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GENERALIZED CAUCHY DETERMINANTS

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Abstract

This paper classifies the sequences that satisfy a generalization of the Cauchy determinant formula. They are the generalized Fibonacci numbers, up to a scalar multiple. Following this, it is determined which of these sequences generate Hankel matrices of unit fractions with integer inverses. As a corollary we obtain another proof that the Filbert Matrix has an inverse with integer entries, complementing proofs using WZ theory and orthogonal polynomials.

1. Introduction

The first goal of this paper is to classify the sequences that satisfy a particular generalization of the Cauchy determinant formula. Let $f : \mathbb{Z} \to \mathbb{C}$, let $x = (x_0, x_1, \ldots x_n) \in \mathbb{Z}^{n+1}$ and $y = (y_0, y_1, \ldots y_n) \in \mathbb{Z}^{n+1}$ be finite sequences of integers, and let $M(f, x, y)$ be the $(n+1)\times(n+1)$ matrix with (i, j) entry given by $1/f(x_i + y_j)$. We classify functions f for which

$$\det(M(f, x, y)) = q^{g_n(x,y)} \frac{\prod_{0\le i<j\le n} f(x_j - x_i) f(y_j - y_i)}{\prod_{0\le i,j\le n} f(x_i + y_j)}, \tag{1}$$

where q is a constant that depends only on f, and g_n is an affine function.

Given a function f that satisfies equation (1), define the matrix $H(f, n, s)$ by $H(f, n, s) = M(f, (0, 1, \ldots, n-1), (s+1, \ldots, s+n))$, where s is a non-negative integer. The second goal of this paper is to determine for which of the functions f that satisfy equation (1) and have integer values does the inverse of $H(f, n, s)$ have integer entries for all n and s?

One motivation for considering this problem is the Filbert matrix, which is the special case of $H(f, n, 0)$ when $f(m)$ is the m^{th} Fibonacci number. The inverse of the Filbert matrix has been shown to have integer entries using WZ theory [6] and orthogonal polynomials [3].

As a result of the second part of this paper, we provide another proof using the generalized Cauchy determinant.

Various generalizations of the Fibonacci numbers have been shown to generate matrices of unit fractions whose inverses have integer entries [6], [3], [4]. The class of sequences which we determine in the second part includes all of these generalizations, and others. In addition, the generalized Cauchy determinant formula applies to the quantum Hilbert matrices studied by Andersen and Berg[2].

2. The Cauchy Determinant Formula

The Cauchy determinant formula says that the determinant of the matrix with (i, j) entry $1/(x_i + y_j)$ is

$$\frac{\prod_{0\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{0\le i,j\le n} x_i + y_j}.$$

For the matrices $M(f, x, y)$ that we consider, the simplest generalization of the Cauchy determinant formula would be $\det(M(f, x, y)) = \frac{\prod_{0\le i<j\le n} f(x_j-x_i)f(y_j-y_i)}{\prod_{0\le i,j\le n} f(x_i+y_j)}$. This generalization is too simple to apply even to the Filbert matrix, since the determinant of the Filbert matrix has an additional factor that is a power of -1. Thus we consider the generalization of equation (1). We want to find all functions f that satisfy this equation for all n and all $x, y \in \mathbb{Z}^{n+1}$.

The affine functions g_n in the exponent of q in equation (1) are dependent on the size of the matrix $M(f, x, y)$. Trivially, we see that $g_0(x_0, y_0) = 0$. We will determine the exact formula for g_n in what follows. By symmetry, we can assume that the coefficient of x_i equals the coefficient of y_i. Thus we can write $g_n(x, y) = a_{n,-1} + a_{n,0}(x_0 + y_0) + a_{n,1}(x_1 + y_1) + \ldots + a_{n,n}(x_n + y_n)$. It would be interesting to investigate the functions that satisfy a generalized Cauchy determinant formula for other forms of g besides affine.

Let f be a function that satisfies equation (1). Observe that since any matrix with a repeated row or column has a zero determinant, f must satisfy $f(0) = 0$.

3. Generalized Fibonacci Numbers and 2×2 Matrices

Continuing to assume that $f : \mathbb{Z} \to \mathbb{C}$ satisfies equation (1), consider the matrix based on the vectors $x = (0, 1)$ and $y = (1, m)$, where $m \neq 0$ and $m \neq -1$. For conciseness in referring to the coefficients of g_1, we let $a_{-1} = a_{1,-1}$, $a_0 = a_{1,0}$, and $a_1 = a_{1,1}$. Computing directly we

have

$$\det(M(f,x,y)) = \frac{f(2)f(m) - f(1)f(m+1)}{f(1)f(2)f(m)f(m+1)}, \tag{2}$$

while equation (1) implies that

$$\det(M(f,x,y)) = q^{a_{-1}+a_0+a_1(m+1)}\frac{f(1)f(m-1)}{f(1)f(2)f(m)f(m+1)}. \tag{3}$$

Equating numerators of equations (2) and (3), and solving for $f(m+1)$ gives

$$f(m+1) = \frac{f(2)}{f(1)}f(m) - q^{a_{-1}+a_0+a_1(m+1)}f(m-1). \tag{4}$$

Next, we consider a number of two by two matrices with $x_i + y_j \leq 6$, and the corresponding equations that result. The determinant formula gives

$$\frac{f(x_0+y_1)f(x_1+y_0) - f(x_0+y_0)f(x_1+y_1)}{f(x_0+y_0)f(x_0+y_1)f(x_1+y_0)f(x_1+y_1)} = \frac{q^{a_{-1}+a_0(x_0+y_0)+a_1(x_1+y_1)}f(x_1-x_0)f(y_1-y_0)}{f(x_0+y_0)f(x_0+y_1)f(x_1+y_0)f(x_1+y_1)}, \tag{5}$$

and equating numerators gives

$$f(x_0+y_1)f(x_1+y_0) - f(x_0+y_0)f(x_1+y_1) = q^{a_{-1}+a_0(x_0+y_0)+a_1(x_1+y_1)}f(x_1-x_0)f(y_1-y_0). \tag{6}$$

Choosing appropriate values for x_0, x_1, y_0 and y_1 gives the eight equations

$$f(2)f(2) - f(1)f(3) = q^{a_{-1}+a_0+3a_1}f(1)f(1) \tag{7}$$
$$f(2)f(3) - f(1)f(4) = q^{a_{-1}+a_0+4a_1}f(1)f(2) \tag{8}$$
$$f(2)f(4) - f(1)f(5) = q^{a_{-1}+a_0+5a_1}f(1)f(3) \tag{9}$$
$$f(2)f(5) - f(1)f(6) = q^{a_{-1}+a_0+6a_1}f(1)f(4) \tag{10}$$
$$f(3)f(3) - f(2)f(4) = q^{a_{-1}+2a_0+4a_1}f(1)f(1) \tag{11}$$
$$f(3)f(4) - f(2)f(5) = q^{a_{-1}+2a_0+5a_1}f(1)f(2) \tag{12}$$
$$f(3)f(3) - f(1)f(5) = q^{a_{-1}+a_0+5a_1}f(2)f(2) \tag{13}$$
$$f(3)f(4) - f(1)f(6) = q^{a_{-1}+a_0+6a_1}f(2)f(3). \tag{14}$$

For reference, the values of x_0, x_1, y_0 and y_1 that give these equations are $x_0 = 0$ for all equations; and, respectively, $(x_1, y_0, y_1) = (1,1,2)$ for equation (7), $(x_1, y_0, y_1) = (1,1,3)$ for equation (8), $(x_1, y_0, y_1) = (1,1,4)$ for equation (9), $(x_1, y_0, y_1) = (1,1,5)$ for equation (10),

$(x_1, y_0, y_1) = (1, 2, 3)$ for equation (11), $(x_1, y_0, y_1) = (1, 2, 4)$ for equation (12), $(x_1, y_0, y_1) = (2, 1, 3)$ for equation (13), and $(x_1, y_0, y_1) = (2, 1, 4)$ for equation (14).

Subtracting equation (9) from (13) gives

$$f(3)f(3) - f(2)f(4) = q^{a_{-1}+a_0+5a_1}(f(2)f(2) - f(1)f(3)). \tag{15}$$

Substituting the right-hand sides of equations (11) and (7) for $f(3)f(3) - f(2)f(4)$ and $f(2)f(2) - f(1)f(3)$, respectively, gives

$$q^{a_{-1}+2a_0+4a_1}f(1)f(1) = q^{a_{-1}+a_0+5a_1}q^{a_{-1}+a_0+3a_1}f(1)f(1). \tag{16}$$

Cancelling $f(1)$ gives

$$q^{a_{-1}+2a_0+4a_1} = q^{2a_{-1}+2a_0+8a_1}, \tag{17}$$

which is equivalent to

$$q^{a_{-1}+4a_1} = 1. \tag{18}$$

Subtracting equation (10) from equation (14) gives

$$f(3)f(4) - f(2)f(5) = q^{a_{-1}+a_0+6a_1}(f(2)f(3) - f(1)f(4)). \tag{19}$$

Substituting the right-hand sides of equations (12) and (8) for $f(3)f(4) - f(2)f(5)$ and $f(2)f(3) - f(1)f(4)$, respectively, gives

$$q^{a_{-1}+2a_0+5a_1}f(1)f(2) = q^{a_{-1}+a_0+6a_1}q^{a_{-1}+a_0+4a_1}f(1)f(2). \tag{20}$$

This gives

$$q^{a_{-1}+2a_0+5a_1} = q^{2a_{-1}+2a_0+10a_1}, \tag{21}$$

which is equivalent to

$$q^{a_{-1}+5a_1} = 1. \tag{22}$$

Equations (18) and (22) now imply that $q^{a_{-1}} = 1$ and $q^{a_1} = 1$. This allows us to rewrite equation (4) as

$$f(m+1) = \frac{f(2)}{f(1)}f(m) - q^{a_0}f(m-1). \tag{23}$$

Now let $k = \frac{f(2)}{f(1)}$, and replace q^{a_0} with q to make equation (23) equivalent to

$$f(m+1) = kf(m) - qf(m-1). \tag{24}$$

The observations about a_{-1} and a_1, and the substitution of q for q^{a_0} imply that we may assume that $g_1((x_0, x_1), (y_0, y_1)) = x_0 + y_0$.

At the beginning of this section we assumed that $m \neq 0$ and $m \neq -1$. Thus we have not shown than equation (24) holds for $m = 0, -1$. We now establish it for those values.

Recall the vectors $x = (0, 1)$ and $y = (1, m)$, and consider $M(f, x, z)$, where $z = (m, 1)$. Equation (1) implies that

$$\det(M(f, x, z)) = q^m \frac{f(1)f(1-m)}{f(1)f(2)f(m)f(m+1)}. \tag{25}$$

Since $M(f, x, z)$ is obtained from $M(f, x, y)$ by a column swap, we also have

$$\det(M(f, x, z)) = -\det(M(f, x, y)) = -q\frac{f(1)f(m-1)}{f(1)f(2)f(m)f(m+1)}. \tag{26}$$

Equating the right-hand sides of equations (25) and (26) and cancelling common factors gives

$$f(1-m) = -\frac{f(m-1)}{q^{m-1}}, \tag{27}$$

for all m such that $m \neq 0$ and $m \neq -1$. This is equivalent to

$$f(-m) = -\frac{f(m)}{q^m}, \tag{28}$$

for all $m \neq -1$ and $m \neq -2$, in particular, for all non-negative m. Now equation (24) for $m \neq 0$ and $m \neq -1$, and equation(28) imply that equation (24) holds for all $m \in \mathbb{Z}$.

This now implies that we can normalize the sequence $f(m)$ to have the form $f(m) = f(1)f_0(m)$, where $f_0(1) = 1$ and f_0 also satisfies equation (24) for all $m \in \mathbb{Z}$. Since $f(0) = 0$, as we previously observed, the sequence $f_0(m)$ for $m \geq 0$ consists of what Kalman and Mena call generalized Fibonacci numbers [5].

Summarizing, we have just proved the following theorem:

Theorem 1. *Assume that $f : \mathbb{Z} \to \mathbb{C}$, and that for all $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$ with $f(x_i + y_j) \neq 0$, the matrix $M(f, x, y)$ is defined by $M(f, x, y)_{i,j} = 1/f(x_i + y_j)$. Assume that there exists a constant q such that f satisfies equation (1). Then there exist constants k and q such that f is defined by $f(0) = 0$, $f(1) \neq 0$, and $f(m+1) = kf(m) - qf(m-1)$.*

4. The Generalized Formula for 2×2 Matrices

Now let k and q be constants, and let $f : \mathbb{Z} \to \mathbb{C}$ be defined by $f(0) = 0$, $f(1) = 1$, and $f(m+1) = kf(m) - qf(m-1)$. We show that f satisfies the generalized Cauchy determinant formula for all 2×2 matrices of the form $M(f, (x_0, x_1), (y_0, y_1))$. This follows from the identity

$$f(x_0 + y_1)f(x_1 + y_0) - f(x_0 + y_0)f(x_1 + y_1) = q^{x_0+y_0} f(x_1 - x_0)f(y_1 - y_0). \tag{29}$$

Equation (29) can be proved using either the Binet formula for generalized Fibonacci numbers, see equation (9) in [5], or the convolution property for generalized Fibonacci numbers, see equation (19) in [5]. We prove it using the convolution formula. The convolution property,

$$f(n) = f(m)f(n-m+1) - qf(m-1)f(n-m), \tag{30}$$

gives us

$$f(x_0+y_1)f(x_1+y_0) = qf(x_0+y_1-1)f(x_1+y_0-1) + f(x_0+y_0+x_1+y_1-1) \tag{31}$$

and

$$f(x_0+y_0)f(x_1+y_1) = qf(x_0+y_0-1)f(x_1+y_1-1) + f(x_0+y_0+x_1+y_1-1). \tag{32}$$

Thus we get

$$\begin{aligned} f(x_0+y_1)f(x_1+y_0) - f(x_0+y_0)f(x_1+y_1) \\ = q(f(x_0+y_1-1)f(x_1+y_0-1) - f(x_0+y_0-1)f(x_1+y_1-1)). \end{aligned} \tag{33}$$

Repeating this x_0+y_0 times we get

$$\begin{aligned} f(x_0+y_1)f(x_1+y_0) - f(x_0+y_0)f(x_1+y_1) \\ =& q^{x_0+y_0}(f(y_1-y_0)f(x_1-x_0) - f(0)f(x_1+y_1-x_0-y_0)) \\ =& q^{x_0+y_0}f(y_1-y_0)f(x_1-x_0), \end{aligned} \tag{34}$$

the last equality following from $f(0) = 0$.

We remark that equation (29) implies that $g_1(f,x,y) = x_0+y_0$, and that in particular g_1 is not just affine, but linear.

5. The Generalized Formula for $n \times n$ Matrices

Now we show that generalized Fibonacci numbers satisfy the generalized Cauchy determinant formula for all $n \times n$ matrices of the form $M(f,x,y)$, using the Dodgson condensation method [1]. We have verified the formula for 1×1 and 2×2 matrices.

Assume that M is an $(n+1)\times(n+1)$ matrix, indexed from 0 to n. Write $M(r\ldots s, t\ldots u)$ for the submatrix of M consisting of the entries $M_{i,j}$ where $r \le i \le s$ and $t \le j \le u$. The determinant identity of Dodgson that we use is

$$\begin{aligned} \det(M)\det(M(1\ldots n-1,1\ldots n-1)) =& \det(M(0\ldots n-1,0\ldots n-1))\det(M(1\ldots n,1\ldots n)) \\ &- \det(M(0\ldots n-1,1\ldots n))\det(M(1\ldots n,0\ldots n-1)). \end{aligned} \tag{35}$$

For the matrix $M(f,x,y)$, this equation becomes

$$\begin{aligned}&\det(M(f,x,y))\det(M(f,(x_1,\ldots,x_{n-1}),(y_1,\ldots x_{n-1})))\\&\quad=\det(M(f,(x_0,\ldots,x_{n-1}),(y_0,\ldots,y_{n-1}))\det(M(f,(x_1,\ldots,x_n),(y_1,\ldots,y_n)))\\&\qquad-\det(M(f,(x_0,\ldots,x_{n-1}),(y_1,\ldots,y_n)))\det(M(f,(x_1,\ldots,x_n),(y_0,\ldots y_{n-1}))).\end{aligned}\tag{36}$$

We will use induction to show both that the Cauchy determinant formula holds for $n+1\times n+1$ matrices $M(f,x,y)$, and that g_n is in fact linear, not just affine. By induction on n, we can apply the Cauchy determinant formula to all the terms in equation (36), except for $\det(M(f,x,y))$, giving

$$\begin{aligned}&\det(M(f,x,y))q^{g_{n-2}((x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1}))}\frac{\prod_{1\le i<j\le n-1}f(x_j-x_i)f(y_j-y_i)}{\prod_{1\le i\le n-1,1\le j\le n-1}f(x_i+y_j)}\\&\quad=q^{g_{n-1}((x_0,\ldots,x_{n-1}),(y_0,\ldots,y_{n-1}))}\frac{\prod_{0\le i<j\le n-1}f(x_j-x_i)f(y_j-y_i)}{\prod_{0\le i\le n-1,0\le j\le n-1}f(x_i+y_j)}\\&\qquad\times q^{g_{n-1}((x_1,\ldots,x_n),(y_1,\ldots,y_n))}\frac{\prod_{1\le i<j\le n}f(x_j-x_i)f(y_j-y_i)}{\prod_{1\le i\le n,1\le j\le n}f(x_i+y_j)}\\&\quad-q^{g_{n-1}((x_0,\ldots,x_{n-1}),(y_1,\ldots,y_n))}\frac{\prod_{0\le i<j\le n-1}f(x_j-x_i)\prod_{1\le i<j\le n}f(y_j-y_i)}{\prod_{0\le i\le n-1,1\le j\le n}f(x_i+y_j)}\\&\qquad\times q^{g_{n-1}((x_1,\ldots,x_n),(y_0,\ldots,y_{n-1}))}\frac{\prod_{1\le i<j\le n}f(x_j-x_i)\prod_{0\le i<j\le n-1}f(y_j-y_i)}{\prod_{1\le i\le n,0\le j\le n-1}f(x_i+y_j)}.\end{aligned}\tag{37}$$

By the inductive hypothesis that g_{n-1} and g_{n-2} are linear, we can combine all the powers of q on the right-hand side to get

$$\begin{aligned}&\det(M(f,x,y))q^{g_{n-2}((x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1}))}\frac{\prod_{1\le i\le n-1,1\le j\le n-1}f(x_j-x_i)f(y_j-y_i)}{\prod_{1\le i\le n-1,1\le j\le n-1}f(x_i+y_j)}\\&=q^{g_{n-1}((x_0,\ldots,x_{n-1})+(x_1,\ldots,x_n),(y_0,\ldots,y_{n-1})+(y_1,\ldots,y_n))}\\&\quad\times\left(\frac{\prod_{0\le i<j\le n-1}f(x_j-x_i)f(y_j-y_i)}{\prod_{0\le i\le n-1,0\le j\le n-1}f(x_i+y_j)}\cdot\frac{\prod_{1\le i<j\le n}f(x_j-x_i)f(y_j-y_i)}{\prod_{1\le i\le n,1\le j\le n}f(x_i+y_j)}\right.\\&\quad\left.-\frac{\prod_{0\le i<j\le n-1}f(x_j-x_i)\prod_{1\le i<j\le n}f(y_j-y_i)}{\prod_{0\le i\le n-1,1\le j\le n}f(x_i+y_j)}\cdot\frac{\prod_{1\le i<j\le n}f(x_j-x_i)\prod_{0\le i<j\le n-1}f(y_j-y_i)}{\prod_{1\le i\le n,0\le j\le n-1}f(x_i+y_j)}\right).\end{aligned}\tag{38}$$

Now paying careful attention to the indices of the products, we get

$$\det(M(f,x,y))q^{g_{n-2}((x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1}))} = q^{g_{n-1}((x_0,\ldots,x_{n-1})+(x_1,\ldots,x_n),(y_0,\ldots,y_{n-1})+(y_1,\ldots,y_n))} \times \left(\frac{\prod_{0<j\le n-1} f(x_j-x_0)\prod_{0<i<j\le n} f(x_j-x_i)\prod_{0<j\le n-1} f(y_j-y_0)\prod_{0<i<j\le n} f(y_j-y_i)}{\prod_{0\le i,j\le n} f(x_i+y_j)}\right) \times \big(f(x_0+y_n)f(x_n+y_0)-f(x_0+y_0)f(x_n+y_n)\big). \quad (39)$$

Again using the convolution formula (or the Binet formula) we have

$$f(x_0+y_n)f(x_n+y_0)-f(x_0+y_0)f(x_n+y_n) = q^{x_0+y_0}f(y_n-y_0)f(x_n-x_0). \quad (40)$$

This results in the formula

$$\det(M(f,x,y)) = q^{g_n(x,y)}\frac{\prod_{0\le i<j\le n} f(x_j-x_i)f(y_j-y_i)}{\prod_{0\le i,j\le n} f(x_i+y_j)}, \quad (41)$$

where

$$\begin{aligned} g_n(x,y) = {} & x_0+y_0+g_{n-1}((x_0,\ldots,x_{n-1})+(x_1,\ldots,x_n),(y_0,\ldots,y_{n-1})+(y_1,\ldots,y_n)) \\ & -g_{n-2}((x_1,\ldots,x_{n-1}),(y_1,\ldots,y_{n-1})). \end{aligned} \quad (42)$$

Since $g_0(x_0,y_0)=0$, and we have already established $g_1((x_0,y_0),(x_1,y_1))=x_0+y_0$, we find that

$$g_n(x,y) = n(x_0+y_0)+(n-1)(x_1+y_1)+(n-2)(x_2+y_2)+\ldots+(x_{n-1}+y_{n-1}). \quad (43)$$

This establishes our next theorem.

Theorem 2. *Assume that $f:\mathbb{Z}\to\mathbb{C}$, and there exist constants k and q such that f is defined by $f(0)=0$, $f(1)\neq 0$, and $f(m+1)=kf(m)-qf(m-1)$. Assume that for all $x=(x_0,\ldots,x_n)$ and $y=(y_0,\ldots,y_n)$ with $f(x_i+y_j)\neq 0$, the matrix $M(f,x,y)$ is defined by $M(f,x,y)_{i,j}=1/f(x_i+y_j)$. Then f satisfies equation (1). Further, the functions g_n in equation (1) take the form of equation (43).*

Combining Theorem 1 and Theorem 2, we get the next theorem, which may be interpreted loosely as "f satisfies the generalized Cauchy determinant formula if and only if f is a sequence of generalized Fibonacci numbers."

Theorem 3. *Assume that $f:\mathbb{Z}\to\mathbb{C}$, and that for all $x=(x_0,\ldots,x_n)$ and $y=(y_0,\ldots,y_n)$ with $f(x_i+y_j)\neq 0$, the matrix $M(f,x,y)$ is defined by $M(f,x,y)_{i,j}=1/f(x_i+y_j)$. Then there exists a constant q such that f satisfies equation (1) if and only if there exist constants k, q such that f is defined by $f(0)=0$, $f(1)\neq 0$, and $f(m+1)=kf(m)-qf(m-1)$, and the functions g_n in equation (1) take the form of equation (43).*

6. Inverses of $M(f,x,y)$ With Integer Entries

We consider the inverses of Hankel matrices of reciprocals of generalized Fibonacci numbers. Let f be defined by $f(0)=0$, $f(1)=1$, and $f(n+1)=kf(n)-qf(n-1)$, where k and q are integer constants. Define the f-factorial $!_f$ by $n!_f = f(1)f(2)\dots f(n)$, and the f-binomial coefficient $\binom{n}{m}_f$ to be $n!_f/m!_f(n-m)!_f$.

Let $H(f,n,s)$ be the $n\times n$ matrix whose (i,j) entry is $\dfrac{1}{f(i+j+s+1)}$. Using the generalized Cauchy determinant, we see that the (i,j) entry of the inverse of $H(f,n,s)$ is

$$\frac{\binom{n+i+s}{n-j-1}_f\binom{n+j+s}{n-i-1}_f\binom{i+j+s}{i}_f\binom{i+j+s}{j}_f f(i+j+s+1)}{q^{\binom{i}{2}+(n-i-1)i+\binom{j}{2}+(n-j-1)j+(n-1)(s+1)}}. \tag{44}$$

To determine whether the (i,j) entry of the inverse of $H(f,n,s)$ is an integer, we need to determine whether the power of q in the denominator of equation (44) divides the product of f-binomial coefficients in the numerator of equation (44).

We exclude certain values of k and q. For example, if $f(n)=0$ for some $n>0$, then $H(f,n,0)$ is undefined, so we exclude any pair of values for k and q that give $f(n)=0$. One example of excluded pairs is $q=k^2$, since that implies $f(3)=0$. We also exclude $q=0$, since then the denominator of equation (44) is zero. (If $q=0$, then f is just a geometric series, and $H(f,n,s)$ is not invertible for $n>1$.)

Theorem 4. *Let f be defined by $f(0)=0$, $f(1)=1$, and $f(n+1)=kf(n)-qf(n-1)$ for $n>1$, where k and q are non-zero integers. Assume that $f(n)\neq 0$ for all $n>0$. Then $H(f,n,s)$ is invertible for all positive integers n and non-negative integers s, and the inverse of $H(f,n,s)$ has integer entries if and only if q divides k^2.*

Proof. We have provided the inverse in equation (44). The $(0,0)$ entry of the inverse of $H(f,2,0)$ is k^2/q, showing the necessity that q divides k^2 for $H(f,n,s)$ to have integer entries.

The proof of the converse is a bit more involved. Assume that q divides k^2. Let t and r be positive integers with $t=r^2$ the maximum square that divides q. Let $k'=k/r$, let $q'=q/t$, and let g be defined by $g(0)=0$, $g(1)=1$, and $g(n+1)=k'g(n)-q'g(n-1)$. Use $\mathrm{diag}(z_0,\dots,z_{n-1})$ to denote the $n\times n$ diagonal matrix with (i,i) entry z_i.Then $H(f,n,s)=\mathrm{diag}(1,r,r^2,\dots,r^{n-1})^{-1}H(g,n,s)\mathrm{diag}(r^s,r^{s+1},r^{s+2},\dots,r^{s+n-1})^{-1}$. Thus it suffices to show that $H(g,n,s)$ has an inverse with integer entries.

Thus we may assume that q is square free. Since q divides k^2 and q is square free, it follows that q divides k. Now observe that the defining recursion for f implies that $q^{\lceil\frac{n-1}{2}\rceil}$ divides $f(n)$.

Next, we want to show that $q^{\lceil m(n-m)/2 \rceil}$ divides $\binom{n}{m}_f$. We prove this inductively using the following recurrence for f-binomial coefficients.

Lemma 1. *The f-binomial coefficients satisfy the recurrence*

$$\binom{n}{m}_f = f(n-m+1)\binom{n-1}{m-1}_f - qf(m-1)\binom{n-1}{m}_f.$$

The lemma is similar to the corresponding recursion for Fibonomial coefficients, with the additional term q given explicitly. It may be proved by induction, or by using the Binet formula. We omit the proof.

Lemma 2. *Let f be defined by $f(0) = 0$, $f(1) = 1$, and $f(n+1) = kf(n) - qf(n-1)$ for $n > 1$, where k and q are integers, and q divides k. Then the f-binomial coefficient $\binom{n}{m}_f$ is divisible by $q^{\lceil m(n-m)/2 \rceil}$.*

Proof. The case $m = 0$ is clear, and from the comment above, the case $m = 1$ follows from the assumption that q divides k. We now proceed inductively on n. Using the recurrence equation of Lemma 1 and the inductive hypothesis, we see that $\binom{n}{m}_f$ is divisible by the minimum of $q^{\lceil (n-m)/2 \rceil + \lceil (m-1)(n-m)/2 \rceil}$ and $q^{1+\lceil (m-2)/2 \rceil + \lceil (m)(n-m-1)/2 \rceil}$. Now $\lceil \frac{n-m}{2} \rceil + \lceil \frac{(m-1)(n-m)}{2} \rceil \geq \frac{m(n-m)}{2}$, and $1 + \lceil \frac{m-2}{2} \rceil + \lceil \frac{m(n-m-1)}{2} \rceil \geq \frac{m(n-m)}{2}$, thus proving the lemma. □

By Lemma 2, the numerator of equation (44),

$$\binom{n+i+s}{n-j-1}_f \binom{n+j+s}{n-i-1}_f \binom{i+j+s}{i}_f \binom{i+j+s}{j}_f f(i+j+s+1),$$

is divisible by $q^{(i+j+s+1)(n-j-1)/2+(i+j+s+1)(n-i-1)/2+(j+s)i/2+(i+s)j/2+(i+j+s)/2}$. Simplifying the exponent of q gives

$$\begin{aligned}
&\frac{(i+j+s+1)(n-j-1)}{2} + \frac{(i+j+s+1)(n-i-1)}{2} + \frac{(j+s)i}{2} + \frac{(i+s)j}{2} + \frac{(i+j+s)}{2} \\
&= (i+j+s+1)(n-1) - \frac{(i+j+s+1)(i+j)}{2} + \frac{(j+s)i}{2} + \frac{(i+s)j}{2} + \frac{(i+j+s)}{2} \\
&= i(n-1) - \frac{(i+1)i}{2} + j(n-1) - \frac{(j+1)j}{2} + (s+1)(n-1) + \frac{(i+j+s)}{2} \\
&= i(n-i-1) + \frac{i(i-1)}{2} + j(n-j-1) + \frac{j(j-1)}{2} + (s+1)(n-1) + \frac{(i+j+s)}{2} \\
&\geq i(n-i-1) + \binom{i}{2} + j(n-j-1) + \binom{j}{2} + (s+1)(n-1).
\end{aligned} \tag{45}$$

This implies that $q^{\binom{i}{2}+(n-i-1)i+\binom{j}{2}+(n-j-1)j+(n-1)(s+1)}$ divides

$$\binom{n+i+s}{n-j-1}_f \binom{n+j+s}{n-i-1}_f \binom{i+j+s}{i}_f \binom{i+j+s}{j}_f f(i+j+s+1),$$

and thus by equation (44), the entries of $H(f, n, s)$ are integers. □

7. Examples

Again, let f be defined by $f(0) = 0$, $f(1) = 1$, and $f(n+1) = kf(n) - qf(n-1)$ for constants k and q. We describe some examples by giving the values of the constants k and q that determine the recursion for f.

Example 1. The Hilbert matrix is the matrix $H(f, n, 0)$ where $k = 2$ and $q = 1$.

Example 2. The Filbert matrix is the matrix $H(f, n, 0)$ where $k = 1$ and $q = -1$.

Example 3. Ismail showed that the inverses of the matrices $H(f, n, 0)$ where f results from k a positive integer and $q = -1$ have integer entries [4].

Example 4. The Quantum Hilbert matrix for the parameter z is the matrix $H(f, n, 0)$ where $k = z^{1/2} + z^{-1/2}$ and $q = 1$. The Quantum Hilbert matrix was introduced in [2]. (The notation in [2] uses q where we have used z, but we are already using q as one of the defining constants for f.)

Example 5. Another q-analog of the Hilbert matrix is the matrix $H(f, n, 0)$ where $k = q+1$.

Example 6. Define f with $k = q_1 + q_2$ and $q = q_1 q_2$, where q_1 and q_2 are constants; the values of q_1 and q_2 are the roots of the characteristic equation of the recurrence relation for f.

Example 7. The Pell numbers are the sequence f for $k = 2$ and $q = -1$. Use $P(n)$ to denote the n^{th} Pell number. By Theorem 4, the inverse of $H(P, n, 0)$ has integer entries.

Example 8. The sequence f for $k = 4$ and $q = 2$ begins $0, 1, 4, 14, 48, 164, 560, 1912, 6528$. By Theorem 4, the inverse of $H(f, n, 0)$ has integer entries.

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