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SUBTRACTION WITH A MULLER TWIST

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Abstract

We consider the usual one-pile subtraction game with an extra feature, called a *Muller twist*. The twist is that the number of stones to be removed from the heap is dictated by our opponent to be either in a prescribed set S or not. We investigate regularity in the sequences of Grundy values for such games.

1. Introduction

Recent papers (see [1], [3], [4]) have focused attention on combinatorial games played with a Muller twist. Such a variation derives its name from the game *Quarto*®, invented by Blaise Muller, in which, on each turn, a player chooses for his opponent the piece he is to play. A combinatorial game with a Muller twist has the feature that each player places a constraint on his opponent's next move.

In this paper, we consider the game of Subtraction played with a Muller twist. Subtraction, as defined in [2], is played with a single heap of stones and a set of positive integers S, called the *subtraction set*. A move consists of removing x stones from the pile where x is an element of S. In our game, the Muller twist is that our opponent dictates that the number of stones that we remove is either an element of S or is not an element of S. In other words,

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our opponent states that we must use either S or $\bar{S} = \mathbf{N} \setminus S$ as the subtraction set. As usual, under normal play, the first player who is unable to move loses.

We assume that the reader is familiar with the standard terminology of combinatorial game theory, as described in [2]. The notations (n, S) and $(n, \bar{S})$ will refer to the games played with n stones with constraints S and $\bar{S}$ respectively. We use $\mathcal{G}(G)$ to denote the Grundy value of the game G and is defined recursively to be the minimum excluded value (or mex) of the set $\{\mathcal{G}(H)|H \text{ is an option of } G\}$. If G has an option with value x then x is said to be an *excluded value* for G. A sequence $\{a_n\}$ is said to be *periodic* if there exist N and p such that $a_{n+p} = a_n$ for all $n \geq N$; it is *arithmetic periodic* if there exist N, p, and s such that $a_{n+p} = a_n + s$ for all $n \geq N$. In such instances, p is called the *length* of the period and s is called the *saltus*.

The purpose of this paper is to investigate the values of $\mathcal{G}(n, S)$ and $\mathcal{G}(n, \bar{S})$ for various sets S. Primarily, we are concerned with periodic or arithmetic periodic behaviour exhibited by the sequences of Grundy values. In Section 2, we study a class of sets S for which the sequence $\{\mathcal{G}(n, S)\}$ may be completely described and $\{\mathcal{G}(n, \bar{S})\}$ exhibits arithmetic periodic behaviour. In Section 3, we focus our attention on subtraction sets of the form $\{x \,|\, x \equiv b \pmod{c}\}$.

We conclude this introductory section with an example of computing Grundy values in Subtraction with a Muller twist. Suppose that $S = \{1, 2, 4\}$. From a pile of 0 stones, no move is possible so $\mathcal{G}(0, S)$ and $\mathcal{G}(0, \bar{S})$ are both equal to mex $\{\} = 0$. Since $1 \in S$, from the position $(1, S)$, we may move to either $(0, S)$ or $(0, \bar{S})$ both of which have value 0. Therefore, $\mathcal{G}(1, S) = \text{mex } \{0\} = 1$. On the other hand, the position $(1, \bar{S})$ has no options so $\mathcal{G}(1, \bar{S}) = 0$. From the position $(2, S)$, we may move to any position having 1 or 0 stones since both 1 and 2 are in S. Each of these four options has value 0 or 1 so $\mathcal{G}(2, S) = \text{mex } \{0, 1\} = 2$. From the position $(2, \bar{S})$ there is no legal move so $\mathcal{G}(2, \bar{S}) = 0$. Continuing, we see that from the position $(3, S)$, we may move to a position having $3 - 1 = 2$ stones or $3 - 2 = 1$ stone. The four possible options have values 0, 1, or 2 so $\mathcal{G}(3, S) = 3$. Since 3 is the smallest element of $\bar{S}$, the only possible move from $(3, \bar{S})$ is to either $(0, S)$ or $(0, \bar{S})$. Both of these options have value 0 so $\mathcal{G}(3, \bar{S}) = 1$.

The reader may find it instructive to verify the entries in the following table.

n	$\mathcal{G}(n, S)$	$\mathcal{G}(n, \bar{S})$
0	0	0
1	1	0
2	2	0
3	3	1
4	4	2
5	5	1
6	3	2
7	0	3

2. Regular Behaviour of Grundy Values

In this section, we establish some results concerning the sequences $\{\mathcal{G}(n,S)\}$ and $\{\mathcal{G}(n,\bar{S})\}$ for certain sets S. In particular, we show that the sequence $\{\mathcal{G}(n,\bar{S})\}$ is arithmetic periodic under certain conditions on S.

Our first lemma establishes a lower bound for the sequence $\{\mathcal{G}(n,\bar{S})\}$.

Lemma 1 *Suppose that $1 \in S$ and that $|x-y| \geq 2$ for any two distinct elements x, y of S. Then $\mathcal{G}(n,\bar{S}) \geq 4$ for all $n \geq 6$.*

Proof. We have $1 \in S$ so $2 \in \bar{S}$. The first 3 rows of the table of $\mathcal{G}$-values may be computed as follows.

n	$\mathcal{G}(n,S)$	$\mathcal{G}(n,\bar{S})$
0	0	0
1	1	0
2	2	1

If $3 \in S$ then $4 \in \bar{S}$ and the table continues as follows.

n	$\mathcal{G}(n,S)$	$\mathcal{G}(n,\bar{S})$
3	3	2
4	4	3

If, on the other hand, $3 \in \bar{S}$ then the table is as follows.

n	$\mathcal{G}(n,S)$	$\mathcal{G}(n,\bar{S})$
3	0	2
4	1	3
5	?	3

In either event, we see that each of the values 0, 1, 2, and 3 appears in consecutive rows among rows 0 through 5.

Now consider the position $(n,\bar{S})$ where $n \geq 6$. At least one of $n-1$ and n must be in $\bar{S}$ so we may move to a position in which the pile size is either 1 or 0. In either case, there is an option that has value 0. Similarly, we see that $(n,\bar{S})$ has options of value 1, 2, and 3 so $\mathcal{G}(n,\bar{S}) \geq 4$. □

The next result determines completely the sequence $\{\mathcal{G}(n,S)\}$ for a particular class of sets S.

Theorem 2.1 *Suppose that $1 \in S$ and that if $x \in S$ then x is odd. Moreover, suppose that $|x - y| \geq 6$ for any two distinct elements x, y of S. Then, for $0 \leq n \leq 2$, $\mathcal{G}(n, S) = n$ and, for $n \geq 3$,*

$$\mathcal{G}(n,S) = \begin{cases} 2 & \text{if } n \in S \\ 3 & \text{if } n-1 \in S \\ 0 & \text{if } n \text{ is odd and } n \notin S \\ 1 & \text{if } n \text{ is even and } n-1 \notin S. \end{cases}$$

Proof. Noting that all $2 \leq i \leq 6$ are in $\bar{S}$, we may compute the initial $\mathcal{G}$-values as follows.

n	$\mathcal{G}(n,S)$	$\mathcal{G}(n,\bar{S})$
0	0	0
1	1	0
2	2	1
3	0	2
4	1	3
5	0	3
6	1	4

We now proceed by induction on n. Suppose that $n \geq 7$ and that the result is true for all smaller pile sizes. We consider the position (n, S).

First, suppose that $n \in S$. We may then move to $(0, S)$ which has value 0. Since $1 \in S$, we may also move to a pile of size $n-1$. Now $n-1$ is even and $(n-1)-1 = n-2 \notin S$ so, by induction, $\mathcal{G}(n-1, S) = 1$. By Lemma 1, $\mathcal{G}(x, \bar{S}) \geq 4$ for all $x \geq 6$. Therefore, the only position with constraint $\bar{S}$ and having value 2 is $(3, \bar{S})$. However, $n-3 \notin S$ so $(3, \bar{S})$ is not an option of (n, S). The positions with constraint S that have value 2 are $(2, S)$ and (m, S) where $m \in S \setminus \{1\}$. But $n-2$ is even as is $n-m$ for such m so these positions are not options of (n, S). Therefore, 2 is an excluded value for (n, S). Since (n, S) does have options of values 0 and 1 as shown above, $\mathcal{G}(n, S) = 2$.

Second, suppose that $n-1 \in S$. From (n, S), we may move to $(1, S)$ or $(1, \bar{S})$ which have value 1 and 0 respectively. We may also move to $(n-1, S)$ which, by induction, has value 2. No position with constraint S has value 3. By Lemma 1, the only positions of value 3 with constraint $\bar{S}$ are $(4, \bar{S})$ and $(5, \bar{S})$. However, since $n-1 \in S$, neither $n-4$ nor $n-5$ are in S so (n, S) does not have an option of value 3. Therefore, $\mathcal{G}(n, S) = 3$.

Third, suppose that n is odd and that $n \notin S$. The positions that have value 0 are $(0, S), (0, \bar{S}), (1, \bar{S})$ and (m, S) where $m \notin S$ is odd. Neither n nor $n-1$ is in S so we cannot move to a position with pile size 0 or 1. Moreover, if m is odd then $n-m$ is even so we cannot move to a position with pile size m. Thus 0 is an excluded value for (n, S) so $\mathcal{G}(n, S) = 0$.

Finally, suppose that n is even and that $n-1 \notin S$. The position (n, S) has $(n-1, S)$ as an option and, by induction, $\mathcal{G}(n-1, S) = 0$. The positions with value 1 are $(1, S), (2, \bar{S})$

and (m, S) where $m \geq 4$ is even and $m-1 \notin S$. Neither $(1, S)$ nor $(2, \bar{S})$ are options of (n, S) since $n-1 \notin S$ and $n-2$, which is even, is not in S. Moreover, if n and m are both even then $n-m \notin S$ so (m, S) with m even is not an option of (n, S). Therefore, 1 is an excluded value for (n, S) so $\mathcal{G}(n, S) = 1$. □

We now turn our attention to the sequence $\{\mathcal{G}(n, \bar{S})\}$. The following theorem shows that, under certain conditions on S, the sequence is arithmetic periodic with period length 2 and saltus 1.

Theorem 2.2 *Suppose that the following conditions hold.*

- $1 \in S$
- $|x-y| \geq 2$ *for any two distinct elements* x, y *of* S
- *there exist nonnegative integers* d, m *such that* $d = \mathcal{G}(m, \bar{S}) = \mathcal{G}(m+1, \bar{S})$
- *for all* $0 \leq i \leq m-1$, $\mathcal{G}(i, \bar{S}) \leq d$
- *for all* $i \geq 0$, $\mathcal{G}(i, S) \leq d$
- *for all* $0 \leq v \leq d-1$, v *appears as a value on two consecutive rows of the table of* $\mathcal{G}$*-values, among rows* 0 *through* $m-1$ *inclusive*

Then, for all $n \geq m$, $\mathcal{G}(n, \bar{S}) = d + \lfloor (n-m)/2 \rfloor$.

Proof. The proof is by induction on n. For $n = m$ and $n = m+1$ the result is true by hypothesis.

Suppose now that $n \geq m+2$ and that the result holds for all pile sizes between m and $n-1$ inclusive. Let $k = \lfloor (n-m)/2 \rfloor$ so $n = m+2k$ or $n = m+2k+1$. We wish to show that $\mathcal{G}(n, \bar{S}) = d+k$.

First, let i be such that $0 \leq i \leq k-1$. By induction, the positions $(m+2i, \bar{S})$ and $(m+2i+1, \bar{S})$ both have value $d+i$. Since S does not contain 2 consecutive elements, at least one of these positions will be an option of $(n, \bar{S})$ so $d+i$ is an excluded value for $(n, \bar{S})$ for all $0 \leq i \leq k-1$.

Now consider a value v with $0 \leq v \leq d-1$. By hypothesis, v appears on two consecutive rows among rows 0 through $m-1$ of the table of $\mathcal{G}$-values so there exists a move from $(n, \bar{S})$ to at least one of these rows. Therefore, v is an excluded value for $(n, \bar{S})$.

Finally, we show that $d+k$ is not an excluded value for $(n, \bar{S})$. We have $n \leq m+2k+1$ and 2 is the smallest element of $\bar{S}$ so the pile size of any option of $(n, \bar{S})$ is at most $(m+2k+1)-2 = m+2(k-1)+1$. By induction, for any $i \leq m+2(k-1)+1$, we have $\mathcal{G}(i, \bar{S}) \leq d+k-1$ and also, by hypothesis, $\mathcal{G}(i, S) \leq d$. Thus no option of $(n, \bar{S})$ has value $d+k$.

Therefore, $\mathcal{G}(n, \bar{S}) = d + k$ as required. □

Corollary 2.3 *Suppose that $1 \in S$ and that if $x \in S$ then x is odd. Moreover, suppose that $|x - y| \geq 6$ for any two distinct elements x, y of S. Then*

$$\mathcal{G}(n, \bar{S}) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{if } 1 \leq n \leq 3 \\ \lfloor n/2 \rfloor + 1 & \text{if } n \geq 4. \end{cases}$$

Proof. The values of $\mathcal{G}(n, \bar{S})$ for $0 \leq n \leq 3$ may be found in the table of initial $\mathcal{G}$-values in the proof of Theorem 2.1.

Moreover, by considering this same table, we see that by letting $d = 3$ and $m = 4$,the hypotheses of Theorem 2.2 are satisifed. Therefore, for $n \geq 4$, we have

$$\mathcal{G}(n, \bar{S}) = 3 + \left\lfloor \frac{n-4}{2} \right\rfloor = 1 + \left\lfloor \frac{n}{2} \right\rfloor$$

as desired. □

Examples There are (infinitely) many natural examples of sets S that satisfy the hypotheses of Theorem 2.1 and Corollary 2.3. We list some of these below.

- $\{x \,|\, x \equiv 1 \pmod{2n}\}$ where $n \geq 3$; for example $\{1, 7, 13, 19, \ldots\}$
- $\{x^p \,|\, p \in T\}$ where x is a fixed odd integer ≥ 7 and T is a set of nonnegative integers that includes 0; for example $\{1, 9, 9^2, \ldots, 9^n\}$
- $\{y^r \,|\, y \in U\}$ where r is a fixed positive integer ≥ 2 and U is a set of odd (positive) integers that includes 1; for example, $\{1, 3^4, 5^4, \ldots, (2n+1)^4\}$

3. Subtraction Sets of the Form b mod c

In this section, we consider subtraction sets S of the form $\{x \,|\, x \equiv b \pmod{c}\}$ where $b \geq 2$. With some additional assumptions, we show (Theorem 3.1) that the sequence $\{\mathcal{G}(n, S)\}$ is eventually periodic with period length c and, in Corollary 3.3, that $\{\mathcal{G}(n, \bar{S})\}$ is an arithmetic periodic sequence with period length $2b$ and saltus b.

We begin by determining the initial segments in both sequences of Grundy values.

Lemma 2 *For $n = 0, 1, \ldots, 2b - 1$, the sequence $\{\mathcal{G}(n, S)\}$ is*

$$\underbrace{0, 0, \ldots 0}_{b}, 1, 2, \underbrace{1, 1, \ldots 1}_{b-2}$$

and, for $n = 0, 1, \ldots, 2b+2$, *the sequence* $\{\mathcal{G}(n, \bar{S})\}$ *is*

$$0, 1, 2, \ldots, b+2, 3, 4, \ldots, b+2 \, .$$

Proof. For $0 \leq i \leq b-1$, no move is possible from (i, S) so these positions have value 0. Moreover, $\mathcal{G}(0, \bar{S}) = 0$ and since $\{1, 2, \ldots, b-1\} \subseteq \bar{S}$, an easy induction argument shows that $\mathcal{G}(i, \bar{S}) = i$ for $1 \leq i \leq b-1$.

The only options from (b, S) are $(0, S)$ and $(0, \bar{S})$, both of which have value 0, so $\mathcal{G}(b, S) = 1$. From $(b, \bar{S})$ we may move to positions of value $0, 1, \ldots, b-1$ but not b so $\mathcal{G}(b, \bar{S}) = b$.

From $(b+1, S)$, we must move to either $(1, S)$ or $(1, \bar{S})$. These positions have value 0 and 1 respectively so $\mathcal{G}(b+1, S) = 2$. From $(b+1, \bar{S})$, we may move to positions of pile size i where $2 \leq i \leq b$ and, for these pile sizes, there are positions of each value from 0 to b inclusive. However, $b+1$ is an excluded value for $(b+1, \bar{S})$ so $\mathcal{G}(b+1, \bar{S}) = b+1$.

Now consider a position $(b+i, S)$ where $2 \leq i \leq b-1$. The only options of such a position are (i, S) which has value 0 and $(i, \bar{S})$ which has value i. Therefore, $\mathcal{G}(b+i, S) = \text{mex}\{0, i\} = 1$. From $(b+2, \bar{S})$, we may move to a position of pile size i where $3 \leq i \leq b+1$. For these pile sizes, there are positions of each value from 0 to $b+1$ inclusive. However, $b+2$ is an excluded value for $(b+2, \bar{S})$ so $\mathcal{G}(b+2, \bar{S}) = b+2$.

Finally, consider positions of the form $(b+i, \bar{S})$ where $3 \leq i \leq b+2$. The only position with pile size smaller than $b+i$ and having value i is $(i, \bar{S})$. Since $b \in S$, this position is not an option of $(b+i, S)$ so i is an excluded value for $(b+i, S)$. It is easy to check that every nonnegative integer less than i is the value of some option of $(b+i, S)$ so $\mathcal{G}(b+i, S) = i$. □

Our next lemma provides a crude lower bound on the sequence $\{\mathcal{G}(n, \bar{S})\}$.

Lemma 3 *For all* $n \geq 2$, $\mathcal{G}(n, \bar{S}) \geq 2$.

Proof. If $n-1 \in \bar{S}$ then we may move from $(n, \bar{S})$ to either $(1, S)$ or $(1, \bar{S})$. By Lemma 2, these options have values 0 and 1 respectively, so $\mathcal{G}(n, \bar{S}) \geq 2$.

Conversely, suppose that $n-1 \in S$. Therefore, $n-1 \equiv b \pmod{c}$ so $n = b+1+mc$ for some $m \geq 0$. Now $mc+1 \equiv 1 \not\equiv b \pmod{c}$ so $mc+1 \in \bar{S}$. We may move then from $(n, \bar{S})$ to $(n-(mc+1), S) = (b, S)$. By Lemma 2, this option has value 1. Moreover, $n \equiv b+1 \not\equiv b \pmod{c}$ so $n \in \bar{S}$. Hence the position $(n, \bar{S})$ is an immediate win so its value is nonzero. Futhermore, as noted above, $(n, \bar{S})$ has an option of value 1 so $\mathcal{G}(n, \bar{S}) \geq 2$. □

To establish that $\{\mathcal{G}(n, S)\}$ is eventually periodic with period length c, we consider "blocks" of the sequence, each of length c, beginning at $n = 2b$. The purpose of the next two lemmas is to show that the initial $\min(b, c-b)$ elements in each block are zeroes, and the final $c-b$ elements are ones. It will be convenient to introduce the following notation: define

$$L_m(r) = \{2b+mc-1, 2b+mc-2, \ldots, 2b+mc-r\}$$

and

$$U_m(r) = \{2b + mc, 2b + mc + 1, \ldots, 2b + mc + r - 1\} .$$

Lemma 4 *Suppose that $m \geq 0$ and let $r = \min(b, c-b)$. Then $\mathcal{G}(n, S) = 0$ for all $n \in U_m(r)$.*

Proof. The proof is by induction on m.

For the base case $m = 0$, we need to show that $\mathcal{G}(2b + i, S) = 0$ for all $0 \leq i \leq \min\{b, c - b\} - 1$. The only options from $(2b + i, S)$ are $(b + i, S)$ and $(b + i, \bar{S})$. Now $\mathcal{G}(b + i, S) \geq 1$ by Lemma 2, and $\mathcal{G}(b + i, \bar{S}) \geq 2$ by Lemma 3. Therefore, $\mathcal{G}(2b + i, S) = 0$ which establishes the base case.

For the induction step, consider a position (n, S) where $n \in U_m(r)$ so that $n = 2b+mc+j$ for some $0 \leq j \leq \min\{b, c - b\} - 1$. The options of this position are (i) a pile size of $n-b = b+mc+j$ with constraint S or $\bar{S}$, and (ii) all the options of $n-c = 2b+(m-1)c+j$ with S. Since $n-c \in U_{m-1}(r)$, we have $\mathcal{G}(n-c, S) = 0$ by induction, so all the options in (ii) have positive value. Moreover, $\mathcal{G}(b+mc+j, \bar{S}) \geq 2$ by Lemma 3. Finally, $\mathcal{G}(b+mc+j, S) > 0$ since there is a move from $(b+mc+j, S)$ to (j, S) which has value 0 by Lemma 2. Thus, all the options of (n, S) have positive value so $\mathcal{G}(n, S) = 0$. □

Lemma 5 *Suppose that $2b \geq c + 2$, and $m \geq 1$. For all $n \in L_m(c - b)$, $\mathcal{G}(n, S) = 1$.*

Proof. Let $n \in L_m(c - b)$ so $n = 2b + mc - j$ for some $1 \leq j \leq c - b$. Any move from (n, S) is to a pile size of the form $b + kc - j$ where $0 \leq k \leq m$. In the case $k = 0$, we have $b - 1 \geq b - j \geq b - (c - b) = 2b - c \geq 2$ so $\mathcal{G}(b - j, S) = 0$ and $\mathcal{G}(b - j, \bar{S}) \geq 2$ by Lemma 2.

Now consider a pile of size $b+kc-j$ where $k \geq 1$. We see that $b+kc-j = 2b+(k-1)c+c-b-j \in U_{k-1}(c-b)$ so, by Lemma 4, $\mathcal{G}(b+kc-j, S) = 0$. Moreover, $\mathcal{G}(b+kc-j, \bar{S}) \geq 2$ by Lemma 3 since $b+kc-j \geq b+c-j \geq b+c-(c-b) = 2b \geq 4$. Finally, since (n, S) has at least one option of value 0, and no option has value exactly 1, $\mathcal{G}(n, S) = 1$. □

Having established the pattern exhibited by the initial and terminal portions of each block of length c, we now turn our attention to the "middle" segments. The next three lemmas show that the number of positions in which successive middle segments agree is strictly increasing.

Lemma 6 *Suppose that $2b \neq c$. If $\mathcal{G}(n + c - b, S) = \mathcal{G}(n - b, S)$ then $\mathcal{G}(n + c, S) = \mathcal{G}(n, S)$.*

Proof. By definition,

$$\mathcal{G}(n + c, S) = \operatorname{mex}(\{\mathcal{G}(n + c - b, S), \mathcal{G}(n + c - b, \bar{S})\} \cup T)$$

where T is the set of the values of all the options of (n, S).

Now $c-b \neq b$ so $c-b \in \bar{S}$. Thus $(n+c-b, \bar{S})$ has (n, S) as an option so $\mathcal{G}(n+c-b, \bar{S}) \neq \mathcal{G}(n, S)$. Moreover, since $b \in S$, (n, S) has $(n-b, S)$ as an option so $\mathcal{G}(n, S) \neq \mathcal{G}(n-b, S) = \mathcal{G}(n+c-b, S)$. Therefore, $\mathcal{G}(n+c, S) = \operatorname{mex} T = \mathcal{G}(n, S)$. □

Lemma 7 *Suppose that $2b \geq c+2$, $m \geq 1$, and $\mathcal{G}(n, S) = \mathcal{G}(n-c, S)$ for all $n \in L_m(r)$. Then $\mathcal{G}(n+c, S) = \mathcal{G}(n, S)$ for all $n \in L_m(r)$.*

Proof. Let $n \in L_m(r)$ so $n = 2b+mc-k$ where $1 \leq k \leq r$. If $k \leq c-b$ then the result follows immediately since both $\mathcal{G}(n, S)$ and $\mathcal{G}(n+c, S)$ are equal to 1 by Lemma 5. If $k > c-b$ then $n+(c-b) \in L_m(r)$ so, by hypothesis, $\mathcal{G}(n+c-b, S) = \mathcal{G}(n-b, S)$. Now applying Lemma 6 gives $\mathcal{G}(n+c, S) = \mathcal{G}(n, S)$ as desired. □

Lemma 8 *Suppose that $2b \geq c+2$. Moreover, suppose that there exist $r \geq c-b$ and $m \geq 1$ such that*

1. *$\mathcal{G}(i, S) = \mathcal{G}(i-c, S)$ for all $i \in L_m(r)$, and*
2. *$\mathcal{G}(n, S) \neq \mathcal{G}(n-c, S)$ where $n = 2b+mc-(r+1)$.*

Then $\mathcal{G}(i, S) = \mathcal{G}(i-c, S)$ for all $i \in L_{m+1}(r+1)$.

Proof. For $i \in L_{m+1}(r)$, the result follows immediately from Lemma 7. Therefore, it remains to show that $\mathcal{G}(n+c, S) = \mathcal{G}(n, S)$ for $n = 2b+mc-(r+1)$.

The options of (n, S) consist of all the options of $(n-c, S)$ as well as $(n-b, S)$ and $(n-b, \bar{S})$. Since $\mathcal{G}(n, S) \neq \mathcal{G}(n-c, S)$, either $(n-b, S)$ or $(n-b, \bar{S})$ has value $\mathcal{G}(n-c, S)$. Now, since $c-b \in \bar{S}$, $(n-b, \bar{S})$ has $(n-c, S)$ as an option so $\mathcal{G}(n-b, \bar{S}) \neq \mathcal{G}(n-c, S)$. Therefore, we must have $\mathcal{G}(n-b, S) = \mathcal{G}(n-c, S)$.

Now $n+(c-b) \in L_m(r)$ so $\mathcal{G}(n+c-b, S) = \mathcal{G}(n-b, S)$. The options of $(n+c, S)$ are $(n+c-b, S)$, $(n+c-b, \bar{S})$, and all the options of (n, S). Now $(n+c-b, \bar{S})$ does not have the same value as (n, S) since the latter position is an option of the former. Moreover,

$$\mathcal{G}(n+c-b, S) = \mathcal{G}(n-b, S) = \mathcal{G}(n-c, S) \neq \mathcal{G}(n, S)$$

so we conclude that $\mathcal{G}(n+c, S) = \mathcal{G}(n, S)$. □

We are now ready to show that the sequence $\{\mathcal{G}(n, S)\}$ is eventually periodic.

Theorem 3.1 *Suppose that $2b \geq c+2$. There exists a positive integer N such that $\mathcal{G}(n, S) = \mathcal{G}(n-c, S)$ for all $n > N$.*

Proof. Define the function $f : \mathcal{N} \to \mathcal{N}$ as follows: Let $f(x) = y$ if y is the largest number such that $\mathcal{G}(i, S) = \mathcal{G}(i - c, S)$ for all $i \in L_x(y)$. Note that, by Lemma 5, $f(x) \geq c - b$ for all $x \geq 1$, and, by Lemma 8, f is a strictly increasing function. The maximum value of f is c and we take N to be the smallest integer at which f attains this value. The result now follows from Lemma 7. □

We now show that the sequence $\{\mathcal{G}(n, \bar{S})\}$ is, starting at $n = b + 3$, arithmetic periodic with period length $2b$ and saltus b. The first step is to establish upper bounds for $\{\mathcal{G}(n, S)\}$.

Lemma 9 *If $n < b + rc$ then $\mathcal{G}(n, S) \leq 2r$.*

Proof. From the position (n, S), we will reduce the pile size by $b + kc$ where $0 \leq k \leq r - 1$ so there are precisely r different pile sizes to which we may move. Since we then select constraint S or $\bar{S}$, the number of options is $2r$ so $\mathcal{G}(n, S) \leq 2r$. □

Lemma 10 *Suppose that $b \geq 5$ and $r \geq 1$. If $n \leq b + 3 + r(2b) + i$ where $0 \leq i \leq 2b - 1$ then $\mathcal{G}(n, S) < 3 + rb + i$.*

Proof. We have

$$n < b + 3 + (r + 1)(2b) \leq b + 3 + 2(r + 1)(c - 1) = b + 1 - 2r + (2r + 2)c < b + (2r + 2)c$$

so $\mathcal{G}(n, S) \leq 2(2r + 2)$ by Lemma 9. Now $(b - 4)r \geq 1$ so $rb \geq 4r + 1$ and thus

$$\mathcal{G}(n, S) \leq 4r + 4 \leq 3 + rb \; .$$

If $i \geq 1$ then the result follows immediately. If $i = 0$ then

$$n \leq b + 3 + r(2b) \leq b + 3 + 2r(c - 1) = b + 3 - 2r - c + (2r + 1)c < b + (2r + 1)c \; .$$

By Lemma 9, $\mathcal{G}(n, S) \leq 2(2r + 1) = 4r + 2 \leq 1 + rb < 3 + rb$ as required. □

We now determine the exact value of $\mathcal{G}(n, \bar{S})$ for $n \geq b + 3$.

Theorem 3.2 *Suppose that $b \geq 5$. If $n = b + 3 + r(2b) + i$ where $r \geq 0$ and $0 \leq i \leq 2b - 1$ then $\mathcal{G}(n, \bar{S}) = 3 + rb + i$.*

Proof. The proof is by induction on r. For the base case, we need to show that $\mathcal{G}(b+3+i, \bar{S}) = 3 + i$ for $i = 0, 1, \ldots, 2b - 1$. For $0 \leq i \leq b - 1$, this follows immediately from Lemma 2. Now consider pile size $m = b + 3 + i$ where $b \leq i \leq 2b - 1$. From $(m, \bar{S})$, we may move to pile sizes $2b+3, 2b+3, \ldots, b+3+i-1$ which, with constraint $\bar{S}$ have values $b+3, b+4, \ldots, i+2$. These then are all excluded values for $(m, \bar{S})$. We now show that 0, 1, and 2 are also excluded values for $(m, \bar{S})$. Both $(0, S)$ and $(1, S)$ have value 0 and at least one of these positions

is an option of $(m, \bar{S})$. Moreover, $(1, \bar{S})$ and (b, S) have value 1 and one of these positions is reachable from $(m, \bar{S})$. Finally $(2, \bar{S})$ and $(b+1, S)$ both have value 2 and one of these positions is an option of $(m, \bar{S})$.

From Lemma 2, we see that each integer from 3 to $b+2$ occurs as the value of two positions, each with constraint $\bar{S}$ and with pile sizes that differ by b. Since at least one of these two positions is an option of $(m, \bar{S})$, $3, 4, \ldots, b+2$ are all excluded values of $(m, \bar{S})$.

Finally, no position with pile size less than $m = b+3+i$ has value greater than $i+2$ so $\mathcal{G}(m, \bar{S}) = i+2$ as desired.

For the induction hypothesis, assume that the result holds for all values less than $n = b+3+r(2b)+i$ where $r \geq 1$.

Consider $m = b+3+k(2b)+j$ and $m-b = b+3+(k-1)(2b)+(b+j)$ where $0 \leq k \leq r-1$ and $0 \leq j \leq b-1$. If $k \geq 1$ then the induction hypothesis may be applied to both $(m, \bar{S})$ and $(m-b, \bar{S})$ to give $\mathcal{G}(m-b, \bar{S}) = 3+(k-1)b+(b+j) = 3+kb+j = \mathcal{G}(m, \bar{S})$. If $k = 0$, we have $\mathcal{G}(m, \bar{S}) = 3+j$ by the induction hypothesis and $\mathcal{G}(m-b, \bar{S}) = 3+j$ by Lemma 2. Thus, in either case, for any integer $3 \leq x \leq 3+(r-1)b+b-1$, there exist $y < n$ for which the positions $(y, \bar{S})$ and $(y-b, \bar{S})$ both have value x. Since at least one of $n-y$ and $n-(y-b)$ is in $\bar{S}$, x is an excluded value for $(n, \bar{S})$.

We now show that 0, 1, and 2 are also excluded values for $(n, \bar{S})$. Both $(0, S)$ and $(1, S)$ have value 0 and at least one of these positions is an option of $(n, \bar{S})$. Moreover, $(1, \bar{S})$ and (b, S) have value 1 and one of these positions is reachable from $(n, \bar{S})$. Finally $(2, \bar{S})$ and $(b+1, S)$ both have value 2 and one of these positions is an options of $(n, \bar{S})$.

We have shown thus far that every nonnegative integer less than $3+rb$ is an excluded value for $(n, \bar{S})$. To complete the proof, we consider two cases; (i) $0 \leq i \leq b-1$, and (ii) $b \leq i \leq 2b-1$.

In case (i), from $(n, \bar{S})$ there is a move to $(b+3+r(2b)+j, \bar{S})$ for any $0 \leq j \leq i-1$. By induction, the value of this option is $3+rb+j$ so $3+rb, 3+rb+1, \ldots, 3+rb+(i-1)$ are all excluded values for $(n, \bar{S})$. Finally, we claim that no option of $(n, \bar{S})$ has value $3+rb+i$. First, by Lemma 10, $\mathcal{G}(m, S) < 3+rb+i$ if $m < n = b+3+r(2b)+i$. Second, the only pile size $m < n$ for which $\mathcal{G}(m, \bar{S}) = 3+rb+i$ is $m = n-b$. However, as $b \in S$, $(m, \bar{S})$ is not an option of $(n, \bar{S})$.

In case (ii), from $(n, \bar{S})$ there is a move to $(b+3+r(2b)+j, \bar{S})$ for any $0 \leq j \leq i-1$ except $j = i-b$. By induction, all values between $3+rb$ and $3+rb+(i-1)$ excepting $3+rb+(i-b)$ are excluded values for $(n, \bar{S})$. We now show that $3+rb+(i-b)$ is also, in fact, an excluded value. Since $2b \in \bar{S}$, $(n-2b, \bar{S})$ is an option of $(n, \bar{S})$ and $\mathcal{G}(n-2b, \bar{S}) = 3+(r-1)b+i = 3+rb+(i-b)$. Finally, we claim that no option of $(n, \bar{S})$ has value $3+rb+i$. First, by Lemma 10, $\mathcal{G}(m, S) < 3+rb+i$ if $m < n = b+3+r(2b)+i$. Second, if $m < n$ then $\mathcal{G}(m, \bar{S}) \leq \mathcal{G}(n-1, \bar{S}) = 3+rb+(i-1)$.

Thus, $\mathcal{G}(n, \bar{S}) = 3 + rb + i$ and the proof is complete. □

The following corollary follows immediately from Theorem 3.2.

Corollary 3.3 *Suppose that $b \geq 5$. For all $n \geq b + 3$, $\mathcal{G}(n + 2b, \bar{S}) = \mathcal{G}(n, \bar{S}) + b$.*

4. Conclusion

The study of combinatorial games with a Muller twist is an exciting and relatively new area of investigation. As noted in [3], much more remains to be discovered and this is true also of subtraction games with a Muller twist. We conclude our preliminary study of these games by offering the following comments and challenges.

- In order to prove the results of Section 3, we often required additional hypotheses such as $2b \geq c + 2$ and $b \geq 5$. It seems likely, however, that these conditions are not necessary as the results appear to hold in other cases. It would be of interest to find proofs of these theorems that apply to a wider class of examples.
- Lemma 10 gives a crude upper bound for $\mathcal{G}(n, S)$ where $S = \{x \,|\, x \equiv b \pmod{c}\}$. Based on the numerical evidence, we conjecture that $\mathcal{G}(n, S) \leq b$ for all n. Indeed, this bound is met in the case $b = 4, c = 5$ but in other instances, the largest value of $\mathcal{G}(n, S)$ is much less than b.
- Suppose that S consists of the Fibonacci numbers. This particular game was studied in [4] and the authors were unable to give a simple rule for determining the outcome class. We calculated the Grundy sequences up to $n = 8000$ but also were not able to discern a pattern even though the sequences exhibited a high degree of regularity.
- Suppose that $S = \{x \,|\, x \equiv a \text{ or } b \pmod{c}\}$. Numerical experiments show that the Grundy values are chaotic and we were unable to prove any general results.

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