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THE FROBENIUS NUMBER OF GEOMETRIC SEQUENCES

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Abstract

The Frobenius problem is about finding the largest integer that is not contained in the numerical semigroup generated by a given set of positive integers. In this paper, we derive a solution to the Frobenius problem for sets of the form $\{m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\}$, where m, n are relatively prime positive integers.

1. Introduction

The *Frobenius number* of a set of positive integers $\{a_1, \ldots, a_k\}$ (known as the generators) is the largest integer that is not in the numerical semigroup generated by the generators. This number is denoted by $g(a_1, \ldots, a_k)$. Finding the Frobenius number without any restrictions on the set of generators is known to be NP-hard [1]. However, James Joseph Sylvester discovered a simple formula for the problem with two generators in 1884 [7]. Efficient algorithms for the solution of the three generator case were discovered by Greenberg [3] in 1988. Also of particular interest is a formula by Roberts for the Frobenius number for arithmetic sequences [5], and a formula by Lewin for almost arithmetic sequences [4]. An extensive list of literature on the problem can be found in [2].

In this note, we investigate the Frobenius number for geometric sequences, that is, sequences of the form $\{a, ar, ar^2, \ldots, ar^k\}$ where a is an initial value and r the common ratio. Since $\gcd(a, ar, ar^2, \ldots, ar^k)$ must equal one[6], then we have that $a = m^k$ and $r = n/m$

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where m, n are relatively prime integers. Our main result is the following:

Theorem. *Let m, n, k be positive integers such that $gcd(m, n) = 1$. Then*

$$g(m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k) = n^{k-1}(mn - m - n) + \frac{(n-1)m^2(m^{k-1} - n^{k-1})}{(m-n)}.$$

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2. Finding the Frobenius Number

We denote by $A(m, n, k)$ the numerical semigroup generated by $\{m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\}$. We will also denote $g(m^k, m^{k-1}n, m^{k-2}n^2, \ldots, n^k)$ by $G(m, n, k)$.

Lemma 1. *For m, n relatively prime and $k \geq 1$, $G(m, n, k+1) \geq (n-1)m^{k+1} + nG(m, n, k)$.*

Proof. We have to show that $(n-1)m^{k+1} + nG(m, n, k)$ is not in $A(m, n, k+1)$. Assume instead that $(n-1)m^{k+1} + nG(m, n, k) \in A(m, n, k+1)$. Then

$$(n-1)m^{k+1} + nG(m, n, k) = \sum_{i=0}^{k+1} c_i m^i n^{k+1-i}, c_i \in \mathbb{Z}_{\geq 0}$$

Taking both sides $\mod n$ we obtain $-m^{k+1} \equiv c_{k+1}m^{k+1}$. Since m, n are relatively prime, we conclude $c_{k+1} \equiv -1 \mod n$. Say that $c_{k+1} = bn - 1$ for some positive integer b. Then we have

$$(n-1)m^{k+1} + nG(m, n, k) = \left[\sum_{i=0}^{k-1} c_i m^i n^{k+1-i}\right] + ((b-1)m + c_k)m^k n + (n-1)m^{k+1}$$

and so

$$G(m, n, k) = \left[\sum_{i=0}^{k-1} c_i m^i n^{k-i}\right] + ((b-1)m + c_k)m^k$$

But this implies $G(m, n, k) \in A(m, n, k)$, which is absurd. Thus we conclude that $(n-1)m^{k+1} + nG(m, n, k) \notin A(m, n, k+1)$, and so $G(m, n, k+1) \geq (n-1)m^{k+1} + nG(m, n, k)$. □

Lemma 2. *For m, n relatively prime and $k \geq 1$, $G(m, n, k+1) \leq (n-1)m^{k+1} + nG(m, n, k)$.*

Proof. We will show that if $y > (n-1)m^{k+1} + nG(m, n, k)$, then $y \in A(m, n, k+1)$. Let $y \equiv dm^{k+1} \mod n$, $d \in [0, n-1]$. Let $z = y - dm^{k+1}$. Since $z \equiv 0 \mod n$, we have $z = nw$ for some non-negative integer w. But $y > (n-1)m^{k+1} + nG(m, n, k)$ implies $z > nG(m, n, k)$, and so $w > G(m, n, k)$, and thus $w \in A(m, n, k)$. But this means that $y = nw + dm^{k+1} \in A(m, n, k+1)$, and so $G(m, n, k+1) \leq (n-1)m^{k+1} + nG(m, n, k)$ □

Proof of Theorem. We prove this by induction on k. For $k = 1$ this reduces to the result of Sylvester in [7], $G(m, n, 1) = mn - m - n$. Suppose that it is true for $k = t$ and thus

$$G(m, n, t) = n^{t-1}(mn - m - n) + \frac{(n-1)m^2(m^{t-1} - n^{t-1})}{m - n}.$$

By Lemmas 1 and 2 we have

$$\begin{aligned} G(m, n, t+1) &= (n-1)m^{t+1} + n\left(n^{t-1}(mn - m - n) + \frac{(n-1)m^2(m^{t-1} - n^{t-1})}{(m-n)}\right) \\ &= n^t(mn - m - n) + (n-1)\left(m^{t+1} + \frac{nm^2(m^{t-1} - n^{t-1})}{(m-n)}\right) \\ &= n^t(mn - m - n) + \frac{(n-1)m^2(m^t - n^t)}{(m-n)}, \end{aligned}$$

which is the theorem for $k = t + 1$. □

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