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KONANE HAS INFINITE NIM-DIMENSION

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Abstract

Berlekamp asked the question "What is the habitat of $*2$?" We generalize this to ask: For a game G, what is the largest n such that $*n$ is a position in G? and answer this for KONANE by presenting a method to construct any nimber.

1. Introduction

In combinatorial game theory, games can be expressed recursively as $G = \{G^L \,|\, G^R\}$ where G^L are the Left options and G^R are the Right options of G. An example of a combinatorial game is the classic game of NIM, first studied by C. Bouton [3]. This game is played with piles of stones. On his turn, each player can remove any number of stones from any pile. The winner is the player who takes the last stone. Many variants of NIM have been studied (see [2] for several). NIM is an example of an impartial game: Left options and Right options are the same for the game and all its followers. The values involved in NIM are called nimbers (stars):

$$*k = \{0, *, ..., *(k-1) \,|\, 0, *, ..., *(k-1)\}$$

It is a surprising fact that all impartial games take only nimbers as values (see [1, 2]).

Many combinatorial games have $G^L \neq G^R$, i.e. the two players have different moves, like in CHESS or GO. These games are are called *partizan*. Berlekamp, [6] **problem 45**, asks *which partizan game have the position* $*2$. We generalize this to ask which impartial games appear in a partizan games? Or: *Given a partizan game, which nimbers occur in the game?*

If all nimbers up to $*2^n$ have been found in a game then the disjunctive sum allows construction of all nimbers up to $*(2^{n+1} - 1)$. This motivates us to define a Nim-dimension.

Definition 1. *A combinatorial game has* Nim-dimension n *if it contains a position* $*2^{n-1}$ *but not* $*2^n$*. A game has* infinite *nim-dimension if all the nimbers can be constructed. It has* null*, or* $\emptyset$*, nim-dimension if* $*$ *cannot be constructed.*

In the game of SHOVE, a player shoves one of his pieces and all other pieces on the left, to the left one square, possibly off the end of the board. For example,

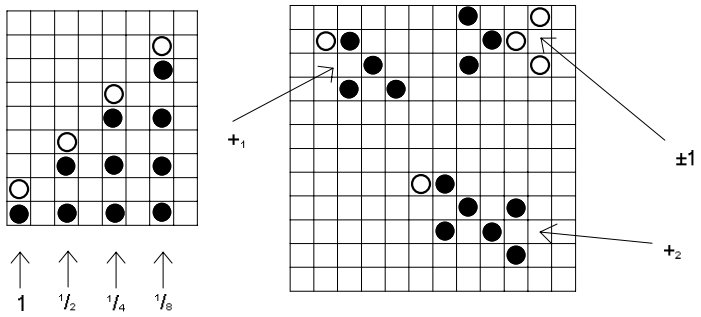
COL is played on a graph with uncolored vertices; a player colors a vertex in their color but cannot color two adjacent vertices the same. TOPPLING DOMINOES is played with a row of black and white dominoes. A player topples, to the left or right, one of their dominoes and it topples all the dominoes in that direction. For example,

- NimDimension(SHOVE) $= \emptyset$ since all the values are numbers (see [1]);
- NimDimension(COL) $= 0$ since all the values are numbers or numbers plus $*$ (see [2]);
- NimDimension(TOPPLING DOMINOES) $= \infty$ since it is easy to show that

We note that even for impartial games, determining the Nim-dimension is still an active question. In all *octal games* — heaps games like NIM except a heap may also be split, see [1, 2]) — is the Nim-dimension finite? (See [6] **problem 2**.)

In [6] **problem 44**, Berlekamp and Ernst ask for any further analysis of KONANE. Indeed, only two mathematical papers [5, 4] have appeared. The first paper restricted the moves to single jumps. The second looked at 1-dimensional boards.

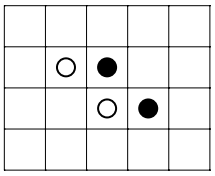
KONANE is an hawaiian game. In the starting position the checkered board is filled in such a way that no two stones of the same color occupy adjacent squares. In the opening, two adjacent pieces of the board are removed. After this, a player moves by taking one of his stones and jumping orthogonally over an opposing stone into an empty square. The jumped stone is removed. A player can make multiple jumps on his turn but cannot change direction mid-turn. Multiple jumps are not mandatory. The winner is the player who makes the last move. KONANE is a very rich combinatorial game with many interesting values. We can see some in the following figure (in this paper, Left plays with black stones):



This paper is about the construction of arbitrary nimbers in KONANE and thereby proving

Theorem 2. *Nim-dimension(*KONANE*)*$= \infty$.

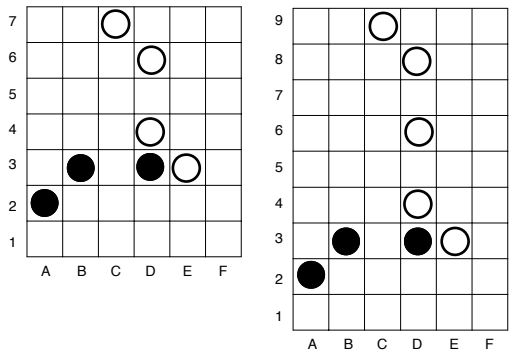
Although we prove that KONANE has infinite nim dimension we do not attempt to find the smallest board size on which $*n$ first occurs. For instance



has value $*2$, but it is not useful in our construction of $*3$ (see the next section).

2. Constructing Nimbers in KONANE

We will see that KONANE has infinite Nim-dimension. To understand the idea behind the process to construct nimbers in KONANE, let us start with the analysis of 2 positions:



In the left position, if Left plays D3-D7 then he goes to a position with value $*$. If Left plays D3-F3 then he goes to a position with value 0. More interesting is the move D3-D5. This option is dominated, because the position after D3-D5 is negative (the position after D3-D5 is $\{*\,|\,0\} = \downarrow$). The Right options are easy: E3-C3 or D4-D2 goes to 0 and E3-A3 goes to $*$. So, the position in the left diagram is a $*2$:

$$\{D3-F3, D3-D7\,|\,E3-C3 \text{ (or } D4-D2), E3-A3\} = \{0, *\,|\,0, *\} = *2$$

In the right diagram, Left has a supplementary option. However, like D3-D5 in previous example, the option is dominated. So the right position is also a $*2$. If we want we can join any number of stones to the column without changing the game value. With this kind of idea we have a process to *gain space on the board*.

If we look at left position, all Left moves must be made by the stone at D3. In similar cases we will say that D3 is the *focal point* and the stone at D3 the *focal stone*.

2.1. Algorithm to Construct a $*n$ in KONANE

We will construct a $*n$ using the previous $*(n-1)$. Let $*2$ be realized by the left diagram in the previous figure. The dimension of the board is 7×6 and the *focal point* has coordinates $(5, 4)$.

Let $l_n \times c_n$ be the dimensions of the $*n$'s board and (a_n, b_n) be the coordinates of the $*n$'s *focal point*. The base case is the $*2$ ($l_2 = 7, c_2 = 6, a_2 = 5, b_2 = 4$). The construction of the $*n$ is described as follows.

`Board dimension`

1. $l_n = l_{n-1} + c_{n-1} + 2$;
2. $c_n = l_{n-1} + 3$;

`Transformation of previous nimber`

3. In the $*(n-1)$, remove the black stone from the *focal point* and add joining white stones in squares with coordinates $(a_{n-1} + 2k - 1, b_{n-1}), k \geqslant 1 \quad \text{and} \quad a_{n-1} + 2k - 1 \leqslant l_{n-1}$. This suppresses all possible moves (item 7 will estipulate the new *focal point*);

`Laying down the pieces`

4. Put the position obtained in 3 on the $*n$'s area (see 1 and 2) on the rectangle with vertices $(1, c_n - c_{n-1} + 1), (1, c_n), (l_{n-1}, c_n - c_{n-1} + 1), (l_{n-1}, c_n)$;
5. Turn black in white and white in black in the position obtained in 3. Rotate 90° and put this on the $*n$'s rectangle with vertices $(l_{n-1}+2, 1), (l_{n-1}+2, l_{n-1}), (l_n - 1, 1), (l_n - 1, l_{n-1})$;

Linking the pieces together

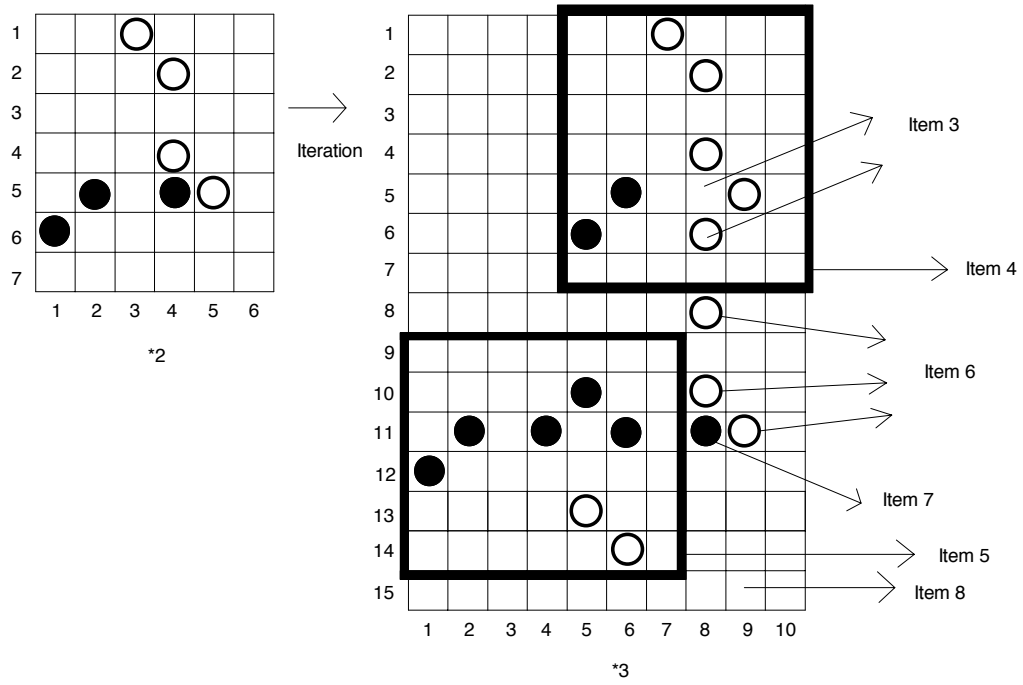
6. Put white stones at

   $(l_{n-1}+1, c_n-2), (l_{n-1}+3, c_n-2), (l_{n-1}+4, c_n-1)$;

7. Put a black stone at $(l_{n-1}+4, c_n-2)$ and this is the new *focal point* $(a_n = l_{n-1}+4; b_n = c_n-2)$;

8. The l_n $^{\text{th}}$ line is empty.

To illustrate the algorithm consider the following figure:



2.2. Proof (correctness of the algorithm)

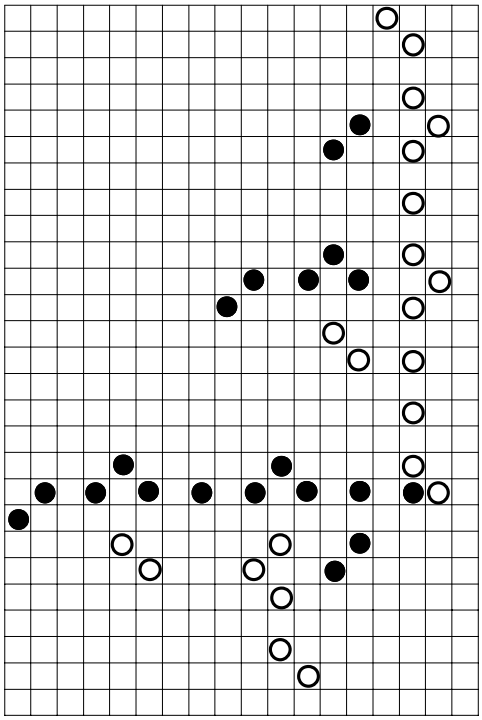
Let's analyze the Left options (the argument for the Right options is similar). The first move must be made from the *focal point*. Suppose we move the *focal stone* to a square (x, b_n) with $x \leqslant a_{n-1}$. Then, by induction, we are in a known situation, because the rectangle defined in the item 4 becomes independent of the rest of the position. By construction, for all n, the last line and the last column are empty, so, there are 3 empty lines between the rectangle and the rest of position (which is sterile) and this justifies the independence. The initial position is fuzzy, so the dominated negative options in the $*(n-1)$ are still dominated.

The mentioned move must go to $*(n-1)$ when it goes to (a_{n-1}, b_n) or to $*, *2, ..., *(n-2)$ (previous $*(n-1)$ options).

Now, suppose we move the *focal stone* to a square (x, b_n) with $x > a_{n-1}$, as Left loses the opportunity to take right, the value becomes negative. So these options are also dominated.

Finally, in the initial position, if Left goes right, then he goes to a position with value 0. The undominated options are $0, *, ..., *(n-1)$. □

An example of a position with value $*4$:



*4

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