#A63 INTEGERS 13 (2013)

HOW TO WRITE A PERMUTATION AS A PRODUCT OF INVOLUTIONS (AND WHY YOU MIGHT CARE)

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Received: 9/10/12, Revised: 3/23/13, Accepted: 8/18/13, Published: 9/26/13

Abstract

It is well-known that any permutation can be written as a product of two involutions. We provide an explicit formula for the number of ways to do so, depending only on the cycle type of the permutation. In many cases, these numbers are sums of absolute values of irreducible characters of the symmetric group evaluated at the same permutation, although apart from the case where all cycles are the same size, we have no good explanation for why this should be so.

1. Introduction

The authors were interested in finding a combinatorial model counted by the following sequence of integers:

$$1, 4, 9, 27, 61, 185, 469, \ldots, \tag{1}$$

And upon putting these terms into The On-Line Encyclopedia of Integer Sequences, we received a match in sequence A164342 [3]. That entry was a stub however, simply

[1]Research partially supported by an NSA Young Investigator grant
[2]Research partially supported by a DePaul University Faculty Summer Research Grant

defined as the row sums of the following table, given in entry A164341:

$n\backslash m$	1	2	3	$\cdots$											
1	1														
2	2	2													
3	3	2	4												
4	4	3	6	4	10										
5	5	4	6	6	6	8	26								
6	6	5	8	12	8	6	20	12	12	20	76				
7	7	6	10	12	10	8	12	18	16	12	20	30	24	52	232

(2)

But this was fantastic news, for our sequence could be refined in exactly the same fashion!

We had generated the terms in (1) by computing the sums of the absolute values of the irreducible characters of the symmetric group S_n, more precisely:

$$\sum_{\lambda,\mu\vdash n} |\chi_\mu^\lambda|,$$

and the entries in array (2) by fixing μ and considering

$$\sum_{\lambda\vdash n} |\chi_\mu^\lambda|.$$

On the other hand, A164341 said entry (n,m) "counts the decompositions into involutions of a permutation that has a cycle structure given by the mth partition of n."

Could this mean there might be a way to compute irreducible characters of S_n by counting involution products in the right way? Well, no. Our refinement of the sequence in (1) and the array in A164341 agreed for $n \leq 7$, but diverged at $n = 8$. Here is the row for $n = 8$ in our table, sitting atop the row for $n = 8$ in A164341.

$\sum_{\lambda\vdash 8} \lvert\chi_\mu^\lambda\rvert$	8	7	12	15	20	12	10	12	**20**	24	20	**12**	24	18	76
A164341	8	7	12	15	20	12	10	12	**24**	24	20	**16**	24	18	76

$\cdots$							
	36	24	**36**	78	**52**	**148**	764
	40	24	**40**	78	**60**	**152**	764

(The differences are highlighted in boldface.) And here are the next few terms in sequence (1), with the next few terms of A164342 below.

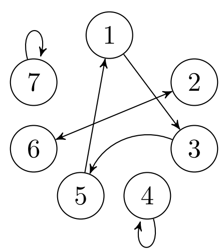
n	8	9	10	11	12	13
$\sum_{\lambda,\mu\vdash n} \lvert\chi_\mu^\lambda\rvert$	1428	4292	14456	50040	186525	724023
A164342	1456	4368	14720	50800	190149	735451
(overcount)	28	76	264	760	3624	11428

In this note, we study the ways in which a permutation can be expressed as a product of two involutions. This puts entries A164341 and A164342 of [3] on solid footing, as there seem to be no references to this question in the literature. Because the motivation for this problem is the study of irreducible characters of the symmetric group, we try to explain what we can of the connection between the two topics, using ideas of White [8] and Stanton and White [7].

Section 2 investigates involution products from a purely combinatorial point of view. Section 3 discusses the relation to irreducible characters.

2. Products of Involutions in S_n

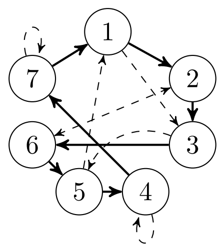
For a permutation $\sigma \in S_n$, we typically use cycle notation in this work. For example, $\sigma = (1236547)$ is the map $1 \mapsto 2 \mapsto 3 \mapsto 6 \mapsto 5 \mapsto 4 \mapsto 7 \mapsto 1$, while $\tau = (135)(26)(4)(7)$ maps $1 \mapsto 3 \mapsto 5 \mapsto 1$, $2 \mapsto 6 \mapsto 2$, and fixes 4 and 7. It is useful to draw our permutations as digraphs on the set $[n] := \{1, 2, \ldots, n\}$, in order to visualize the cycle structure. For example,

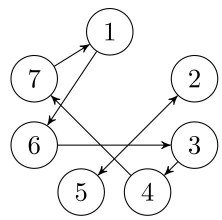
$\tau =$ 

The cycle structure of a permutation is encoded by a *partition*. A partition $\lambda \vdash n$ is a collection of positive integers whose sum is n. We usually order the parts of $\lambda = (\lambda_1, \lambda_2, \ldots)$ in nonincreasing order; that is, $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum \lambda_i = n$. An alternate notation is to let j_i denote the number of parts of size i in λ, and to write $\lambda = 1^{j_1} 2^{j_2} \cdots$, often suppressing any $j_i = 1$. For example, τ above has two one-cycles, one two-cycle, and one three-cycle; we encode this information with the partition $\lambda = (3, 2, 1, 1) = 1^2 2^1 3^1 = 1^2 23$.

The product of two permutations can be found by superimposing two digraphs of the form described above. For example, if we wish to compute the product of σ and τ from above, we draw the edges of σ and τ distinctly, say with solid and dashed edges, respectively. Then $\sigma\tau(i) = j$ if there is a directed path from i to $\tau(i)$

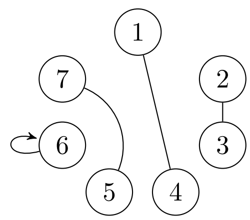
along a dashed edge, and from $\tau(i)$ to j along a solid edge. For example,

$\sigma\tau =$ 

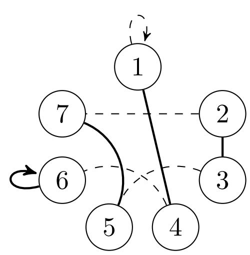
$=$ 

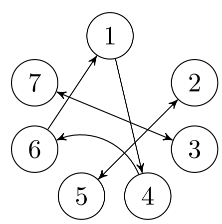
$= (16347)(25)$

We now consider the special case of *involutions*; that is, permutations whose squares are the identity. In an involution, each cycle must have size one or two, meaning that we can represent such elements graphically by *partial matchings*. For example,

$\iota = (14)(23)(57)(6) =$ 

The product of two involutions in S_n can then be expressed as a decorated graph on $[n]$ (with loops) for which every vertex has precisely one edge of each decoration, counting a loop as a single edge. We call such a graph an *involution product graph*. For example, with the involution ι as above, and $\kappa = (1)(27)(35)(46)$, we have

$\iota\kappa =$ 

$=$ 

.

We see from this example that the set of involutions is not closed under multiplication. Indeed, it is well-known that any permutation can be written as a product of involutions (see [5, Exercise 10.1.17]), often in many different ways. Our goal is to describe, and to count, all the ways in which this can be done.

If an involution product graph defines a permutation ρ, we say it is an *involution graph for ρ*. In what follows, we identify the set of pairs of involutions whose product is ρ with the set of involution graphs for ρ. Let $N(\rho)$ denote the number of ways $\rho \in S_n$ can be written as a product of involutions. To be clear,

$$N(\rho) = |\{(\sigma,\tau) \in S_n^2 : \sigma^2 = \tau^2 = 1, \sigma\tau = \rho\}| = |\{\text{involution graphs for } \rho\}|.$$

Example 2.1. $N((123)(456)) = 12$, as depicted in Figure 1.

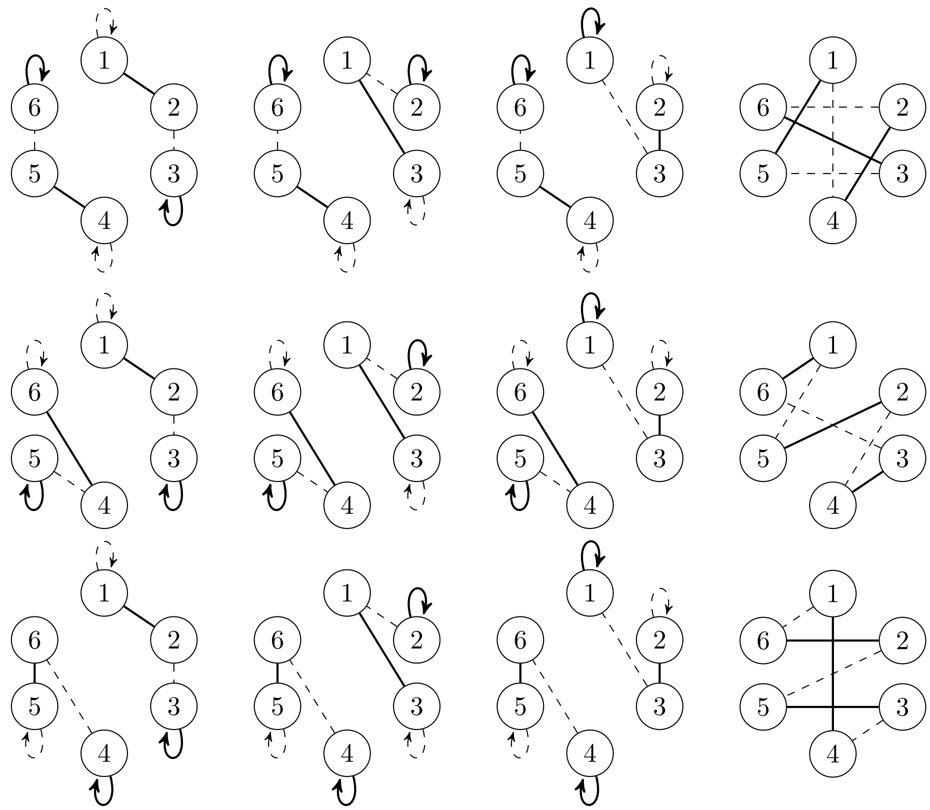


Figure 1: The involution product graphs depicting the twelve ways to write the permutation (123)(456) as the product of two involutions, $\sigma\tau$, where σ is denoted with solid marks and τ with dashed marks.

Suppose $\rho' = g\rho g^{-1}$ for some $g \in S_n$. Let $\rho = \sigma\tau$ for some involutions σ and τ. Then $\sigma' = g\sigma g^{-1}$ and $\tau' = g\tau g^{-1}$ are also involutions, and we have $\rho' = \sigma'\tau'$. Hence, $N(\rho) = N(\rho')$, and we see that the number of ways to write an element of S_n as a product of involutions depends only on its conjugacy class (we could make the same observation in any finite group). Partitioning S_n by conjugacy class is equivalent to partitioning it by cycle type. Thus, for any partition $\lambda \vdash n$, we define $N(\lambda)$ to be the number of ways any particular permutation with cycle type λ can be written as a product of involutions. Carter [1, Theorem C] shows that any element in a finite Weyl group can be written as a product of two involutions, so one could study the same question for conjugacy classes of Weyl groups.

Define

$$R_m(k) = \sum_{0\le i\le m/2} \frac{k^{m-i}}{i!}\binom{m}{\underbrace{2,2,\dots,2}_{i}} = \sum_{0\le i\le m/2} \frac{k^{m-i}m!}{2^i i!(m-2i)!}. \tag{3}$$

We will prove the following formula for $N(\lambda)$.

Theorem 2.2. *Let $\lambda = 1^{j_1}2^{j_2}\cdots$. Then*

$$N(\lambda) = \prod_{i=1}^{n} R_{j_i}(i). \tag{4}$$

Before proving the theorem, we need some preliminary results.

Lemma 2.3. *Suppose an involution product graph for ρ has an alternating path of the form:*

$$1 \dashrightarrow v_1 \to 2 \dashrightarrow v_2 \to 3 \dashrightarrow v_3 \to \cdots \to k \dashrightarrow v_k \to 1,$$

where the first, third, fifth, ... edges in this path are dashed and the second, fourth, sixth, ... edges are solid. Then both

$$(12\cdots k) \quad \textit{and} \quad (v_1\, v_k \cdots v_2)$$

are (possibly identical) k-cycles in ρ.

Proof. Fix an involution product graph of ρ with the alternating path described. Then obviously $(12\cdots k)$ is a k-cycle in ρ. Consider now $(v_1\, v_k \cdots v_2)$.

Since the involution graph has precisely one edge of each decoration at every vertex, we see that $v_i \ne v_j$ if $i \ne j$. That is, the path cannot have both $i \dashrightarrow v_i \to (i+1)$ and $j \dashrightarrow v_i \to (j+1)$ if $i \ne j$, where the first edges in each of these excerpts is dashed and the second is solid. Hence, $|\{v_1,\dots,v_k\}| = k$, and because all arrows in the product graph are reversible, we get another alternating path that shows $(v_1 v_k v_{k-1}\cdots v_2)$ is a k-cycle in ρ:

$$v_1 \dashrightarrow 1 \to v_k \dashrightarrow k \to v_{k-1} \dashrightarrow (k-1) \to \cdots \to v_2 \dashrightarrow 2 \to v_1.$$

$\square$

If differently decorated cycles in an involution product graph form an alternating path as described in Lemma 2.3, then those cycles are *interlaced*. If a cycle does not interlace with any other, then that cycle is *isolated*.

A first consequence of Lemma 2.3 is the following.

Corollary 2.4. *The connected components of an involution product graph of a permutation ρ each either describe a cycle in ρ or two disjoint cycles of the same size. In particular,*

1. *three or more cycles cannot be interlaced in an involution product graph, and*
2. *if $k \neq k'$, then a k-cycle and a k'-cycle cannot be interlaced in an involution product graph.*

The next consequence is the key to our enumeration of involution product graphs.

Corollary 2.5. *For all $k \geq 3$,*

1. *There are precisely k ways to write a k-cycle as a product of involutions.*
2. *There are precisely k ways to write two disjoint k-cycles as a product of involutions, given that their elements are in the same connected component of the involution graph.*

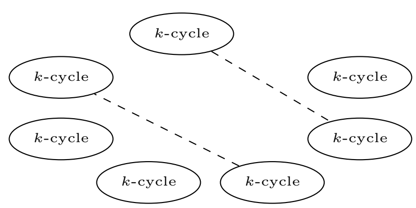
Proof. If the cycles from Lemma 2.3 are fixed, then the designation of the label v_1 uniquely determines all the edges of this connected component of the involution graph. □

Proposition 2.6. *Suppose $\lambda = k^m = (k, k, \ldots, k) \vdash km$. Then,*

$$N(k^m) = R_m(k),$$

where $R_m(k)$ is defined in Equation (3).

Proof. Let ρ be a fixed permutation of cycle type λ. By Corollary 2.4, an involution graph for ρ either has all the k-cycles in disjoint connected components, or some pairs of the cycles are connected. For example, with $m = 7$ k-cycles, perhaps two pairs of them are interlaced, while the other three are isolated, as in the following picture.



There are precisely

$$\frac{1}{i!}\binom{m}{2, 2, \ldots, 2, m-2i} = \frac{m!}{2^i i!(m-2i)!}$$

ways to choose a matching of i pairs of the k-cycles, leaving $m - 2i$ of the k-cycles isolated. Given such a matching, Corollary 2.5 tells us there are k ways to draw each connected component, giving $k^i \cdot k^{m-2i} = k^{m-i}$ choices. Hence, the number of involution product graphs for ρ that have i pairs of k-cycles is:

$$\frac{k^{m-i} m!}{2^i i!(m-2i)!}.$$

To count all possible involution graphs for ρ, we sum over all i to obtain the desired result:

$$N(\rho) = \sum_{0 \le i \le m/2} \frac{k^{m-i} m!}{2^i i!(m-2i)!} = R_m(k).$$

□

We are now able to prove Theorem 2.2.

Proof of Theorem 2.2. Let ρ be a permutation of type $\lambda = 1^{j_1} 2^{j_2} \cdots$. By Corollary 2.4, two cycles of ρ are in different connected components if they are of different size. Hence, for $k \neq k'$, the k-cycles and k'-cycles are mutually independent, and to count the number of involution product graphs for ρ, we simply multiply the number of possibilities for each cycle size:

$$N(\lambda) = N(1^{j_1}) \cdot N(2^{j_2}) \cdots = \prod_{j=1}^{n} R_{j_i}(i).$$

□

3. Connections with the Character Table of S_n

Now that we have enumerated the number of pairs of involutions with a fixed product, we will describe why understanding such involution products may be related to characters of the symmetric group. Our tone is primarily expository here, built upon substantial work of White [8] and Stanton and White [7]. (A more recent approach to many of the same topics can be found in [2].) Omitted details may be found in those papers.

To begin, we recall that the well-known Murnaghan-Nakayama rule provides the following formula for an irreducible character of the symmetric group S_n:

$$\chi^{\lambda}_{\mu} = \sum_{T} (-1)^{\mathrm{ht}(T)},$$

where the sum is taken over all *rim-hook tableaux* T of shape λ and content μ, and $\mathrm{ht}(T)$ is the sum of the heights of the hooks in T minus the number of hooks. In general, $(-1)^{\mathrm{ht}(T)}$ gives the *sign* of a tableau T. For example,

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3 \\ \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 6 \\ \cline{1-4} 2 & 4 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$$

is a rim-hook tableau with content $\mu = (5,4,4,3,1,1)$, its height is calculated by $\mathrm{ht}(T) = 2+3+2+2+1+1-6 = 5$, and $(-1)^{\mathrm{ht}(T)} < 0$ so T is a negative tableau. See either [4, Sec. 4.10] or [6, Sec. 7.17] for precise definitions and further discussion.

This formula is lovely in that it gives an explicit combinatorial description for the characters. However, it involves tremendous cancellation. For instance, consider the character table for S_6 in Figure 2. There are 40 rim-hook tableau of content $(2,2,1,1)$, yet in the column indexed by $\mu = (2,2,1,1)$ we see that only 12 terms are non-cancelling; that is, the sum of the absolute values of the entries in this column is $1+1+1+2+1+2+1+1+1+1 = 12$.

$\lambda \backslash \mu$	6	51	42	411	33	321	3111	222	2211	21111	111111
6	1	1	1	1	1	1	1	1	1	1	1
51	-1	0	-1	1	-1	0	2	-1	1	3	5
42	0	-1	1	-1	0	0	0	3	1	3	9
411	1	0	0	0	1	-1	1	-2	-2	2	10
33	0	0	-1	-1	2	1	-1	-3	1	1	5
321	0	1	0	0	-2	0	-2	0	0	0	16
3111	-1	0	0	0	1	1	1	2	-2	-2	10
222	0	0	-1	1	2	-1	-1	3	1	-1	5
2211	0	-1	1	1	0	0	0	-3	1	-3	9
21111	1	0	-1	-1	-1	0	2	1	1	-3	5
111111	-1	1	1	-1	1	-1	1	-1	1	-1	1
$N(\mu)$	6	5	8	8	12	6	12	20	12	20	76

Figure 2: The character table of S_6, along with $N(\mu)$, the number of ways to write a permutation of cycle type μ as a product of involutions.

On the other hand, there are precisely 12 rim hook tableaux of content (3,3), and so there is no cancellation in this column. As we will see shortly, it follows from work of White [8] that, when all parts of μ are the same size, the Murnaghan-Nakayama formula for χ^λ_μ is *cancellation-free* (Corollary 3.6). Such a partition μ is called *rectangular*. Our feeling is that understanding involution products may lead to a similar cancellation-free formula for any content μ.

The idea begins with the following observation. For any λ, the Murnaghan-Nakayama rule gives

$$\chi^\lambda_{1^n} = |\{\text{standard Young tableaux of shape } \lambda\}|.$$

On the other hand, the Schensted insertion algorithm gives a bijection between the set of standard Young tableaux with n boxes and the set of involutions in S_n. If we let $I^\lambda = \{\sigma \in S_n : \sigma^2 = 1, \mathrm{sh}(\sigma) = \lambda\}$, where $\mathrm{sh}(\sigma)$ denotes the shape of the image of σ upon the Schensted insertion, then

$$\chi^\lambda_{1^n} = |I^\lambda|.$$

To generalize this idea to $\mu \neq 1^n$, we first need to discuss a more general insertion algorithm.

In [8], White defines a map Sch that generalizes the usual Schensted insertion algorithm. Given a pair of rim hook tableaux (P, Q) of the same shape and content, we say the pair is *positive* if P and Q have the same sign; the pair is *negative* otherwise. The map Sch provides a bijection that maps a positive pair (P, Q) of content μ to either a negative pair (P', Q') of tableaux of content μ or to a "hook permutation" of content μ.

Definition 3.1. Fix a positive integer m. A *hook permutation* $(H_1, H_2, \dots, H_m)$ of *type* $\rho = (\rho_1, \rho_2, \dots, \rho_m) = 1^{j_1} 2^{j_2} \cdots k^{j_k}$ and shapes $(\tau^{(1)}, \tau^{(2)}, \dots, \tau^{(m)})$ has the properties

- the H_i are each hook tableaux of shape $\tau^{(i)}$,
- $\rho_i = |\tau^{(i)}|$, and
- if $\rho_i > \rho_j$, then the content of H_i is less than the content of H_j.

Using the notation of Definition 3.1, define the intervals $I_1 = [1, j_k]$, $I_2 = [j_k + 1, j_k + j_{k-1}]$, $I_3 = [j_k + j_{k-1} + 1, j_k + j_{k-1} + j_{k-2}]$, and so on. Then a hook permutation maps each I_ℓ bijectively onto itself, and the image of $i \in I_\ell$ is assigned a hook consisting of $k + 1 - \ell$ boxes, whose shape is $\tau^{(i)}$. We can think of this as a sequence of m hooks, where the lengths of the hooks is nonincreasing, and the entries of the k-hooks are I_1, the entries of the $(k-1)$-hooks are I_2, and so on.

Example 3.2. The following is an example of a hook permutation of type $\rho = 1^3 3 4^3 5$.

$$\mathcal{H} = \left(\begin{array}{cccccccc} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1\\\cline{1-1} \end{array} & \begin{array}{|c|}\hline 4\\\hline 4\\\hline 4\\\hline 4\\\hline \end{array} & \begin{array}{|c|c|}\hline 2&2\\\hline 2\\\cline{1-1} 2\\\cline{1-1} \end{array} & \begin{array}{|c|}\hline 3\\\hline 3\\\hline 3\\\hline 3\\\hline \end{array} & \begin{array}{|c|c|}\hline 5&5\\\hline 5\\\cline{1-1} \end{array} & \begin{array}{|c|}\hline 8\\\hline \end{array} & \begin{array}{|c|}\hline 7\\\hline \end{array} & \begin{array}{|c|}\hline 6\\\hline \end{array} \end{array} \right)$$

It is easy to see that the number of hook permutations of type $(\rho_1, \rho_2, \dots, \rho_m) = 1^{j_1} 2^{j_2} \cdots k^{j_k}$ is $1^{j_1} j_1! 2^{j_2} j_2! \cdots k^{j_k} j_k!$, since there are j_i different hooks of length i.

We will choose to think of Sch as a map from the set of hook permutations of a given content to the non-cancelling positive pairs of rim-hook tableaux (P, Q) having that same content. The algorithm involved in which turns a hook permutation $(H_1, \dots, H_m)$ into a pair of rim hook tableaux (P, Q) builds the tableaux by successively inserting the hooks $H_1, \dots, H_m$. The first tableau is the insertion tableau, and the second is the recording tableau. We briefly describe the process here, and the full details can be found in [8].

The first hook H_1 is inserted into a pair of empty tableaux (P_0, Q_0), yielding (P_1, Q_1). Subsequently, the hook H_i is inserted into (P_{i-1}, Q_{i-1}) to form (P_i, Q_i).

When inserting the hook H_i with content a and size ℓ, the tableau P_{i-1} is divided into an *inside part* consisting of all entries less than a, and an *outside part* consisting of all entries greater than a.

The hook H_i is first placed so that its upper left corner coincides with the upper left corner of the inside part of P_{i-1}, and it slides (changing shape, in the process) down the first column of P_{i-1}, ℓ units at a time, until it is completely beyond the inside part of the tableau. It then slithers up the outer rim of the inside part of P_{i-1}, ℓ units at a time, until it reaches a position so that the union of this rim hook and the inside part is a partition. This new rim hook is now considered part of the inside part of the tableau.

Now the rim hooks in the outside part of P_{i-1} successively move into the inside part, in increasing order of content. If the rim hook being added does not overlap the inside part at all, then it joins the inside part without changing position. If it overlaps the inside part but does not lie completely inside the inside part, then it joins the inside part after the overlapping entries each move one step southeast, and the non-overlapping entries do not move. Finally, if the rim hook being added, of length ℓ', overlaps the inside part completely, then it slithers up the outer rim of the inside part, ℓ' units at a time, until it reaches a position for which the union of this rim hook and the inside part is a partition. Once the entire outside part has been moved to the inside, the result is (P_i, Q_i), where Q_i records the newly added boxes. Finally, $(P, Q) = (P_m, Q_m)$.

Example 3.3. Given the rim hook permutation $\mathcal{H}$ of Example 3.2, the insertion algorithm produces the following sequence of tableaux:

$$P_1 = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & & & \end{array}, \quad P_2 = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 4 & & \\ 4 & 4 & & \\ 4 & & & \end{array}, \quad P_3 = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & \\ 2 & 2 & 4 & \\ 4 & 4 & 4 & \end{array}, \quad P_4 = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 4 & \\ 3 & 3 & 3 & & \end{array},$$

$$P_5 = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 4 & \\ 3 & 3 & 3 & & \\ 5 & 5 & 5 & & \end{array}, \quad P_6 = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 4 & \\ 3 & 3 & 3 & & \\ 5 & 5 & 5 & & \\ 8 & & & & \end{array}, \quad P_7 = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 4 & \\ 3 & 3 & 3 & & \\ 5 & 5 & 5 & & \\ 7 & 8 & & & \end{array},$$

$$P_8 = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 4 & 4 \\ 2 & 2 & 3 & 4 & \\ 3 & 3 & 3 & & \\ 5 & 5 & 5 & & \\ 6 & 7 & 8 & & \end{array}$$

and thus the algorithm Sch produces the following pair of tableaux:

$$\mathsf{Sch}(\mathcal{H}) = \left(\begin{array}{|c|c|c|c|c|} \hline 1&1&1&1&4 \\ \hline 1&2&2&4&4 \\ \hline 2&2&3&4 \\ \cline{1-4} 3&3&3 \\ \cline{1-3} 5&5&5 \\ \cline{1-3} 6&7&8 \\ \cline{1-3} \end{array} \; , \; \begin{array}{|c|c|c|c|c|} \hline 1&1&1&1&4 \\ \hline 1&2&3&4&4 \\ \hline 2&2&3&4 \\ \cline{1-4} 2&3&3 \\ \cline{1-3} 5&5&5 \\ \cline{1-3} 6&7&8 \\ \cline{1-3} \end{array} \right)$$

The following facts describe the aspects of the Sch construction that are most relevant to our work here.

Proposition 3.4. *Fix a positive integer m and a composition $\rho = (\rho_1, \dots, \rho_m) = 1^{j_1} \cdots k^{j_k}$. Let $\mathcal{H} = (H_1, \dots, H_m)$ be a hook permutation of type ρ.*

- *(a) If the content of all the hooks $\{H_1, \dots, H_i\}$ is less than the content of all the hooks $\{H_{i+1}, \dots, H_m\}$, then the tableaux built from the hook permutation $(H_1, \dots, H_i)$ are stable for the duration of the procedure. That is, if $\mathsf{Sch}(H_1, \dots, H_i) = (P', Q')$, and $\mathsf{Sch}(H_1, \dots, H_m) = (P, Q)$, then the rim hook tableau P' is a subtableau of the rim hook tableau P, and similarly Q' is a subtableau of Q.*

- *(b) In Sch, the shape being constructed propagates down and to the right, never upward; that is, once the top row of the shape has been established, it is maintained as the top row for the duration of the procedure.*

White [8] shows many of the properties of Sch are particularly nice in the special case of rectangular content; that is, when all rim hooks have the same length. In [7], Stanton and White consider this case in further detail.

Proposition 3.5 ([8], Corollaries 9 and 10)**.** *If $\mu = k^m$, then the inverse procedure Sch^{-1} is a bijection from hook permutations of content μ to the set of all pairs (P, Q) of rim hook tableaux of the same shape and content μ. In particular, every pair of tableaux (P, Q) is positive.*

The following is immediate.

Corollary 3.6. *Any two rim hook tableaux of the same shape and the same rectangular content have the same sign. Hence,*

$$|\chi^{\lambda}_{k^m}| = |\{P : P \text{ has shape } \lambda \text{ and content } k^m\}|.$$

In other words, there is no cancellation in the Murnaghan-Nakayama formula for χ^{λ}_{μ} when μ is rectangular.

We now show how this result is related to involution products.

Definition 3.7. A hook permutation $\mathcal{H}$ of type ρ is a *hook-block involution* if

- the underlying permutation is an involution, and
- the hooks associated to the letters in each cycle of the involution are the same.

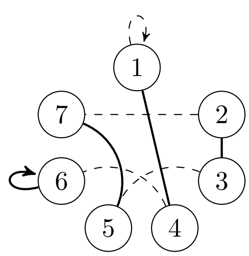
Example 3.8. The hook permutation

$$\left(\begin{ytableau} 1 & 1 & 1 \\ 1 \\ 1 \end{ytableau} \quad \begin{ytableau} 4 \\ 4 \\ 4 \end{ytableau} \quad \begin{ytableau} 3 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 \\ 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 5 \\ 5 \end{ytableau} \quad \begin{ytableau} 7 & 7 \end{ytableau} \quad \begin{ytableau} 6 & 6 \end{ytableau} \quad \begin{ytableau} 8 \end{ytableau}\right)$$

is a hook-block involution. In this case, the underlying permutation is the involution $(1)(24)(3)(5)(67)(8)$, and the shape equalities are $\tau^{(2)} = \tau^{(4)}$ and $\tau^{(6)} = \tau^{(7)}$.

To any pair of involutions we may assign a hook-block involution as follows. We look at all of the k-cycles in the involution product graph, in decreasing order of k. Suppose that we have examined the k'-cycles for all $k' > k$, and so far created a hook-block involution on the letters $\{1, \ldots, M\}$. Now suppose that there are r k-cycles in the involution product graph. Label these $\{M+1, \ldots, M+r\}$ in order of the smallest element appearing in each. We thus inherit an involution of $\{M+1, \ldots, M+r\}$, based on which k-cycles are interlaced. To each letter in this involution, we associate a hook, where the choice of v_1 in Lemma 2.3 determines the height of the first column in each hook of the 2-cycle: if v_1 is in the ith position when the cycle is written with the smallest letter in the first position, then the height of that first column in the hook is i. Note that any pair of letters that are each other's images under the involution get assigned the same hook shape, by Lemma 2.3. We demonstrate with an example.

Example 3.9.



$$\longleftrightarrow \left(\begin{ytableau} 1 & 1 & 1 \end{ytableau} \quad \begin{ytableau} 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 \\ 2 \end{ytableau}\right)$$

First of all, the product is $(146)(25)(37)$, which has cycle type $(3, 2, 2)$. Hence we know the hook-block involution will have a 3-hook and two 2-hooks. Since 1 is the smallest element of the 3-cycle, we see where its dashed arrow points to determine the height of the 3-hook. It points to itself, so we choose the hook of height 1. (If the 1 had had a dashed arrow to the 4, then the 3-hook would have had height

2. If it had pointed to the 6 it would have had height 3.) There are two 2-cycles, and they lie in the same connected component of the matching graph. Hence, the corresponding 2-hooks are transposed in the hook permutation. The shape of the hooks is determined by where the 2 is connected with its dashed edge. If it had been connected to the 3, then the 2-hooks would be horizontal. However, the dashed edge connects to the 7 (which is the second-largest element in its cycle), and so the hooks have height 2.

A hook-block involution in which all hooks are the same size k is a "k-partial involution" [7].

Example 3.10. 1. The hook permutation

$$\begin{pmatrix} \ytableaushort{3,3,3} & \ytableaushort{22,2} & \ytableaushort{1,1,1} & \ytableaushort{55,5} & \ytableaushort{44,4} & \ytableaushort{666} \end{pmatrix}$$

is a 3-partial involution because the underlying permutation is $(13)(2)(45)(6)$, which is an involution, and it has the shape equalities $\tau^{(1)} = \tau^{(3)}$ and $\tau^{(4)} = \tau^{(5)}$.

2. The hook permutation

$$\begin{pmatrix} \ytableaushort{3,3,3} & \ytableaushort{55,5} & \ytableaushort{1,1,1} & \ytableaushort{22,2} & \ytableaushort{44,4} & \ytableaushort{666} \end{pmatrix}$$

is not a 3-partial involution because the underlying permutation $(13)(254)(6)$ is not an involution.

3. The hook permutation

$$\begin{pmatrix} \ytableaushort{3,3,3} & \ytableaushort{22,2} & \ytableaushort{1,1,1} & \ytableaushort{55,5} & \ytableaushort{4,4,4} & \ytableaushort{666} \end{pmatrix}$$

is not a 3-partial involution because, although the underlying permutation is the involution $(13)(2)(45)(6)$, this hook permutation has the shape inequality $\tau^{(4)} \neq \tau^{(5)}$.

Indeed, k-partial involutions have nice properties.

Theorem 3.11 ([7])**.** *A hook permutation $\mathcal{H}$ of type $\mu = k^m$ is a k-partial involution if and only if $\mathsf{Sch}(\mathcal{H}) = (P, P)$. That is, we have a bijection,*

$$\{ \text{ k-partial involutions } \} \overset{\mathsf{Sch}}{\longleftrightarrow} \{ \text{ rim-hook tableaux P of content k^m } \}.$$

Now let

$$I_k^\lambda = \{\mathcal{H} : \mathcal{H} \text{ is a } k\text{-partial involution and } \mathsf{Sch}(\mathcal{H}) \text{ has shape } \lambda\}.$$

Then, together with Corollary 3.6, we obtain the following.

Corollary 3.12. *Let k and m be positive integers. Then for all $\lambda \vdash n$,*

$$|\chi^\lambda_{k^m}| = |I_k^\lambda|.$$

Because $N(\mu)$ counts all hook-block involutions of content μ, Theorem 2.2 implies the following result.

Corollary 3.13. *Let k and m be positive integers. Then,*

$$\sum_{\lambda \vdash km} |\chi^\lambda_{k^m}| = N(k^m) = \sum_{0 \le i \le m/2} \frac{k^{m-i} m!}{2^i i! (m-2i)!}.$$

This leads one to wonder whether something similar is true for non-rectangular μ.

Question 1. *For which $\mu \vdash n$ is it true that*

$$\sum_{\lambda \vdash n} |\chi^\lambda_\mu| = N(\mu)? \tag{5}$$

Note that the sums of the characters themselves, not the absolute values of the characters, have been studied in other contexts, and have been shown in [2] to equal the number of certain hook-block involutions. Not taking the absolute values of characters allows the possibility of canceling terms, a possibility we are interested in avoiding.

Corollary 3.12 shows that Equation (5) holds for $\mu = k^m$ and by computer we have checked that it holds for all $\mu \vdash n$ when $n \le 7$. However, when $n = 8$, we begin to see some discrepancies. In each of these cases $N(\mu)$ is greater than $\sum_{\lambda \vdash n} |\chi^\lambda_\mu|$. See Table 1, where for $n \le 10$ we have listed all the shapes $\mu \vdash n$ for which Equation (5) does not hold.

If we check all shapes $\mu \vdash n$ with $n \le 16$, we find that there are indeed a number of cases in which Equation (5) holds. We summarize the number of agreements ($N(\mu) = \sum_{\lambda \vdash n} |\chi^\lambda_\mu|$) and discrepancies ($N(\mu) > \sum_{\lambda \vdash n} |\chi^\lambda_\mu|$) in Table 2. We have also included the number of rectangular partitions $\mu = k^m$ since in these cases we know that equality holds. For example, when $n = 16$ there are 64 shapes $\mu \vdash 16$ for which $N(\mu) = \sum_{\lambda \vdash n} |\chi^\lambda_\mu|$, yet we can only explain 5 of these: $1^{16}, 2^8, 4^4, 8^2$, and 16^1.

If we are to move beyond the rectangular case, we need to understand how to associate involution products, or hook-block involutions, to a collection of rim-hook tableaux. Theorem 3.11 does this in the case when μ is rectangular. In the more

μ	$N(\mu) - \sum_{\lambda\vdash n} \lvert\chi_\mu^\lambda\rvert$
$2^2 4$	4
$1^2 24$	4
$1^4 4$	4
$1^2 2^3$	4
$1^4 2^2$	8
$1^6 2$	4
234	4
$1^2 34$	4
$12^2 4$	4
$1^3 24$	4
$1^5 4$	4
$1^3 3^2$	8
$2^3 3$	4
$1^2 2^2 3$	4
$1^4 23$	4
$1^6 3$	20
$1^3 2^3$	4
$1^5 2^2$	8
$1^7 2$	4

μ	$N(\mu) - \sum_{\lambda\vdash n} \lvert\chi_\mu^\lambda\rvert$
235	4
$1^2 35$	4
$1^3 25$	4
$1^5 5$	8
$2^3 4$	8
$1^2 2^2 4$	8
$1^4 24$	8
$1^6 4$	8
$2^2 3^2$	8
$1^4 3^2$	8
$1^3 2^2 3$	8
$1^5 23$	16
$1^7 3$	56
$1^2 2^4$	8
$1^4 2^3$	28
$1^6 2^2$	48
$1^8 2$	32

Table 1: When Equation (5) does not hold.

n	≤ 7	8	9	10	11	12	13	14	15	16
Discrepancies:	0	6	13	17	24	46	60	83	114	167
Agreements:	all	16	17	25	32	31	41	52	62	64
Known agreements ($\mu = k^m$):	-	4	3	4	2	6	2	4	4	5
Partitions of n:	-	22	30	42	56	77	101	135	176	231

Table 2: Counting the number of shapes μ for which $N(\mu) = \sum_{\lambda\vdash n} \lvert\chi_\mu^\lambda\rvert$.

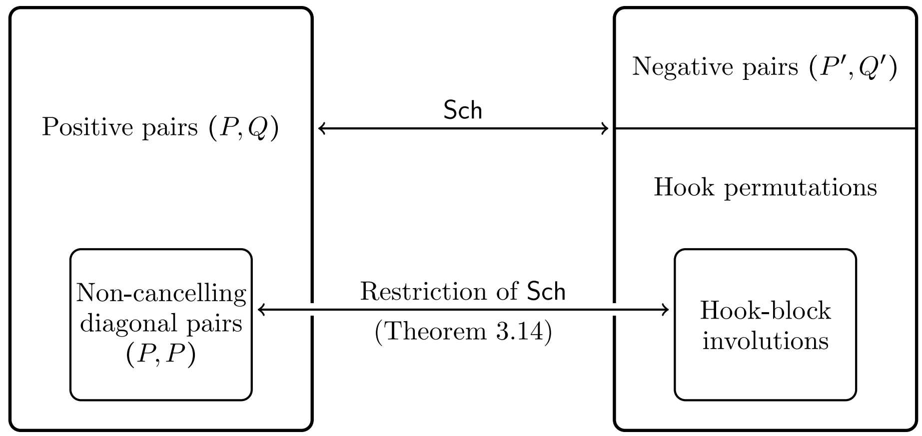


Figure 3: Schematic diagram for the Sch bijection with fixed shape λ and content μ.

general case, Theorem 3.14 below shows that Sch does something similar, giving a bijection between hook-block involutions and the set of tableaux P for which (P, P) is a non-cancelling pair. See Figure 3 for an illustration of these results and their scopes.

Theorem 3.14. *A hook permutation $\mathcal{H}$ of content μ is a hook-block involution if and only if $\mathsf{Sch}(\mathcal{H}) = (P, P)$. That is, the non-cancelling pairs (P, P) of content μ are in bijection with the set of hook-block involutions.*

Proof of Theorem 3.14. Fix an integer $m > 0$ and a composition $\mu = (\mu_1, \ldots, \mu_m) = 1^{j_1} \cdots k^{j_k}$. Let $\mathcal{H} = (H_1, \ldots, H_m)$ be a hook permutation of type μ.

Suppose that $\mathsf{Sch}(\mathcal{H}) = (P, P)$. By Theorem 3.11 and Proposition 3.4(a), the initial j_k letters and hooks in $\mathcal{H}$ must form a k-partial involution. Inductively, suppose now that all hooks $\{H_1, \ldots, H_r\}$ of length greater than ℓ have been inserted, forming the pair (P', P') of shape ν. Note that the entries of P' are all less than or equal to r. Let $\pi = \nu \cup \alpha$ be the shape obtained by appending ℓ sufficiently long rows, each of length at least $|\nu| + \ell j_\ell$, on top of ν, and such that $|\pi| = \ell t$ for some t.

Let $\widehat{P}$ be a rim hook tableau of shape π, and content ℓ^t, whose entries are $\{r-t+1, r-t+2, \ldots, r\}$ (allowing entries to be non-positive). By Proposition 3.5 and Theorem 3.11, $\mathsf{Sch}^{-1}(\widehat{P}, \widehat{P})$ equals some hook permutation $\widehat{\mathcal{H}} = (\widehat{H}_{r-t+1}, \widehat{H}_{r-t+2}, \ldots, \widehat{H}_r)$ which is in fact an ℓ-partial involution. (We are allowing permutations of sets that include non-positive values due to the content of $\widehat{P}$.)

Now define the hook permutation

$$\mathcal{H}^+ = (\widehat{H}_{r-t+1}, \widehat{H}_{r-t+2}, \ldots, \widehat{H}_r, H_{r+1}, H_{r+2}, \ldots, H_{r+j_\ell}),$$

where $\{H_{r+1},\dots,H_{r+j_\ell}\}$ are the hooks of size ℓ in $\mathcal{H}$. The point of Proposition 3.4(b) is that adding ℓ rows on top of the shape ν to form π does not affect the insertion of the ℓ-hooks $\{H_{r+1},\dots,H_{r+j_\ell}\}$ via Sch, except to allow us to create a shape with rectangular content. Because all content in the hooks $\{H_{r+j_\ell+1},\dots,H_m\}$ is less than the content of the hooks $\{H_{r+1},\dots,H_{r+j_\ell}\}$, we must have that $\mathsf{Sch}(\mathcal{H}^+) = (P^+,P^+)$ by Proposition 3.4(a). Moreover, the hooks with content $\{r+1,\dots,r+j_\ell\}$ in P^+ without α, its top ℓ rows, occupy the same positions that they do in P. Theorem 3.11 implies that $\mathcal{H}^+$ must be an ℓ-partial involution. Because $\widehat{H}$ was an ℓ-partial involution as well, this means that $(H_{r+1},H_{r+2},\dots,H_{r+j_\ell})$ itself is an ℓ-partial involution. Consequently, by the inductive hypothesis, $\mathcal{H}$ must be a hook-block involution.

Now suppose that $\mathsf{Sch}(\mathcal{H}) = (P,Q)$ where $P \neq Q$. Then the tableaux first diverge during the insertion of the hooks of length i for some i. A similar argument to the above shows that this means that the hooks of length i do not form a i-partial involution. Thus the full hook permutation $\mathcal{H}$ is not a hook-block involution. $\square$

Theorem 3.14 shows it is possible to associate a collection of $N(\mu)$ rim-hook tableaux of content μ to the set of all hook-block involutions of content μ. Unfortunately, it is not obvious how to deduce when $\sum_{\lambda\vdash n}|\chi^\lambda_\mu|$ and $N(\mu)$ coincide from this result. Moreover, as the following example shows, even when $N(\mu)$ and $\sum_{\lambda\vdash n}|\chi^\lambda_\mu|$ are equal, applying Sch to the set of hook-block involutions does not necessarily yield a result like Corollary 3.12.

Example 3.15. While $N(2211) = 12$, applying Sch to the corresponding twelve hook-block involutions yields no tableaux of shape 6 or shape 51, while $\chi^6_{2211} = \chi^{51}_{2211} = 1$. Compare Figures 4 and 5 with the column corresponding to 2211 in Figure 2.

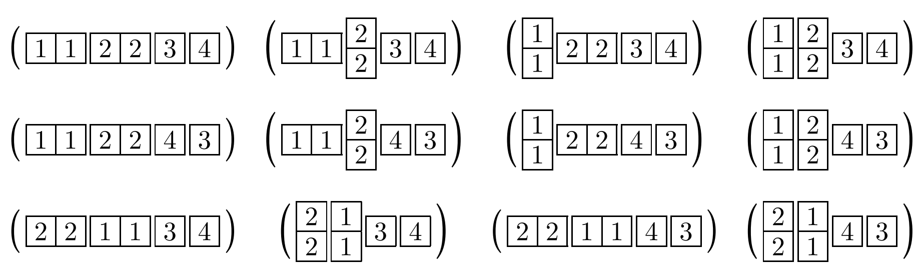


Figure 4: The hook-block involutions corresponding to the involution product graphs for $\sigma = (12)(34)(5)(6)$.

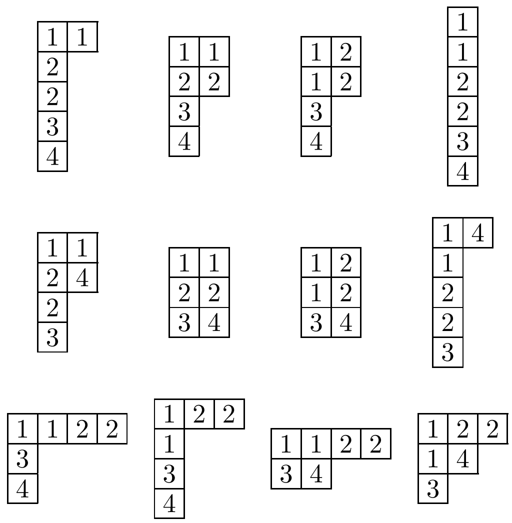


Figure 5: The image under Sch of the hook-block involutions of content $(2,2,1,1)$.

3.1. Connection with commuting pairs

As a final thought, we communicate an observation of Allan Trojan, shared with us via private communication, in response to an early draft of this paper.

For a positive integer n, let $C(n)$ denote the set of all ordered pairs of commuting elements of S_n,

$$C(n) = \{(\sigma,\tau) \in S_n^2 : \sigma\tau = \tau\sigma\}.$$

Define an equivalence relation on $C(n)$ by conjugacy acting diagonally; that is, say $(\sigma,\tau) \sim (g\sigma g^{-1}, g\tau g^{-1})$ for each $g \in S_n$. For $\mu \vdash n$, define

$$C(\mu) = \{(\sigma,\tau) \in S_n^2 : \sigma\tau = \tau\sigma \text{ and } \operatorname{sh}(\sigma) = \mu\} \subseteq C(n).$$

Let $c(\mu)$ denote the number of conjugacy classes in $C(\mu)$.

Example 3.16. There are five conjugacy classes of commuting pairs in S_4 whose first element has cycle type $(2,2)$, so $c((2,2)) = 5$. Taking $\sigma = (12)(34)$ in each class, they are:

$$(\sigma,1),(\sigma,(12)),(\sigma,(12)(34)),(\sigma,(13)(24)),(\sigma,(1324)).$$

Let $\operatorname{nz}(\mu)$ count the nonzero entries in column μ of the character table of S_n. Trojan observes that $c(\mu)$ is linked to $\operatorname{nz}(\mu)$ in much the same way that our numbers $N(\mu)$ are linked to the sum of absolute values of the entries in column μ. Namely,

- $c(\mu) = \operatorname{nz}(\mu)$ for all $\mu \vdash n$ where $n \le 7$,

- if $\mu = k^m$ is rectangular, then $c(\mu) = \text{nz}(\mu)$, and
- if $n = 8, 9$, or 10, the partitions μ for which $c(\mu)$ and $\text{nz}(\mu)$ differ are precisely those listed in Table 1.

This certainly suggestions some interesting connections to explore.

Acknowledgments The authors wish to thank Allan Berele, Ira Gessel, and John Stembridge for conversations that helped prompt us to better understand characters of S_n. Thanks to Dennis White for helping us understand Sch, and thanks to Peter Winkler for helpful discussions related to the formula for $N(\mu)$. We are also grateful to Allan Trojan for sharing his comments, and to the suggestions of an anonymous referee. Lastly, we should mention the role played by the OEIS, without which we would never have guessed at a connection between involution products and characters of the symmetric group.

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