the array* $\mathbf{T}_m$ *is*

$$\mathbf{T}_m(n,k) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} ((m+1)^{j-k+1}-1)/m, \qquad m \neq 0.$$

Proof. By the last proposition, we have

$$\mathbf{T}_m = \mathbf{B}^{-1} * \left(\frac{1}{1-x} \cdot \frac{1}{1-(m+1)x}, x\right).$$

The general term of $\mathbf{B}^{-1} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right)$ is $(-1)^{n-k}\binom{n}{k}$ while that of the second Riordan array is $\frac{(m+1)^{n-k+1}-1}{(m+1)-1}$. The result follows from the product formula for matrices. $\square$

Corollary 7.

$$\mathbf{T}_{m-1}(n,k) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \frac{m^{j-k+1}-1}{m-1} = \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n}{j+k} \frac{m^{j+1}-1}{m-1}.$$

Equivalently,

$$(m-1)\binom{n}{k}\mathbf{T}_{m-1}(n,k) = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n}{j+k} (m^{j+1}-1).$$

This last result makes evident the link between the Riordan array $\mathbf{T}_{m-1}$ and the weight distribution of MDS codes. Thus, based on equation (3), we have

$$T_{q-1}(i, d_{min}) = \sum_{j=d_{min}}^{i} (-1)^{i-j} \binom{i}{j} \frac{q^{j-d_{min}+1} - 1}{q-1}.$$

Hence

$$\begin{aligned} (q-1)\binom{n}{i} T_{q-1}(i, d_{min}) &= \binom{n}{i} \sum_{j=d_{min}}^{i} (-1)^{i-j} \binom{i}{j} (q^{j-d_{min}+1} - 1) \\ &= A_i. \end{aligned}$$

and thus

$$A_i = (q-1)\binom{n}{i} T_{q-1}(i, d_{min}). \tag{8}$$

We now find a number of alternative expressions for the general term of $\mathbf{T}_m$ which will give us a choice of expressions for the weight distribution of an MDS code.

Proposition 8.

$$\begin{aligned} \mathbf{T}_m(n,k) &= \sum_{j=0}^{n-k} (-1)^j \binom{j+k-2}{j} m^{n-k-j} \\ &= \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-j-2}{n-j-k} m^j \\ &= \sum_{j=k}^{n} \binom{n-1}{n-j} (-1)^{n-j} (m+1)^{j-k}. \end{aligned}$$

Proof. The first two equations result from

$$\begin{aligned} \mathbf{T}_m(n,k) &= [x^n] \frac{1+x}{1-mx} \left(\frac{x}{1+x}\right)^k \\ &= [x^{n-k}](1-mx)^{-1}(1+x)^{-(k-1)} \\ &= [x^{n-k}] \sum_{i \geq 0} m^i x^i \sum_{j \geq 0} \binom{-(k-1)}{j} x^j \\ &= [x^{n-k}] \sum_{i \geq 0} \sum_{j \geq 0} \binom{k+j-2}{j} (-1)^j m^i x^{i+j}. \end{aligned}$$

An alternative proof for the second identity if furnished by using the convolution rule (rule

4KE - **conv** in [7]) to get:

$$\begin{aligned}[x^n]\frac{1+x}{1-mx}\cdot\frac{x^k}{(1+x)^k} &= [x^{n-k}]\frac{1}{1-mx}\cdot\frac{1}{(1+x)^{k-1}}\\ &= \sum_{j=0}^{n-k}\binom{-k+1}{n-k-j}m^j\\ &= \sum_{j=0}^{n-k}\binom{n-j-2}{n-k-j}(-1)^{n-k-j}m^j,\end{aligned}$$

while the first identity is obtained when we apply the convolution rule in the symmetric way, i.e., with $j\mapsto n-k-j$. The third equation is a consequence of the factorization

$$\mathbf{T}_m=\left(1,\frac{1}{1+x}\right)*\left(\frac{1}{1-(m+1)x},x\right)$$

since $\left(1,\frac{1}{1+x}\right)$ has general term $\binom{n-1}{n-k}(-1)^{n-k}$. □

Thus we have, for instance,

$$(m-1)\binom{n}{k}\mathbf{T}_{m-1}(n,k)=(m-1)\binom{n}{k}\sum_{j=k}^{n}\binom{n-1}{n-j}(-1)^{n-j}m^{j-k}.$$

Using the standard notation for weight distributions, we obtain, from equation (8),

$$\begin{aligned}A_i &= (q-1)\binom{n}{i}T_{q-1}(i,d_{min})\\ &= (q-1)\binom{n}{i}\sum_{j=d_{min}}^{i}\binom{i-1}{i-j}(-1)^{i-j}q^{j-d_{min}}.\end{aligned}$$

5 Applications to MDS codes

We begin this section with an example.

Example 9. *The dual of the* $[7,2,6]$ *Reed Solomon code over* $GF(2^3)$ *is an MDS* $[7,5,3]$ *code, also over* $GF(2^3)$*. Thus the code parameters of interest to us are* $q=8$, $n=7$, $k=5$ *and* $d_{min}=n-k+1=3$*. Let* $\mathbf{D}=diag(\binom{7}{0},\binom{7}{1},\ldots,\binom{7}{7},0,0,\ldots)$ *denote the infinite square matrix all of whose entries are zero except for those indicated. We form the matrix product*

$(q-1)\mathbf{DT}_{q-1}$, with $q=8$, to get

$$7diag\left\{\binom{7}{j}\right\}\left(\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
56 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
392 & 49 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
2744 & 343 & 43 & 5 & 1 & 0 & 0 & 0 & 0 & \dots \\
19208 & 2401 & 300 & 38 & 4 & 1 & 0 & 0 & 0 & \dots \\
134456 & 16807 & 2101 & 262 & 34 & 3 & 1 & 0 & 0 & \dots \\
941192 & 117649 & 14706 & 1839 & 228 & 31 & 2 & 1 & 0 & \dots \\
6588344 & 823543 & 102943 & 12867 & 1611 & 197 & 29 & 1 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$$= \left(\begin{array}{cccccccccc}
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
392 & 49 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
8232 & 1029 & 147 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
96040 & 12005 & 1470 & \mathbf{245} & 0 & 0 & 0 & 0 & 0 & \dots \\
672280 & 84035 & 10535 & \mathbf{1225} & 245 & 0 & 0 & 0 & 0 & \dots \\
2823576 & 352947 & 44100 & \mathbf{5586} & 588 & 147 & 0 & 0 & 0 & \dots \\
6588344 & 823543 & 102949 & \mathbf{12838} & 1666 & 147 & 49 & 0 & 0 & \dots \\
6588344 & 823543 & 102942 & \mathbf{12873} & 1596 & 217 & 14 & 7 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

Column 3 (starting from column 0) of this matrix then yields the weight distribution of the $[7,5,3]$ code. That is, we obtain the vector $(1,0,0,245,1225,5586,12838,12873)$, where we have made the adjustment $A_0 = 1$. Thus we obtain

A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	0	0	245	1225	5586	12838	12873

We moreover notice that the numbers $(0,0,0,1,5,38,262,1839)$, which correspond to the ratios $A_i/((q-1)\binom{n}{i})$, are elements of the sequence with ordinary generating function $\frac{1+x}{1-7x}(\frac{x}{1+x})^3 = \frac{x^3(1+x)}{1-4x-18x^2-20x^3-x^4}$. Hence they satisfy the recurrence

$$a_n = 4a_{n-1} + 18a_{n-2} + 20a_{n-3} + 7a_{n-4}.$$

This last result leads us to define the *weight ratios* of a q-ary $[n,k,d]$ MDS code to be the ratios $A_i/((q-1)\binom{n}{i})$.

We are now in a position to summarize the results of this paper.

Theorem 10. *Let C be a q-ary $[n,k,d]$ MDS code. The weight distribution of C, adjusted for A_0=1, is obtained from the d-th column of the matrix*

$$(q-1)Diag\left\{\binom{n}{j}\right\}\left(\frac{1+x}{1-(q-1)x}, \frac{x}{1+x}\right).$$

Moreover, the weight ratios of the code satisfy a recurrence defined by the ordinary generating function $\frac{1+x}{1-(q-1)x}(\frac{x}{1+x})^d$.

Proof. Inspection of the expressions for the general term $\mathbf{T}_{q-1}$ and the formulas for A_i yield the first statement. The second statement is a standard property of the columns of a Riordan array. $\square$

Thus the weight ratios satisfy the recurrence

$$a_n = ((q-1)\tbinom{d}{0} - \tbinom{d}{1})a_{n-1} + ((q-1)\tbinom{d}{1} - \tbinom{d}{2})a_{n-2} + \cdots + ((q-1)\tbinom{d}{d-1} - \tbinom{d}{d})a_{n-d} + (q-1)a_{n-d-1}.$$

Letting $R_0 = 0$, and $R_i = A_i/((q-1)\binom{n}{i})$ for $i > 0$, we therefore have

$$R_l = \sum_{j=0}^{d}((q-1)\binom{d}{j} - \binom{d}{j+1})R_{l-j-1}$$

where $d = d_{min} = n - k + 1$. In terms of the A_i, this therefore gives us

$$A_i = \binom{n}{i}\sum_{j=0}^{d}\frac{(q-1)\binom{d}{j} - \binom{d}{j+1}}{\binom{n}{i-j-1}}A_{i-j-1}.$$

For instance, we have

$$12873 = \binom{7}{7}\left[\frac{4}{7}\cdot 12838 + \frac{18}{21}\cdot 5586 + \frac{20}{35}\cdot 1225 + \frac{7}{35}\cdot 245\right].$$

and

$$12838 = \binom{7}{6}\left[\frac{4}{21}\cdot 5586 + \frac{18}{35}\cdot 1225 + \frac{20}{35}\cdot 245 + \frac{7}{21}\cdot 0\right].$$

6 Acknowledgements

The author would like to thank an anonymous reviewer for their careful reading of this manuscript and for their constructive remarks.

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2000 *Mathematics Subject Classification*: Primary 11B83; Secondary 94B05, 11B37, 11B65.

(Concerned with sequences A007318, A113310, and A113311.)

Received June 21 2007; revised version received November 16 2007. Published in *Journal of Integer Sequences*, November 20 2007.

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