SOLITAIRE CLOBBER PLAYED ON HAMMING GRAPHS

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Abstract

The one-player game *Solitaire Clobber* was introduced by Demaine *et al.* Beaudou *et al.* considered a variation called *SC2*. Black and white stones are located on the vertices of a given graph. A move consists in picking a stone to replace an adjacent stone of the opposite color. The objective is to minimize the number of remaining stones. The game is interesting if there is at least one stone of each color. In this paper, we investigate the case of Hamming graphs. We prove that game configurations on such graphs can always be reduced to a single stone, except for hypercubes. Nevertheless, hypercubes can be reduced to two stones.

1. Introduction and Definitions

We consider the one-player game *SC2* that was introduced in [3]. This game is a variation of the game *Solitaire Clobber* defined by Demaine et al. in [2]. Note that both solitaire games come from the two-player game *Clobber*, that was created and studied in [1]. One can have a look to [4] for more information about *Clobber*.

The game *SC2* is a solitaire game whose rules are described in the following. Initially, black and white stones are placed on the vertices of a given graph G (one per vertex), forming what we call a *game configuration*. A move consists in picking a stone and "clobbering" (i.e.

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removing) another one of the opposite color located on an adjacent vertex. The clobbered stone is removed from the graph and is replaced by the picked one. The goal is to find a succession of moves that minimizes the number of remaining stones. A game configuration of *SC2* is said to be *k-reducible* if there exists a succession of moves that leaves at most k stones on the board. The *reducibility value* of a game configuration C is the smallest integer k for which C is k-reducible.

In [3], the game was investigated on cycles and trees. It is proved that in these cases, the reducibility value can be computed in quadratic/cubic time. In this paper, we play *SC2* on Hamming graphs.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *cartesian product* $G_1 \square G_2$ is the graph $G = (V, E)$ where $V = V_1 \times V_2$ and $(u_1u_2, v_1v_2) \in E$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. One generally depicts such a graph with $|V_2|$ vertical copies of G_1, and $|V_1|$ horizontal copies of G_2, as shown on Fig. 1.

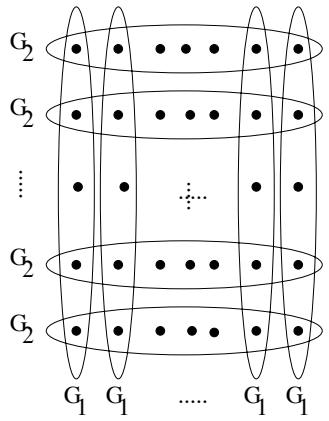

Figure 1: The cartesian product of two graphs G_1 and G_2

A *Hamming graph* is a multiple cartesian product of cliques. $K_2 \square K_3$ and $K_4 \square K_5 \square K_2$ are examples of Hamming graphs. Hypercubes, defined by $\square^n K_2$, constitute a well-known class of Hamming graphs.

For the convenience of the reader, we may often mix up a vertex and the stone that it supports. The label/color of a vertex will thus define the color of the stone on it. We may also say that "a vertex clobbers another one", instead of talking of the corresponding stones.

Given a game configuration C on a graph G, we say that a label/color c is *rare* on a subgraph S of G if there exists a unique vertex $v \in S$ such that v is labeled c. On the contrary, c is said to be *common* if there exist at least two vertices of this color in S. A configuration is said to be *monochromatic* if all the vertices have the same color. A monochromatic game configuration does not allow any move, so we now assume that a game configuration is never monochromatic.

Given v a vertex of G, the color of the stone on v will be denoted by $c(v)$. For a color c (black or white), we denote by $\overline{c}$ the other color.

In this paper, we prove that we can reduce any game configuration (non monochromatic) on a Hamming graph to one or two stones. Moreover, we assert that we can choose the color and the location of the remaining stones. To facilitate the proofs, we make the following three definitions.

Definition We say that a graph G is *strongly 1-reducible* if: for any vertex v, for any arrangement of the stones on G (provided $G \setminus v$ is not monochromatic), for any color c (black or white), there exists a way to play that yields a single stone of color c on v.

A joker move consists of changing the color of any stone at any time during the game. It can be used only once.

Therefore, a graph G is *strongly 1-reducible joker* if: for any vertex v, for any color c, for any arrangement of the stones on G (provided $c(v)$ is not rare or $c(v) = c$), there exists a way to play that yields a single stone of color c on v, with the possible use of a joker move.

Definition A graph G is said to be *strongly 2-reducible* if: for any vertex v, for any arrangement of the stones on G (provided $G \setminus v$ is not monochromatic), for any two colors c and c' (provided there exist two different vertices u and u' such that $c(u) = c$ and $c(u') = c'$), there exists a way to play that yields a stone of color c on v, and (possibly) a second stone of color c' somewhere else.

Definition Let G be a graph, v_i and v_j two vertices of G, c and c' two colors belonging to $\{0, 1\}$. A game configuration C on G is said to be *1-reducible on v_i with c* or $(1, v_i, c)$-*reducible* if there exists a way to play that yields only one stone of color c on G, located on v_i. A configuration C is said to be *2-reducible on v_i with c and c'* or $(2, v_i, c, c')$-*reducible* if there exists a way to play that yields a stone of color c on v_i, and (possibly) a second stone of color c' on some other vertex. C is said to be $(2, v_i, c, v_j, c')$-*reducible* if there exists a way to play that yields a stone of color c on v_i and a second stone of color c' on v_j.

In the next section, we solve the case of *SC2* played on cliques. We prove in Proposition 1 that any clique of size at least 3 is strongly 1-reducible.

In Section 3, we play the game on hypercubes. We prove in Theorem 5 that hypercubes are both strongly 1-reducible joker and strongly 2-reducible, the proofs are intertwined. We also prove in Proposition 6 that any hypercube has a non-monochromatic configuration for which it is not 1-reducible. This somehow stresses the relevance ot Theorem 5.

Finally, in Section 4, we prove in Theorem 12 that all the Hamming graphs except hypercubes and $K_2 \Box K_3$ are strongly 1-reducible. To prove this, we use a slightly stronger result in Theorem 8; we prove that if G is a strongly 1-reducible graph containing at least 4 vertices, then the Cartesian product of G with any clique is strongly 1-reducible.

2. *SC2* Played on Cliques

It is not very surprising that every game configuration on a clique is 1-reducible. Furthermore, we also prove that we can choose the color and the location of the single remaining stone.

Proposition 1. *Cliques of size $n \geq 3$ are strongly 1-reducible.*

When $n < 3$, note that cliques are 1-reducible, but we can't decide where and with which color we finish.

Proof. Let C be a game configuration on K_n ($n \geq 3$). Let v be a vertex of K_n such that $K_n \backslash v$ is not monochromatic. Let c be any color in $\{0, 1\}$. We prove that C is $(1, v, c)$-reducible:

First assume that C contains no rare color. We consider two cases:

* if $c = c(v)$. By hypothesis, there exists a vertex w labeled $\overline{c(v)}$. Since $c(v)$ and $c(w)$ are not rare, there exist two vertices v' and w' such that $c(v') = c(v)$ and $c(w') = c(w)$. The succession of moves leading to a single remaining stone is the following: w clobbers v, w' clobbers all the vertices with the label $c(v)$ except v', and finally, v' clobbers all the vertices labeled $\overline{c(v)}$, and ends on v.
* if $c = \overline{c(v)}$. As previously, there exist w labeled $\overline{c(v)}$ and v' labeled $c(v)$. v' clobbers all the vertices labeled $\overline{c(v)}$ except w. Then w clobbers all the vertices labeled $c(v)$ and ends on v.

Now assume that C has a rare color located on a vertex $v_r \neq v$. If $c = c(v_r)$, then it is enough to have v_r clobber all the vertices and finish on v. If $c = \overline{c(v_r)}$, have v_r clobber all the vertices except one (call it $v' \neq v$) and finish on v. Then have v' clobber v and this concludes the proof. □

3. *SC2* Played on Hypercubes

In this section, we study *SC2* on hypercubes. We prove that these graphs are strongly 2-reducible.

Let $n > 2$. Note that Q_n is defined recursively as the product $K_2 \square Q_{n-1}$, Q_0 being a single vertex. This means that Q_n is made of two copies Q_n^l and Q_n^r of Q_{n-1}, where each vertex of Q_n^l is adjacent to its copy in Q_n^r. Let $N = 2^{n-1}$. For each $i > 1$, it is well known that Q_i admits a Hamiltonian cycle. Denote by $v_1, \ldots, v_N$ the vertices of Q_n^l, ordered such that $(v_1, \ldots, v_N)$ form a Hamiltonian cycle. Denote by $v'_1, \ldots, v'_N$ the vertices of Q_n^r, such

that v_i is adjacent to v'_i for all i. Note that $(v'_1, \ldots, v'_N)$ forms a Hamiltonian cycle of Q^r_n. Here is the diagram of the hypercube Q_n that will be used in the rest of the paper:

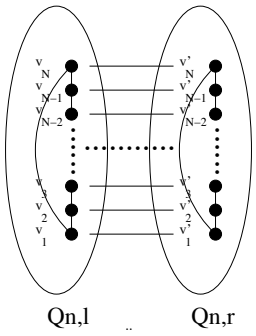

Figure 2: The hypercube Q_n

Let v_i be a vertex of Q_n. Note that when referring to v_{i+j}, where $(i+j)$ is not in $[1, N]$, then use the appropriate subscript $i + j \pm N$ instead.

The following lemmas describe the successions of moves used to reduce a game configuration to a certain form:

Lemma 2. *Let C be a game configuration on a Hamiltonian graph G with n vertices ($n > 2$). Let $(v_1, \ldots, v_n)$ be the list of the vertices ordered according to a Hamiltonian cycle of G. If there exists a vertex v_i such that $c(v_i)$ is rare on G, then C is both $(1, v_{i\pm1}, c(v_i))$-reducible and $(1, v_{i\pm2}, \overline{c(v_i)})$-reducible.*

Proof. The first reduction is obtained when v_i clobbers all the stones along the Hamiltonian cycle $(v_1, \ldots, v_N)$. According to the direction in which we move around the cycle, we end either on v_{i+1} or on v_{i-1}.

To get the second reduction, v_i clobbers all the stones along the Hamiltonian cycle, except the last one. This means that v_i finishes on v_{i+2} or v_{i-2}, and is then clobbered by v_{i+1} or v_{i-1} respectively. □

Lemma 3. *Let C be a game configuration on Q_n, with $n > 3$. If there exists a rare color on Q^r_n, and if Q^l_n is not monochromatic, then there exists a way to play that yields no stones on Q^r_n and N stones on Q^l_n, both colors being common on Q^l_n. If $n = 3$, there may be a rare color on Q^l_n, but we can choose its location on two distinct vertices.*

Proof. Let c be the rare color on Q^r_n and denote by v'_i the vertex such that $c(v'_i) = c$. We consider three cases for the stones on Q^l_n:

- $\overline{c}$ is rare on Q^l_n. Thanks to its Hamiltonian cycle and by Lemma 2, we know that Q^r_n is $(1, v'_{i\pm2}, \overline{c})$-reducible. If $n > 3$, v'_{i+2} and v'_{i-2} are distinct vertices. Also

since $\overline{c}$ is rare on Q_n^l, this means that either v_{i+2} or v_{i-2} is labeled with the color c. Without loss of generality, suppose that v_{i+2} is labeled c; hence we apply a $(1, v'_{i+2}, \overline{c})$-reduction of Q_n^r. Then v'_{i+2} clobbers v_{i+2}, so that Q_n^l contains at least two stones of each color afterwards.

If $n = 3$ and $c(v_{i+2}) = \overline{c}$, this proof is no longer valid. In that case, there are two ways to play, each of them leaving the rare color $\overline{c}$ either on v_{i+1} (diagram 1) or on v_{i-1} (diagram 2).

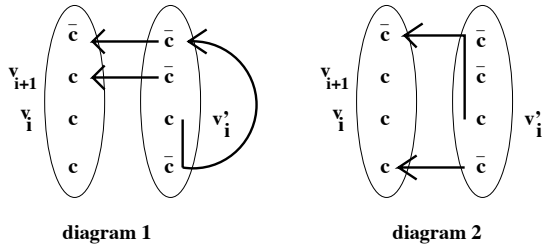

Figure 3: Lemma 3: special instance of the case $n = 3$

- c is rare on Q_n^l. By Lemma 2, Q_n^r is $(1, v'_{i\pm1}, c)$-reducible. We know that at least one of both vertices v_{i+1} and v_{i-1} has the common label $\overline{c}$. Without loss of generality, assume v_{i+1} does. Last, we apply a $(1, v'_{i+1}, c)$-reduction of Q_n^r, and then we play from v'_{i+1} to v_{i+1}.
- Both colors are common on Q_n^l. We consider the four cases for the labels of v_{i+1} and v_{i+2}:
  - $c(v_{i+1}) = c$ and $c(v_{i+2}) = \overline{c}$. Use a Hamiltonian cycle of Q_n^r to have v'_i clobber all the vertices except v'_{i+1}. This operation yields two stones on Q_n^r: v'_{i+1} labeled $\overline{c}$, and v'_{i+2} labeled c. Play now from v'_{i+1} to v_{i+1} and from v'_{i+2} to v_{i+2}.
  - $c(v_{i+1}) = \overline{c}$ and $c(v_{i+2}) = c$. If $n > 3$, c or c' appears more than twice in Q_n^l. If it is the case of $\overline{c}$, then apply a $(1, v'_{i+1}, c)$-reduction of Q_n^r, and play from v'_{i+1} to v_{i+1}. If c appears more than twice in Q_n^l, then apply a $(1, v'_{i+2}, \overline{c})$-reduction of Q_n^r, and play from v'_{i+2} to v_{i+2}. If $n = 3$, there are two possible arrangements of the stones on Q_n^l. In both cases, there exists a way to play that yields a rare color on Q_n^l, with two possible locations:
  - $c(v_{i+1}) = c$ and $c(v_{i+2}) = c$. If c appears more than twice in Q_n^l, then apply a $(1, v'_{i+2}, \overline{c})$-reduction of Q_n^r, and play from v'_{i+2} to v_{i+2}. Then play from v'_{i+2} to v_{i+2}. Otherwise, and if $n > 3$, this means that the color $\overline{c}$ appears more than twice, in particular on v_{i-1}. Then apply a $(1, v'_{i+1}, c)$-reduction of Q_n^r, and play from v'_{i-1} to v_{i-1}. If $n = 3$, this implies $c(v_i) = c(v_{i-1}) = \overline{c}$. It then suffices to invert the order of the vertices (v_{i+1} becomes v_{i-1}...) to reduce to the previous case.
  - $c(v_{i+1}) = \overline{c}$ and $c(v_{i+2}) = \overline{c}$. This case is similar to the previous one. □

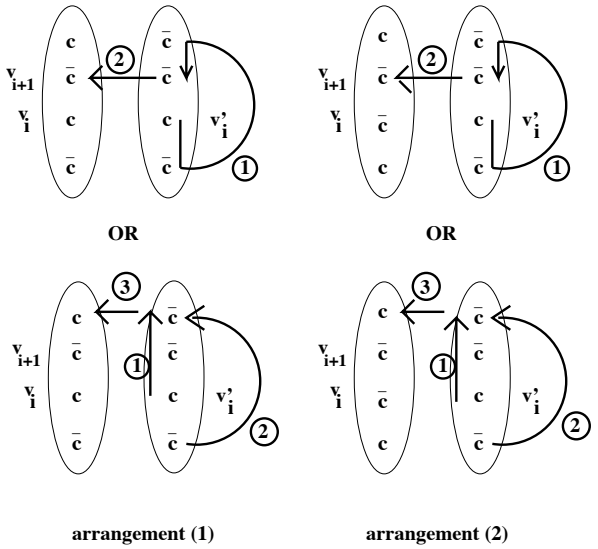

Figure 4: Lemma 3: special instances of the case $n = 3$ (2)

Lemma 4. *Let C be a game configuration on Q_n, with $n > 2$. If there exists a rare color on Q_n^r, and if Q_n^l is monochromatic, then there exists a way to play that yields no stones on Q_n^r and N stones on Q_n^l, which is not monochromatic. Also, if this operation yields a rare label on Q_n^l, we can choose its location on two distinct vertices.*

Proof. Let c be the rare color on Q_n^r and denote by v_i' the vertex such that $c(v_i') = c$. We consider two cases about Q_n^l:

- All the vertices of Q_n^l have the color c. Use a Hamiltonian cycle of Q_n^r to have v_i' clobber all the vertices except v_{i+1}' and v_{i+2}'. It ends on v_{i+3}'. Then v_{i+2}' clobbers v_{i+3}'. This operation yields two stones labeled $\overline{c}$ on v_{i+1}' and v_{i+3}'. Then play from v_{i+1}' to v_{i+1} and from v_{i+3}' to v_{i+3}. Both colors now appear at least twice on Q_n^l.
- All the vertices of Q_n^l have the color $\overline{c}$. By Lemma 2, we can apply a $(1, v_{i\pm1}', c)$-reduction of Q_n^r. Then play from v_{i+1}' or v_{i-1}' to the corresponding vertex in Q_n^l. In that case, the color c is rare on Q_n^l, but it can be located either on v_{i+1} or on v_{i-1}.

□

We now give the main result of this section about the "strong reducibility" of the hypercube.

Theorem 5. *Hypercubes are strongly 1-reducible joker and strongly 2-reducible.*

Of course, the most interesting property concerns the 2-reducibility of the hypercube. However, this result is tightly linked to the strong 1-reducibilty joker. One can notice

that the conditions defining the strong 2-reduction and the strong 1-reduction joker are a bit different. Indeed, the "vertex" condition of strong 2-reducibility (i.e. $G \setminus v$ must not be monochromatic) is contained in the condition of strong 1-reducibility joker. But monochromatic hypercubes and hypercubes with a rare color on v_r such that $c = c(v_r)$ are also strongly 1-reducible joker, although they are not strongly 2-reducible. This explains why the conditions of strong 1-reducibility joker are "larger".

Proof. We proceed via induction on the dimension of the hypercube. The reader can verify that these results are true on the hypercube Q_2 (the square). Note that only four arrangements of the stones must be considered:

Assume that the theorem is true for the hypercube Q_{n-1} and consider the hypercube Q_n.

Q_n `is strongly 1-reducible joker.`

Without loss of generality, assume that the vertex that will support the last stone is v_1. Let c be any color in $\{0, 1\}$. We consider any arrangement of the stones on Q_n such that $c(v_1)$ is not rare or $c(v_1) = c$. Our objective consists in finding a way to yield a single stone of color c on v_1. We are allowed to use a joker. Five cases are considered:

1. Suppose Q_n^l is $(1, v_1, c)$-reducible joker, and the joker is used to change the color of some vertex v_j from the color $d \in \{0, 1\}$ to $\overline{d}$. Also, we suppose that Q_n^r is $(1, v'_j, \overline{d})$-reducible joker.

   We first apply the $(1, v'_j, \overline{d})$-reduction joker on Q_n^r, which yields a stone of color $\overline{d}$ on v'_j. We may have used a joker to do this. Then we apply a $(1, v_1, c)$-reduction joker on Q_n^l with a small modification: instead of using the joker on v_j, we play from v'_j to v_j. This move is indeed equivalent to the use of the joker, since v'_j has the color $\overline{d}$ at this moment. At the end of the play, the joker has been used at most once.

2. Q_n^l is $(1, v_1, c)$-reducible joker, and the joker is used to change the color of some vertex v_j from the color $d \in \{0, 1\}$ to $\overline{d}$. Moreover, Q_n^r is not $(1, v'_j, \overline{d})$-reducible joker. From the conditions of the strong 1-reduction joker, this means that $c(v'_j) = d$, and $c(v'_i) = \overline{d}$ for all $i \neq j$.

   Since d is rare on Q_n^r, we can apply both Lemma 3 and 4. If this yields a rare color on Q_n^l, we choose a location different from v_1 for it. Hence $c(v_1)$ is never rare and we can apply a $(1, v_1, c)$-reduction joker on Q_n^l.

3. Q_n^l is $(1, v_1, c)$-reducible joker, but the joker is not used. We consider any arrangement of the stones on Q_n^r.

   We consider a succession of moves resulting from a $(1, v_1, c)$-reduction of Q_n^l. In this sequence, there exists a vertex v_i that clobbers at least two other vertices before being

(or not) clobbered. Indeed, if each vertex clobbers at most once, then Q_n^l would be a star, which is not the case. Denote by v_j and v_k the first two vertices clobbered by v_i. When the moves from v_i to v_j and then to v_k are made, let y be the color of v_i, and $\overline{y}$ the color of v_j and v_k. We consider four cases about the colors of v_i' and v_j':

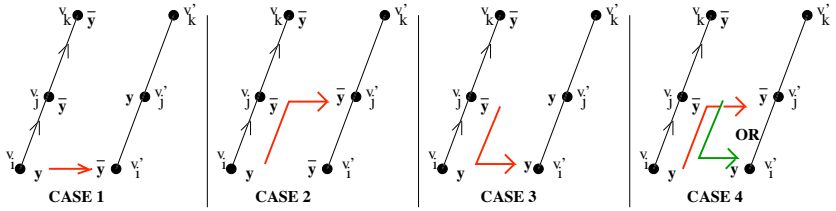

Figure 5: Q_n^l is 1-reducible on v_1 with c

- CASE 1: $c(v_i') = \overline{y}$ and $c(v_j') = y$. Apply a $(1, v_1, c)$-reduction of Q_n^l, and when the time comes to play from v_i to v_j, play to v_i' instead. At this moment, y is not rare on Q_n^r, so we can apply a $(1, v_j', y)$-reduction joker on Q_n^r. Play then from v_j' to v_j and continue the $(1, v_1, c)$-reduction of Q_n^l.
- CASE 2: $c(v_i') = c(v_j') = \overline{y}$. Begin a $(1, v_1, c)$-reduction of Q_n^l up to the move from v_j to v_k (not included). Play to v_j' instead. Since $c(v_k')$ is not rare, apply a $(1, v_k', y)$-reduction joker on Q_n^r. Then play from v_k' to v_k and continue the $(1, v_1, c)$-reduction of Q_n^l.
- CASE 3: $c(v_i') = c(v_j') = y$. Apply a $(1, v_1, c)$-reduction of Q_n^l up to the move from v_i to v_j (not included). Instead of it, have v_j clobber v_i and then v_i'. The rest of the play is identical to the previous case.
- CASE 4: $c(v_i') = y$ and $c(v_j') = \overline{y}$. If $c(v_k') = y$, then play as in the second case. Otherwise, play as in the third case.

4. Q_n^l is not $(1, v_1, c)$-reducible joker, and Q_n^r is $(2, v_1', c, \overline{c})$-reducible.

   This implies that $c(v_1) = \overline{c}$ and $c(v_i) = c$ for all $i > 1$. If Q_n^r is $(1, v_1', c)$-reducible, we apply this reduction and then play from v_1' to v_1. Q_n^l becomes monochromatic and the $(1, v_1, c)$-reduction joker can now be applied on it. If Q_n^r is $(2, v_1', c, \overline{c})$-reducible, then choose the second remaining stone of color $\overline{c}$. Let v_j' be the vertex on which this stone is left. Play now from v_1' to v_1, and from v_j' to v_j. Q_n^l now satisfies the right conditions to apply a $(1, v_1, c)$-reduction joker.

5. Q_n^l is not $(1, v_1, c)$-reducible joker, and Q_n^r is not $(2, v_1', c, \overline{c})$-reducible.

   There are four possible arrangements of the stones on Q_n corresponding to these conditions:

   - The arrangement (A) does not have to be considered. Indeed, this arrangement is not allowed by the conditions of the 1-reduction joker, since $c(v_1)$ is rare on Q_n and $c(v_1) \neq c$.

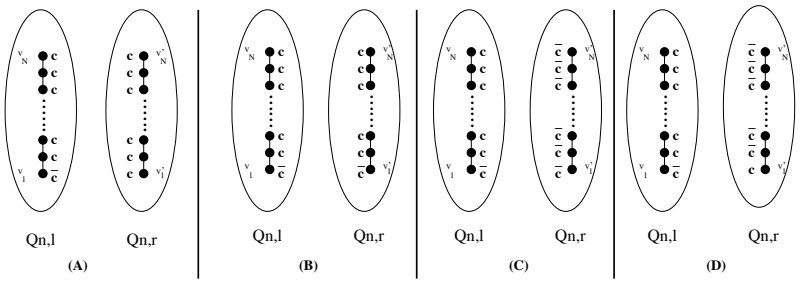

Figure 6: Strong 1-reducibility joker: case 5

- If the arrangement of the stones is (B), have v'_1 clobber all the vertices of Q^r_n and end on v'_N. Then v'_N clobbers v_N, and the conditions of a $(1, v_1, c)$-reduction joker are fulfilled on Q^l_n.
- If the arrangement of the stones is (C), have v_i clobber v'_i for all $2 < i < N$. Apply now a $(1, v'_1, \overline{c})$-reduction joker of Q^r_n. Finally, v_1 is clobbered by v_2, v'_1 and v_N in this order.
- If the stones are placed as in (D), use Lemma 2 to apply a $(1, v'_{N-1}, \overline{c})$-reduction of Q^r_n. Then v'_{N-1} clobbers v_{N-1}, and we can apply a $(1, v_1, c)$-reduction joker of Q^l_n.

Q_n is strongly 2-reducible.

Without loss of generality, assume that the vertex that will support the last stone is v_1. We consider any arrangement of the stones on Q_n such that $Q_n \setminus v_1$ is not monochromatic. Let c and c' be any two colors in $\{0, 1\}$ such that there are two distinct vertices of Q_n labeled with these values. Our objective consists in finding a way to leave a stone of color c on v_1, and possibly another one of color c' somewhere else. We consider eleven cases, starting with those where Q^r_n is monochromatic (cases 1 to 5):

1. Q^r_n is monochromatic of color $y \in \{0, 1\}$, and Q^l_n is $(1, v_1, c)$-reducible. Consider a succession of moves resulting from a $(1, v_1, c)$-reduction of Q^l_n. First suppose that there exists a move from a stone of color $\overline{y}$ on some vertex v_i clobbering a stone of color y on the vertex v_j. Replace this move by having v_i clobber v'_i. There exists an Hamiltonian cycle of Q^r_n where v'_i and v'_j are consecutive. Have v'_i clobber all the stones of Q^r_n and end on v'_j with the color $\overline{y}$. Finally v'_j clobbers v_j, and we can continue the $(1, v_1, c)$-reduction of Q^l_n.

   Suppose now that there exist no moves clobbering a vertex labeled y when applying a $(1, v_1, c)$-reduction of Q^l_n. Necessarily this means that $c = y$. Also, this implies that all the vertices of Q^l_n are labeled $\overline{y}$, except one, namely v_i. The $(1, v_1, c)$-reduction of Q^l_n thus consists in having v_i clobber all the vertices of Q^l_n and end on v_1. Without loss of generality, suppose that v_2 is the penultimate vertex which is clobbered when

applying the $(1, v_1, c)$-reduction of Q_n^l. The following diagram shows how to apply the $(1, v_1, c)$-reduction of Q_n:

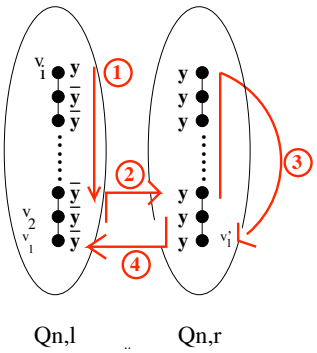

Figure 7: Strong 2-reducibility: specific instance of case 1

2. Q_n^r is monochromatic of color $y \in \{0, 1\}$, and Q_n^l is $(2, v_1, c, \overline{y})$-reducible.

   If Q_n^l is $(1, v_1, c)$-reducible, then we are in case 1. Suppose then that the reduction yields two stones, the second one being located on some vertex v_i. In that case, apply a $(2, v_1, c, v_i, \overline{y})$-reduction of Q_n^l and play from v_i to v_i'. Then use Lemma 2 to yield a stone of color c' either on v_{i+1}' (if $c' = \overline{y}$) or on v_{i+2}' (if $c' = y$).

   In cases 3, 4 and 5, we suppose that Q_n^l is not $(2, v_1, c, \overline{y})$-reducible. If Q_n^l is not $(2, v_1, c, \overline{y})$-reducible, then either $Q \setminus v_1$ is monochromatic, or $c = \overline{y}$ and $\overline{y}$ is rare in Q_n^l. But from our initial assumption that $Q_n \setminus v_1$ is not monochromatic, we know that there is at least one stone colored in $\overline{y}$ in $Q \setminus v_1$. So either $Q \setminus v_1$ is monochromatic of color $\overline{y}$ (see cases 4 and 5), or $\overline{y}$ is rare in Q_n^l and $c(v_1) \neq \overline{y}$ (see case 3).

3. Q_n^r is monochromatic of color $y \in \{0, 1\}$, and $\overline{y}$ is rare on Q_n^l with $c(v_1) \neq \overline{y}$. If Q_n^l is not $(2, v_1, c, \overline{y})$-reducible, then $c = \overline{y}$ and $c' = y$ (by our initial assumption that there are two distinct vertices of color c and c' respectively in Q_n). Let v_i be the vertex of Q_n^l such that $c(v_i) = \overline{y}$. See Fig.8 for the diagram of such a configuration.

   Since $c = \overline{y}$ and $c' = y$, Q_n^l is $(2, v_1, c, c')$-reducible. Consider the first move of this 2-reduction: it is a move from v_i to some v_j since $c(v_i)$ is rare. Instead of playing it, play from v_i to v_i', and then have v_i' clobber all the stones of Q_n^r and end on v_j'. Then play from v_j' to v_j and continue the $(2, v_1, c, c')$-reduction of Q_n^l to conclude this part of the proof.

4. Q_n^r is monochromatic of color $y \in \{0, 1\}$ and $c(v_1) = y$ is rare on Q_n^l (see Fig. 9).

   We first consider the case $c = y$. For all $2 \leq i \leq N$, play from v_i to v_i'. Then use an Hamiltonian cycle of Q_n^r to yield the second stone of the right color c' (on v_N or v_{N-1} according to c') after having clobbered all the other vertices of Q_n^r.

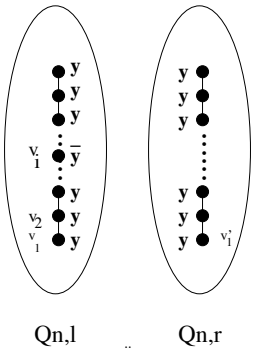

Figure 8: Strong 2-reducibility: case 3

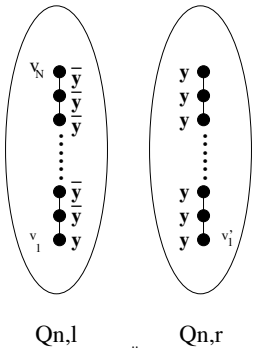

Figure 9: Strong 2-reducibility: case 4

If $c = \overline{y}$, then first v_N clobbers v_1. Then v_i clobbers v'_i for all $3 \leq i \leq N-1$. We apply a $(2, v'_1, y, c')$-reduction of Q_n^r. The last two moves are v'_1 to v_1, and v_2 to v_1.

5. Q_n^r is monochromatic of color $y \in \{0, 1\}$ and Q_n^l is monochromatic of color $\overline{y}$.

   We first consider the case when $c = y$. Play from v_N to v'_N and from v'_{N-1} to v_{N-1}. Then use a Hamiltonian cycle of Q_n^r to clobber all its vertices and yield a stone of color c' on Q_n^r. Finally, have v_{N-1} clobber all the stones of Q_n^l and end on v_1.

   If $c = \overline{y}$, play from v'_1 to v_1, and then from v_2 to v_1. Have v_i clobber v'_i for all $2 < i \leq N$. Use a Hamiltonian cycle to reduce Q_n^r to a single stone of color c'.

   In the next cases, we suppose that Q_n^r is not monochromatic.

6. Q_n^l is $(1, v_1, c)$-reducible, and Q_n^r has a rare color.

   Apply a $(1, v_1, c)$-reduction of Q_n^l and use a Hamiltonian cycle to reduce Q_n^r to a single stone of color c' on v'_{i+1} or v'_{i+2}.

7. Q_n^l is $(1, v_1, c)$-reducible and both colors are common on Q_n^r.

   We consider a sequence of moves resulting from a $(1, v_1, c)$-reduction of Q_n^l. In this sequence, there exists a vertex v_i that clobbers at least two other vertices before being (or not) clobbered. Denote by v_j and v_k the first two vertices clobbered by v_i. When considering the moves from v_i to v_j and then to v_k, let y be the color of v_i, and $\overline{y}$ the color of v_j and v_k. We consider four cases according to the colors of v'_i and v'_j:

   - CASE 1: $c(v'_i) = \overline{y}$ and $c(v'_j) = y$. Apply a $(1, v_1, c)$-reduction of Q_n^l until the move from v_i to v_j (not included). Play now from v_i to v'_i, and from v_j to v'_j instead. After this operation, both colors are still common on Q_n^r, so that we can apply a $(2, v'_k, y, c')$-reduction. Then play from v'_k to v_k, and continue the $(1, v_1, c)$-reduction of Q_n^l.
   - CASE 2: $c(v'_i) = c(v'_j) = \overline{y}$. Apply a $(1, v_1, c)$-reduction of Q_n^l, and when the time comes to play from v_j to v_k, play to v'_j instead. Since y is not rare on

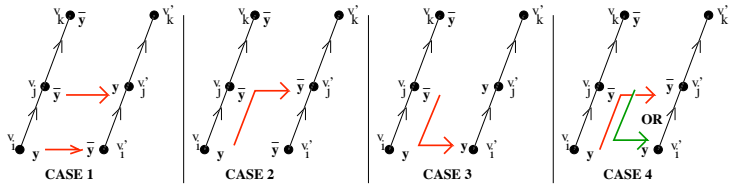

Figure 10: Strong 2-reducibility: case 7

Q_n^r after this operation, apply a $(2, v'_k, y, c')$-reduction of Q_n^r. After this, play from v'_k to v_k and continue the $(1, v_1, c)$-reduction of Q_n^l.

- CASE 3: $c(v'_i) = c(v'_j) = y$. Apply a $(1, v_1, c)$-reduction of Q_n^l until the move from v_i to v_j (not included). Instead of it, have v_j clobber v_i and then v'_i. If y is not rare on Q_n^r after this operation, then apply a $(2, v'_k, y, c')$-reduction of Q_n^r. If y is rare on Q_n^r, then use a Hamiltonian path of Q_n^r starting on v'_j and ending on v'_k to yield a stone of color y on v'_k.

  After this, play from v'_k to v_k and continue the $(1, v_1, c)$-reduction of Q_n^l.
- CASE 4: $c(v'_i) = y$ and $c(v'_j) = \overline{y}$. If the color $\overline{y}$ appears more than twice in Q_n^r, or if $c(v'_k) = y$, then play as in the second case. Otherwise, this means that $c(v'_j) = c(v'_k) = \overline{y}$ and the other vertices of Q_n^r have the color y. Play thus as in the third case.

In the next two cases, we suppose that $c(v_1)$ is not rare on Q_n^l (which may be monochromatic). Hence Q_n^l is $(1, v_1, c)$-reducible joker. If this reduction does not use the joker, then refer to case 6 or 7. Otherwise, assume that the joker is used to change the color of some vertex v_j from d to $\overline{d}$.

8. If Q_n^r is $(2, v'_j, \overline{d}, c')$-reducible, we first apply a $(2, v'_j, \overline{d}, c')$-reduction of Q_n^r. We then apply a $(1, v_1, c)$-reduction joker of Q_n^l, and when the time comes to use the joker, we play from v'_j to v_j instead.

9. Suppose that Q_n^r is not $(2, v'_j, \overline{d}, c')$-reducible. By our earlier assumption, Q_n^r is not monochromatic, so this can occur in only three kinds of arrangements of the stones on Q_n^r, all with a rare color. The case when Q_n^l is monochromatic is studied in case 10, we assume in this section that Q_n^l is not monochromatic.

   - $c(v'_j) \neq \overline{d}$, $\overline{d}$ is rare on Q_n^r and $c' = \overline{d}$. If $n > 3$, then use Lemma 3 to empty Q_n^r and yield N stones on Q_n^l where both colors are common. Then we can apply a $(2, v_1, c, c')$-reduction of Q_n^l.

     If $n = 3$, the lemma can not be used. We thus have to consider all the configurations on Q_3 satisfying these conditions. Figure 11 details these five configurations (the final colors c and c' are detailed under each diagram):
   - $c(v'_j) = \overline{d}$, and $\overline{d}$ is rare on Q_n^r. If $n > 3$, we play as in the previous case. When $n = 3$, here are the configurations that must be considered:

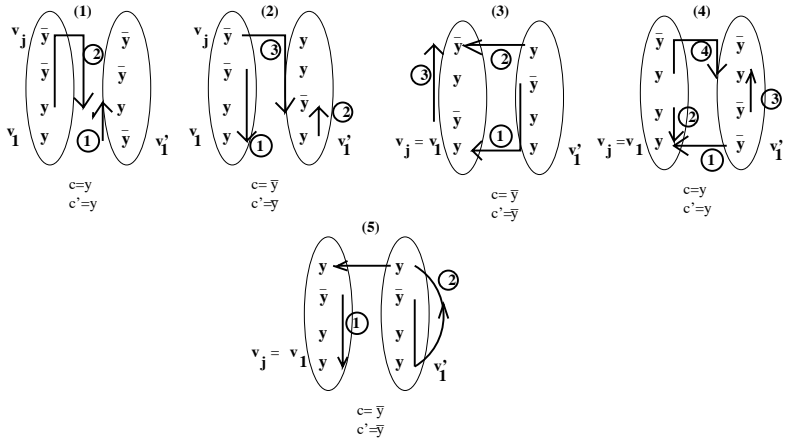

Figure 11: Case 9: arrangements on Q_3 (1)

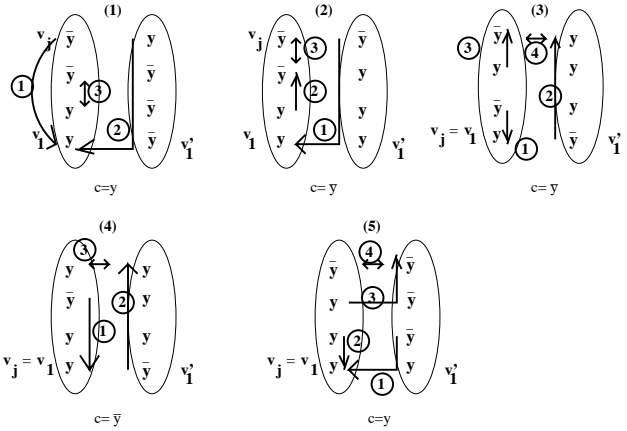

Figure 12: Case 9: arrangements on Q_3 (2)

- d is rare on Q_n^r and $c(v_j') = d$. If $n > 3$, we play as in the previous case. If $n = 3$, here are the configurations that must be considered:

10. Assume that $c(v_1) = \overline{y}$ is rare on Q_n^l or that Q_n^l is monochromatic, and that Q_n^r has a rare label. This induces four possible cases:

    - CASE 1: We suppose that $c(v_1) = \overline{y}$ is rare on Q_n^l and Q_n^r. Let v_i' be the vertex such that $c(v_i') = \overline{y}$. Either v_{i+1}' or v_{i-1}' (or both) is different from v_1'. Without loss of generality, assume v_{i+1}' is. Apply a $(1, v_{i+1}', \overline{y})$-reduction of Q_n^r in the way of Lemma 2. Then play from v_{i+1}' to v_{i+1}. Both colors are now common on Q_n^l, which becomes $(2, v_1, c, c')$-reducible.
    - CASE 2: $c(v_1) = \overline{y}$ is rare on Q_n^l and y is rare on some vertex v_i' of Q_n^r. By Lemma 2, apply a $(1, v_{i\pm2}', \overline{y})$-reduction of Q_n^r (choose to finish on a vertex different from v_1'). Play then as in the previous case. This operation is not possible if $n = 3$ and when the arrangement of the stones is the following:

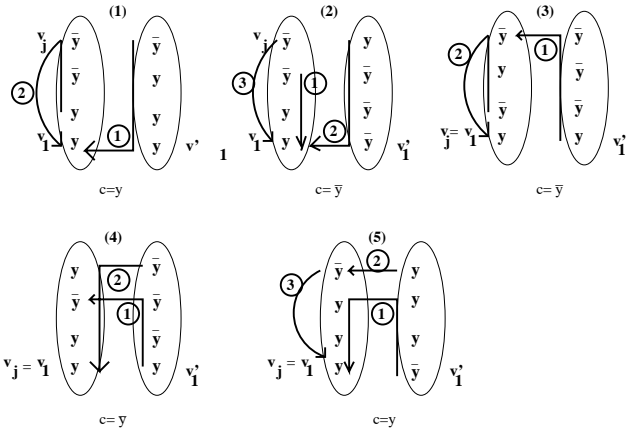

Figure 13: Case 9: arrangements on Q_3 (3)

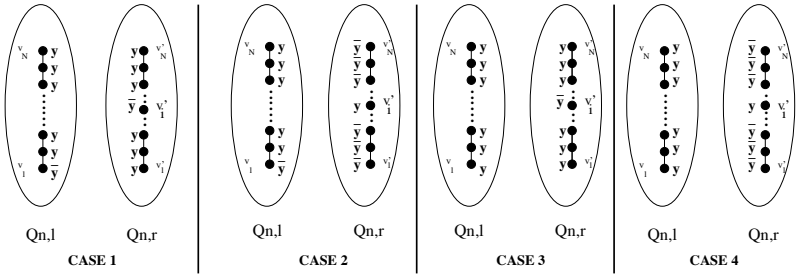

Figure 14: Possible arrangements in case 10

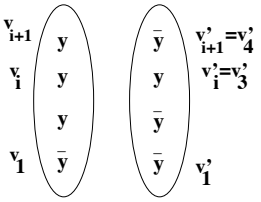

Figure 15: Special instance of the case 10.2

In that case, if $(c, c') \neq (y, y)$, then consider the following succession of moves: v'_{i+1} to v_{i+1}, v'_i to v'_2, v'_1 to v'_2, v'_2 to v_2. Use then a Hamiltonian cycle of Q^l_n to conclude. If $(c, c') = (y, y)$, then play like this: Use a Hamiltonian cycle of Q^r_n to apply a $(1, v'_1, \overline{y})$-reduction. Then move from v_2 to v_1, from v'_1 to v_1, and from v_N to v_1.

- CASE 3: Q^l_n is monochromatic of color y and $\overline{y}$ is rare on some v'_i of Q^r_n. This case is identical to the first case (note that $c = c' = \overline{y}$ is not allowed since $\overline{y}$ is rare on Q_n).
- CASE 4: Q^l_n is monochromatic of color y and y is rare on some v'_i. Have v'_i clobber all the vertices of Q^r_n except v'_{i+1} and v'_{i+2},and end on v'_{i+3}. Then play from v'_{i+2} to v'_{i+3}, from v'_{i+3} to v_{i+3}, and from v'_{i+1} to v_{i+1}. All the stones

of Q_n^r have been removed and both colors are now common on Q_n^l. Apply now a $(2, v_1, c, c')$-reduction of Q_n^l.

11. Assume that $c(v_1) = \overline{y}$ is rare on Q_n^l and that both colors are common on Q_n^r.

    If Q_n^r is $(1, v'_{N-1}, \overline{y})$-reducible, then apply this reduction and move from v'_{N-1} to v_{N-1}. Both colors are now common on Q_n^l, and we can conclude to the right result.

    Otherwise, Q_n^r is 2-reducible on v'_{N-1} with $\overline{y}$, and $\overline{y}$ on some other vertex called v'_i. Apply this reduction. If $v'_i \neq v'_1$, move from v'_{N-1} to v_{N-1}, and from v'_i to v_i. If $n > 3$, then both colors are common on Q_n^l, and we can conclude the proof. If $n = 3$, then y is rare on Q_n^l, and located either on v_2, or on v_N. Clobbering along the Hamiltonian cycle of Q_n^l permits a 2-reduction.

    If $v'_i = v'_1$, we distinguish two cases. If $c = y$, then play from v_2 to v_1, v'_1 to v_1 and v_N to v_1. Then have v'_{N-1} clobber v_{N-1} and follow a Hamiltonian cycle of Q_n^l to leave the last stone of color c'. If $c = \overline{y}$, then play from v_N to v_1, and from v'_1 to v_1. Have v'_{N-1} clobber v_{N-1} and use a Hamiltonian cycle of Q_n^l to leave the last stone of color c'. □

This theorem ensures that hypercubes are 2-reducible. Besides, as next proposition shows, non 1-reducible configurations exist. We use to prove it the invariant δ given by Demaine et al. in [2], defined below.

Proposition 6. *For each integer n, there exists a non-monochromatic configuration on Q_n which is not 1-reducible.*

Proof. We prove this result thanks to the invariant defined by Demaine et al. in [2]. On a bipartite graph G, vertices of both partitions are respectively labeled '0' and '1'. Now consider a game configuration C of Solitaire Clobber on G, with stones labeled '0' and '1'. A stone is said to be "clashing" if its label differs from the label of the vertex it occupies. Denote by $\delta(C)$ the following quantity:

$$\delta(C) = \text{number of stones plus number of clashing stones.}$$

In their paper, Demaine et al. proved that $\delta(C) \pmod 3$ never changes during the game.

Let $n > 1$ and consider $Q_n = Q_{n-1} \square K_2$. As previously, denote by Q_n^l and Q_n^r both copies of Q_{n-1}. Hypercubes are bipartite graphs. Choose a bipartition of Q_n such that half the vertices of Q_n^l are labeled '0', and the other ones are labeled '1'. Ditto for Q_n^r. Now choose an arrangement of the stones on Q_n such that all the stones labeled '0' belong to Q_n^l, and all the stones labeled '1' belong to Q_n^r. In that case, we have

$$\delta(C) = 2^n + 2^{n-1} = 3 \cdot 2^{n-1}$$

Hence $\delta(C)\ (mod\ 3) = 0$. Since a single stone configuration never satisfies $\delta(C) \pmod 3 = 0$ (see [2]), this concludes the proof. □

Proposition 6 shows that our result is sharp. Nevertheless, it is still an open problem to determine if a given configuration in a hypercube satisfying $\delta = 1$ is 1-reducible.

4. On the Other Hamming Graphs

Hypercubes are strongly 2-reducible. In this section, we prove that *almost* all the other Hamming graphs are strongly 1-reducible. This induction is initialized by Lemmas 10 and 11, and the property is proved to be hereditary by Theorem 8.

In the following, we prove that the cartesian product of a strongly 1-reducible graph G with a clique K_n is strongly 1-reducible. This product contains n copies of G, that we denote by $G_1, \ldots, G_n$. For any vertex v of G, we denote by v_i the corresponding vertex in the copy G_i. Then, denote by v_1 any vertex of G_1.

Lemma 7. *Let G be a strongly 1-reducible graph containing at least 4 vertices. $K_2 \square G$ is strongly 1-reducible.*

Proof. Let G be a strongly 1-reducible graph with at least 4 vertices. Without loss of generality, assume that the vertex on which we will leave the last stone is v_1. Let c be any color in $\{0, 1\}$. We consider any arrangement of the stones on $K_2 \square G$ such that $K_2 \square G \setminus v_1$ is not monochromatic. Let us prove that $K_2 \square G$ is $(1, v_1, c)$-reducible. We split the problem into three cases.

1. **G_2 is not monochromatic.**
   Since G is of size at least 4, there exist 2 vertices of the same color in $G_1 \setminus v_1$. We denote them by a_1 and b_1. Similarly, $c(a_2)$ or $c(b_2)$ (or both) is common in G_2. Without loss of generality, we suppose $c(a_2)$ is. One applies a $(1, a_2, \overline{c(a_1)})$-reduction of G_2, and then have a_2 clobber a_1. G_2 is now empty. a_1 and b_1 are now of different colors on G_1, so we can apply a $(1, v_1, c)$-reduction of G_1.

2. **G_2 is monochromatic of color y and $G_1 \setminus v_1$ is not monochromatic.**

   This means that G_1 is $(1, v_1, c)$-reducible. We consider two cases:

   - Suppose that when one applies a $(1, v_1, c)$-reduction of G_1, there exists a vertex a_1 colored in $\overline{y}$ clobbering another vertex b_1 of color y. We then choose to apply this reduction, and when the time comes to play from a_1 to b_1, play to a_2 instead. We then apply a $(1, b_2, \overline{y})$-reduction of Q_2. b_2 then clobbers b_1 and we can continue the $(1, v_1, c)$-reduction of G_1.
   - Otherwise, there is exactly one vertex a_1 colored in y in G_1. Since there are at least 4 vertices in G_1, a_1 has to clobber consecutively 2 vertices during the $(1, v_1, c)$-reduction of G_1. Denote them by b_1 and c_1. We replace these two consecutive moves by these ones: b_1 clobbers a_1 and then a_2. We then apply a $(1, c_2, y)$-reduction of G_2. It finally suffices to play from c_2 to c_1, and continue the $(1, v_1, c)$-reduction of G_1.

3. **G_2 is monochromatic of color y and $G_1 \setminus v_1$ is monochromatic.**
Since $K_2 \square G \setminus v_1$ is not monochromatic, $G_1 \setminus v_1$ is necessarily colored $\overline{y}$. Let a_1 be any vertex of G_1 different from v_1. Act now as if a_1 was colored y. We can thus consider a $(1, v_1, c)$-reduction of G_1. The first step of such a reduction would be "a_1 clobbers some vertex b_1." We use this reduction, replacing this step by "a_1 (which is actually colored $\overline{y}$) clobbers a_2, then we do a $(1, b_2, y)$-reduction of G_2, followed by b_2 clobbers b_1". □

Theorem 8. *Let G be a strongly 1-reducible graph containing at least 4 vertices. Then for any positive integer n, $K_n \square G$ is strongly 1-reducible.*

Proof. Let G be a strongly 1-reducible graph with at least 4 vertices. We prove the theorem by induction on n. If $n = 2$, see Lemma 7. Suppose $n \geq 3$ and $K_{n-1} \square G$ is strongly 1-reducible. Without loss of generality, assume that the vertex on which we will leave the last stone is v_1. Let c be any color in $\{0, 1\}$. We consider any arrangement of the stones on $K_2 \square G$ such that $K_2 \square G \setminus v_1$ is not monochromatic. Let us give a $(1, v_1, c)$-reduction of $K_n \square G$.

We consider 3 different cases:

1. **There exists $i \in [2 \dots n]$ such that G_i is not monochromatic.**
Since G contains at least 4 vertices, there are 2 vertices a_i and b_i such that $G_i \setminus \{a_i, b_i\}$ is not monochromatic. For the same reasons, in any other copy G_j, $c(a_j)$ or $c(b_j)$ (or both) is not rare. Without loss of generality, we can suppose that $c(a_j)$ is common on G_j. Start by applying a $(1, a_i, \overline{c(a_j)})$-reduction of G_i, and then play from a_i to a_j. We can proceed with a $(1, v_1, c)$-reduction of the remaining non monochromatic $K_{n-1} \square G$.

2. **For all $i \in [2 \dots n]$, G_i is monochromatic of color y.**
If G_n is deleted from the graph, then the configuration is $(1, v_1, c)$-reductible according to the induction hypothesis. In this reduction, there exists a move from some a_i to some b_i of color y, where $1 < i < n$. When considering the graph with G_n, we apply the $(1, v_1, c)$-reduction as if G_n was not there. And when the time comes to play from a_i to b_i, we play to a_n instead. We then do a $(1, b_n, \overline{y})$-reduction of G_n and have b_n clobber b_i. We can finally continue the execution of the $(1, v_1, c)$-reduction.

3. **For all $i \in [2 \dots n]$, G_i is monochromatic, but all the copies do not have the same color.**
Let y be the color of some vertex of $G_1 \setminus v_1$. Let G_i $(i > 1)$ be a copy of color y and G_j $(j > 1)$ a copy of color $\overline{y}$. We start by having all the vertices of G_j clobber the corresponding vertices of G_i. Hence there remains a $K_{n-1} \square G$ where $K_{n-1} \square G \setminus v_1$ is not monochromatic. We can apply the induction hypothesis to conclude the proof. □

With these results, we can assert that any Hamming graph containing a K_4 is strongly 1-reducible. What about Hamming graphs that are the product of K_2 and K_3 only?

We begin by studying configurations on $K_2 \square K_3$. Such a graph will be considered as two adjacent copies G_1 and G_2 of K_3.

Lemma 9. *Let $G = K_3 \square K_2$ and $i \in \{1, 2\}$. For any vertex a_i of G, for any color $c \in \{0, 1\}$ and for any configuration C on G such that: (i) $c(a_i)$ is not rare on G_i and (ii) $K_3 \square K_2 \setminus a_i$ is not monochromatic, C is $(1, a_i, c)$-reducible.*

Proof. For $i \in \{1, 2\}$, let v_i, u_i, and w_i be the vertices of each copy G_i. Without loss of generality, assume that we will leave the last stone on v_1. By (i), one may assume that v_1 and u_1 have the same color y. Let $c \in \{0, 1\}$. Our goal is now to prove that any configuration satisfying (i) and (ii) is $(1, v_1, c)$-reducible. We consider several cases:

- $c(w_1) = y$ and G_2 is not monochromatic. By Proposition 1, G_2 is either $(1, u_2, \overline{y})$-reducible, or $(1, w_2, \overline{y})$-reducible. Without loss of generality, suppose that G_2 is $(1, u_2, \overline{y})$-reducible. Apply this reduction and play from u_2 to u_1. The conditions are now fulfilled on the clique G_1 to apply a $(1, v_1, c)$-reduction.
- $c(w_1) = y$ and G_2 is monochromatic. From (ii), G_2 is monochromatic of color $\overline{y}$. According to c, play as shown on diagrams (a) $(c = y)$ or (b) $(c = \overline{y})$ of Figure 16.
- $c(w_1) = \overline{y}$ and G_2 is $(1, v_2, \overline{y})$-reducible. Apply this reduction, and then play from v_2 to v_1. Now G_1 is $(1, v_1, c)$-reducible by Proposition 1.
- $c(w_1) = \overline{y}$ and G_2 is monochromatic. Play according to Figure 16. On diagrams (c) and (e), we have $c = y$. On diagrams (d) and (f), we end with the color $c = \overline{y}$.
- $c(w_1) = \overline{y}$ and $c(v_2)$ is rare on G_2. In both cases, we play from v_2 either to u_2 or to w_2, such that $c(u_2) \neq c(u_1)$ and $c(w_2) \neq c(w_1)$ after this operation. We then play from u_2 to u_1, and from w_2 to w_1. Use Proposition 1 to apply a $(1, v_1, c)$-reduction of G_1.

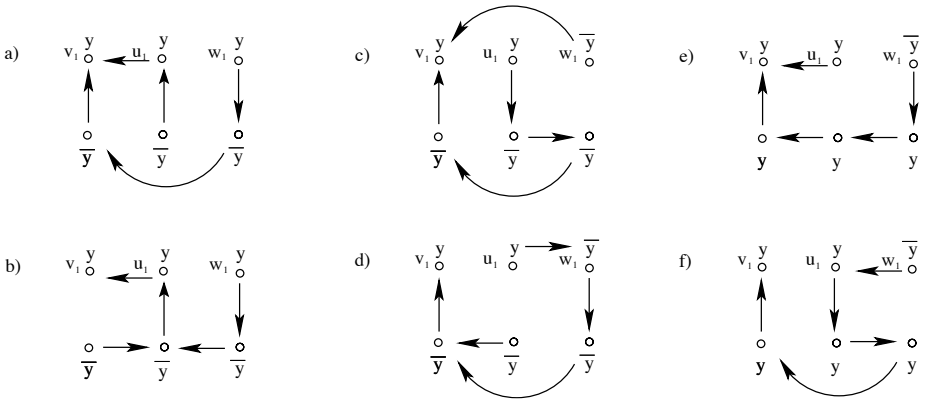

Figure 16: reduction of $K_2 \square K_3$

□

Lemma 10. *$K_3 \square K_3$ is strongly 1-reducible.*

Proof. Let us consider the graph $K_3 \square K_3$, v_1 being any vertex of it. Assume that we will leave the last stone on v_1. Let $c \in \{0, 1\}$. We consider any arrangement of the stones such that $K_3 \square K_3 \setminus v_1$ is not monochromatic. Let us prove that this configuration is $(1, v_1, c)$-reducible.

Among the six copies of K_3 constituting the product $K_3 \square K_3$ (three horizontal and three vertical), one of them is not monochromatic and does not contain v_1: call it G_3. Denote by G_1 the parallel copy of G_3 containing v_1, and G_2 the last parallel copy. G_3 is then 1-reducible with any color on two possible vertices: a_3 and b_3. At least one of these is different from v_3 (v_3 being the copy of v_1 in G_3). Without loss of generality, assume $a_3 \neq v_3$.

If $G_1 \setminus v_1$ is not monochromatic, we apply a $(1, a_3, \overline{c(a_2)})$-reduction of G_3 and then play from a_3 to a_2. Otherwise, we apply a $(1, a_3, \overline{c(a_1)})$-reduction of G_3 and then play from a_3 to a_1. In both cases, we finally get a configuration on $K_2 \square K_3$ that we can reduce from Lemma 9. □

Lemma 11. *$K_3 \square K_2 \square K_2$ is strongly 1-reducible.*

Proof. Consider the graph $K_3 \square K_2 \square K_2$. Let v_1 be any vertex of it and let c be any color. Assume that we will leave the last stone on v_1. We consider any arrangement of the stones such that $K_3 \square K_2 \square K_2 \setminus v_1$ is not monochromatic.

Let G_1 be the copy of K_3 containing v_1. We call G_2, G_3, and G_4 the other copies of K_3, G_3 being the copy containing no neighbour of v_1. We distinguish two cases:

- **The graph without G_1 is not monochromatic**
  There exists a non monochromatic copy of $K_2 \square K_3$ that does not contain G_1. Without loss of generality, suppose it is the one made of G_3 and G_4. We can 1-reduce it to various places.

  We first suppose that both vertices a_1 and b_1 of $G_1 \setminus v_1$ have the same color. At least one of the corresponding vertex a_4 and b_4 in G_4 has a common color in G_4. Assume it is the case of a_4. The conditions of Lemma 9 are fulfilled so that we are able to apply a $(1, a_4, \overline{c(a_1)})$-reduction of $G_3 \cup G_4$; then we have a_4 clobber a_1. Now, $G_1 \cup G_2 \setminus v_1$ is not monochromatic, and $c(v_1)$ is common on G_1. By Lemma 9, $G_1 \cup G_2$ is $(1, v_1, c)$-reducible.

  Suppose now that the vertices a_1 and b_1 of $G_1 \setminus v_1$ have different colors. At least one vertex of a_3 and b_3 has a common color in G_3. Assume it is a_3. The conditions of Lemma 9 are fulfilled to apply a $(1, a_3, \overline{c(a_2)})$-reduction of $G_3 \cup G_4$; then have a_3 clobber a_2. Now, $G_1 \cup G_2 \setminus v_1$ is not monochromatic, and $c(v_1)$ is common on G_1. By Lemma 9, $G_1 \cup G_2$ is $(1, v_1, c)$-reducible.

- **The graph without G_1 is monochromatic of color y**
  Then $G_1 \setminus v_1$ contains a stone of color $\overline{y}$. Denote by z the initial color of v_1. We describe the way to play on Figure 17.

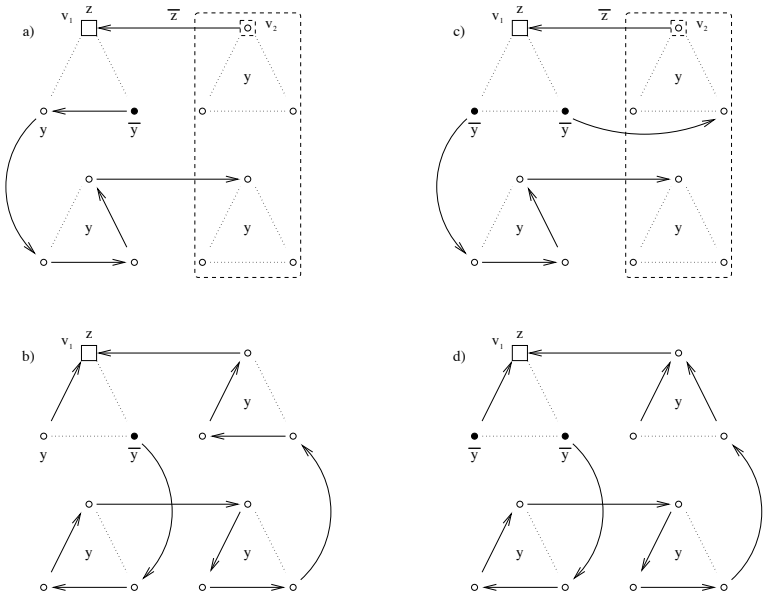

Figure 17: 1-reduction of $K_3 \square K_2 \square K_2$

In cases (a) and (c), we have $c = \overline{z}$. We execute the moves described by the figure, leaving v_1 and a copy of $K_2 \square K_3$. We can apply a $(1, v_2, \overline{z})$-reduction of this copy (from Lemma 9), and conclude by playing from v_2 to v_1. In cases (b) and (d), we have $c = z$. Just follow the moves on the figure as soon as they are possible. □

From all these results, we can deduce the following theorem about Hamming graphs.

Theorem 12. *Any Hamming graph that is neither $K_2 \square K_3$ nor a hypercube is strongly 1-reducible.*

Note that $K_2 \square K_3$ is 1-reducible for any coloration, and is also strongly 1-reducible joker.

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