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# ON UNIVERSAL BINARY HERMITIAN FORMS

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**Abstract**

Earnest and Khosravani, Iwabuchi, and Kim and Park recently gave a complete classification of the universal binary Hermitian forms. We give a unified proof of the universalities of these Hermitian forms, relying upon Ramanujan's list of universal quadratic forms and the Bhargava-Hanke 290-Theorem. Our methods bypass the *ad hoc* arguments required in the original classification.

## 1. Introduction

The question of representing integers by quadratic forms dates back to the time of Fermat, whose *Two Squares Theorem* solved the question of which primes could be represented by the form $x^2 + y^2$ (see [6, p. 219]). This theorem was later generalized by Lagrange, who showed in his *Four Squares Theorem* [11] that every positive integer can be written as a sum of four squares of integers.

Lagrange's theorem has led to the modern study of *universal forms*, those forms which represent all positive integers. In the first half of the twentieth century, Ramanujan [13] identified the universal positive-definite classically integral quaternary diagonal quadratic forms, up to equivalence. Maass [12] and Chan, Kim, and Raghavan [3] gave analogous classification results leading to the full classification of the positive-definite classically integral ternary quadratic forms which are universal over real quadratic fields.

Motivated by the work on universal quadratic forms over real fields, Earnest and Khosravani [5] sought a classification of universal binary Hermitian forms over imaginary quadratic fields. Recently, Iwabuchi [7] and Kim and Park [10] finished Earnest and Khosravani's program, completing the list of universal binary Hermitian forms.

A different direction of recent research has focused on the search for *universality criteria*, simple tests which characterize the universality of positive-definite quadratic forms. The earliest-discovered result in this vein is Conway and Schneeberger's surprising *15-Theorem* (see [4] for statement and history and [1] for a proof):

**15-Theorem.** *A positive-definite classically integral quadratic form is universal if and only if it represents the nine "critical numbers"*

$$\{1, 2, 3, 5, 6, 7, 10, 14, 15\}.$$

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More recently, Bhargava and Hanke [2] showed an analogous criterion for the universality of positive-definite nonclassically integral quadratic forms:

**290-Theorem.** *A positive-definite nonclassically integral quadratic form is universal if and only if it represents the numbers*

$$S_{290} = \{1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, \\ 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290\}.$$

While the criterion theorems reduce testing a form's universality to a simple computation, they have rarely been applied in practice. The reason for this somewhat curious fact is that the proofs of both the 15- and 290-Theorems rely on independent identification of many universal forms of low rank, called the *universal escalators*.

The results on Hermitian forms, however, give us a chance to greatly simplify prior work through an application of the 290-Theorem. Specifically, we apply the 290-Theorem to reduce the most difficult universality verifications in the classification of universal binary Hermitian forms to simple, finite computations.

## 2. Preliminaries

We let $E$ be an imaginary quadratic field over $\mathbb{Q}$ and let $m > 0$ be a squarefree integer for which $E = \mathbb{Q}(\sqrt{-m})$. We denote the $\mathbb{Q}$-involution of $E$ by $\bar{}$ and the ring of integers of $E$ by $\mathcal{O}_E$.

We let $V/E$ be an $n$-dimensional Hermitian space over $E$ with nondegenerate Hermitian form $H$. As shown by Jacobson [8], we may consider $(V, H)$ as a $2n$-dimensional quadratic space $(\widetilde{V}, B)$ with the bilinear form $B$ defined by the trace map

$$B(v, w) = \frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(H(v, w)).$$

An $\mathcal{O}_E$-lattice $L$ is a finitely generated $\mathcal{O}_E$-module on the Hermitian space $(V, H)$. We consider only positive-definite integral $\mathcal{O}_E$-lattices $L$, that is, those for which $H(v, w) \in \mathcal{O}_E$ for all $v, w \in L$ and $H(v, v) > 0$ for all $L \ni v \neq 0$. If an $\mathcal{O}_E$-lattice $L$ is of the form $L = L_1 \oplus L_2$ for sublattices $L_1$, $L_2$ of $L$ with $H(v_1, v_2) = 0$ for all $v_1 \in L_1$ and $v_2 \in L_2$, then we write $L \cong L_1 \perp L_2$.

When $E$ has class number 1, the ring $\mathcal{O}_E$ is a principal ideal domain whereby every $\mathcal{O}_E$-lattice $L$ is free. In this case, we may think of the Hermitian form $H$ acting on $L$ as a function $f : \mathcal{O}_E^n \to \mathbb{Z}$ defined by

$$f(x_1, \ldots, x_n) = H\left(\sum_{i=1}^{n} x_i v_i, \sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} H(v_i, v_j) x_i \bar{x}_j$$

for some suitable basis $\{v_i\}_{i=1}^n$ of $L$. If the basis $\{v_i\}_{i=1}^n$ is orthogonal, we write $L \cong \langle H(v_1), \ldots, H(v_n)\rangle$. (For example, the form $x\bar{x} + 2y\bar{y}$ is associated to the lattice $\langle 1, 2\rangle$.)

Similarly, we may associate a quadratic lattice $\widetilde{L}$ with every Hermitian $\mathcal{O}_E$-lattice $L$. The ring $\mathcal{O}_E$ has a basis $\{1, \omega_m\}$ as a $\mathbb{Z}$-module, where

$$\omega_m = \begin{cases} \frac{1+\sqrt{-m}}{2}, & m \equiv 3 \bmod 4, \\ \sqrt{-m}, & \text{otherwise.} \end{cases}$$

Then, $\tilde{f}(x_1, y_1, \ldots, x_n, y_n) = f(x_1 + \omega_m y_1, \ldots, x_n + \omega_m y_n)$ is a quadratic form in $2n$ variables corresponding to the lattice $\widetilde{L}$. From this construction, it is clear that the Hermitian form $f$ is universal if and only if the quadratic form $\tilde{f}$ is. We write $\sim$ to denote the correspondence between a Hermitian lattice and its associated quadratic form.

## 3. Classification of Universal Hermitian Forms

Earnest and Khosravani [5], Iwabuchi [7], and Kim and Park [10] identified all potentially universal Hermitian forms over imaginary quadratic fields. This "screening process" is the more straightforward part of the classification, relying on a uniform computational method (see [5]).

The universality of the candidates identified was then shown by a variety of methods. Indeed, a total of eight different approaches were used. Six of these methods were "*ad hoc*" arguments, each an intricate method developed to prove the universality of an individual Hermitian form.

We give a unified proof of the universalities of the forms in the classification, relying upon Ramanujan's list of universal forms [13] and the 290-Theorem [2]. The universalities of the twenty-five universal binary Hermitian forms follow directly from our methods.

**Main Theorem.** *Up to equivalence, the integral positive-definite universal binary Hermitian lattices in imaginary quadratic fields are exactly the lattices in* (1)*:*

| $\mathbb{Q}(\sqrt{-m})$ | *universal binary lattices* |
|---|---|
| $\mathbb{Q}(\sqrt{-1})$ | $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle,$ |
| $\mathbb{Q}(\sqrt{-2})$ | $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle,$ |
| $\mathbb{Q}(\sqrt{-3})$ | $\langle 1,1\rangle, \langle 1,2\rangle,$ |
| $\mathbb{Q}(\sqrt{-5})$ | $\langle 1,2\rangle, \langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_5 \\ -1+\bar{\omega}_5 & 3 \end{pmatrix},$ |
| $\mathbb{Q}(\sqrt{-6})$ | $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_6 \\ \bar{\omega}_6 & 3 \end{pmatrix},$ |
| $\mathbb{Q}(\sqrt{-7})$ | $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle,$ |
| $\mathbb{Q}(\sqrt{-10})$ | $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{10} \\ \bar{\omega}_{10} & 5 \end{pmatrix},$ |
| $\mathbb{Q}(\sqrt{-11})$ | $\langle 1,1\rangle, \langle 1,2\rangle,$ |
| $\mathbb{Q}(\sqrt{-15})$ | $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{15} \\ \bar{\omega}_{15} & 2 \end{pmatrix},$ |
| $\mathbb{Q}(\sqrt{-19})$ | $\langle 1,2\rangle,$ |
| $\mathbb{Q}(\sqrt{-23})$ | $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{23} \\ \bar{\omega}_{23} & 3 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{23} \\ -1+\bar{\omega}_{23} & 3 \end{pmatrix},$ |
| $\mathbb{Q}(\sqrt{-31})$ | $\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{31} \\ \bar{\omega}_{31} & 4 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{31} \\ -1+\bar{\omega}_{31} & 4 \end{pmatrix}.$ |

(1)

*Proof.* Earnest and Khosravani [5], Iwabuchi [7], and Kim and Park [10] showed that no binary Hermitian forms not in the list (1) can be universal over an imaginary quadratic field $E$. Therefore, we must only show the universality of each of these candidate forms.

First, we identify the diagonal lattices in the list (1) which correspond to diagonal quaternary quadratic forms:

$$\begin{aligned}
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-1}) &\sim w^2+x^2+y^2+z^2,\\
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) &\sim w^2+x^2+2y^2+2z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-1}) &\sim w^2+x^2+2y^2+2z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) &\sim w^2+2x^2+2y^2+4z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-5}) &\sim w^2+2x^2+5y^2+10z^2,\\
\langle 1,3\rangle \text{ in } \mathbb{Q}(\sqrt{-1}) &\sim w^2+x^2+3y^2+3z^2,\\
\langle 1,3\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) &\sim w^2+3x^2+3y^2+6z^2,\\
\langle 1,4\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) &\sim w^2+2x^2+4y^2+8z^2,\\
\langle 1,5\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) &\sim w^2+2x^2+5y^2+10z^2.
\end{aligned} \tag{2}$$

The universality of each of the forms on the right-hand side of (2) was shown by Ramanujan [13]. Thus, we have the universality of the Hermitian forms on the left-hand side of (2) immediately.

This leaves only eight other diagonal Hermitian lattices in (1),

$$\begin{array}{lcl}
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-3}) & \sim & w^2+wx+x^2+y^2+yz+z^2,\\
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-7}) & \sim & w^2+wx+2x^2+y^2+yz+2z^2,\\
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-11}) & \sim & w^2+wx+3x^2+y^2+yz+3z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-3}) & \sim & w^2+wx+x^2+2y^2+2yz+2z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-7}) & \sim & w^2+wx+2x^2+2y^2+2yz+4z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-11}) & \sim & w^2+wx+3x^2+2y^2+2yz+6z^2,\\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-19}) & \sim & w^2+wx+5x^2+2y^2+2yz+10z^2,\\
\langle 1,3\rangle \text{ in } \mathbb{Q}(\sqrt{-7}) & \sim & w^2+wx+2x^2+3y^2+3yz+6z^2.
\end{array} \tag{3}$$

We may invoke the 290-Theorem to show the universality of the eight quadratic forms in (3); the check that each of these forms represents all of $S_{290}$ is an easy computation. It then follows directly that the eight Hermitian forms in (3) are all universal.

Now, we turn to the non-diagonal Hermitian lattices in (1). These are the remaining eight lattices,

$$\begin{array}{rcl}
\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_5 \\ -1+\bar{\omega}_5 & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-5}) & \sim & w^2+2x^2+2xy+3y^2+5z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_6 \\ \bar{\omega}_6 & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-6}) & \sim & w^2+2x^2+3y^2+6z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{10} \\ \bar{\omega}_{10} & 5 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-10}) & \sim & w^2+2x^2+3y^2+10z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{15} \\ \bar{\omega}_{15} & 2 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-15}) & \sim & w^2+2x^2+xy+2y^2+wz+4z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{23} \\ \bar{\omega}_{23} & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-23}) & \sim & w^2+2x^2+xy+3y^2+wz+6z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{23} \\ -1+\bar{\omega}_{23} & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-23}) & \sim & w^2+2x^2+xy+3y^2+wz+6z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{31} \\ \bar{\omega}_{31} & 4 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-31}) & \sim & w^2+2x^2+xy+4y^2+wz+8z^2,\\
\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{31} \\ -1+\bar{\omega}_{31} & 4 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-31}) & \sim & w^2+2x^2+xy+4y^2+wz+8z^2.
\end{array} \tag{4}$$

Now, all of the diagonal quadratic forms in (4) are found in the list of universal forms obtained by Ramanujan [13]. Furthermore, the universalities of the non-diagonal quadratic forms in (4) follow from the 290-Theorem. It then follows immediately that all the Hermitian forms in (4) are universal. □

## 4. Remarks

Kim, Kim, and Park [9] have recently announced a criterion which completely characterizes the universality of Hermitian forms.

**15-Theorem for Hermitian Lattices.** *A positive-definite integral Hermitian form is universal if and only if it represents the ten integers* $\{1, 2, 3, 5, 6, 7, 10, 13, 14, 15\}$.

Unfortunately, the proof of this result cites the original proof of the classification of universal binary Hermitian forms. Consequently, Kim, Kim, and Park's 15-Theorem for Hermitian Lattices cannot give a direct, unified proof of our Main Theorem.

Kim, Kim, and Park note that the 290-Theorem can be used to simplify some of the arguments in their proof of the 15-Theorem for Hermitian Lattices. Such simplifications would take the same form as those we have presented here to unify the classification of universal binary Hermitian forms.

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