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ON P, Q-BINOMIAL COEFFICIENTS

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Abstract

In this paper, we develop the theory of a p, q-analogue of the binomial coefficients. Some properties and identities parallel to those of the usual and q-binomial coefficients will be established including the triangular, vertical, and the horizontal recurrence relations, horizontal generating function, and the orthogonality and inverse relations. The construction and derivation of these results give us an idea of how to handle complex computations involving the parameters p and q. This may be a good start in developing the theory of p, q-analogues of some special numbers in combinatorics. Furthermore, several interesting special cases will be disclosed which are analogous to some established identities of the usual binomial coefficients.

Introduction

The binomial coefficients, denoted by $\binom{n}{k}$, play an important role in enumerative combinatorics. These numbers appear as the coefficients in the expansion of the binomial expression $(x+y)^n$. That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This identity is known as the *Binomial Theorem* [5][6][16][19]. If $y = 1$, this identity becomes

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1}$$

which is the horizontal generating function for the binomial coefficients. The explicit value of the binomial coefficients is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{if } 0 \le k \le n, \text{ and } \binom{n}{k} = 0, \quad \text{otherwise.}$$

A more general definition of the binomial coefficients is given in [11] where n can be a complex number; in particular the binomial theorem (1) holds in this more general setting if $r \geq 0$ is an integer or $|x/y| < 1$. This can be interpreted as the number of possible k-subsets out of a set of n distinct elements or the number of ways to choose k elements from the set of n distinct elements. The binomial coefficients are also known as *combinations* or *combinatorial numbers*.

Almost all books in combinatorics devote one chapter for the discussion of binomial coefficients (see, e.g., [5] [6][16][19]). This shows that the binomial coefficients are already well-studied and well-discussed in the past resulting to possession of numerous properties and identities. To mention a few, we have the triangular recurrence relation

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{2}$$

the identity

$$\sum_{k\ even} \binom{n}{k} = \sum_{k\ odd} \binom{n}{k}, \tag{3}$$

the inverse relation

$$f_n = \sum_{k=0}^{n} \binom{n}{k} g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k, \tag{4}$$

Chu Shih-Chieh's identity

$$\binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k},$$

and the orthogonality relation

$$\sum_{j=i}^{n} (-1)^{n-j} \binom{n}{j} \binom{j}{i} = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j}{i} = \delta_{ni} = \begin{cases} 0 & n \neq i \\ 1 & n = i \end{cases} \tag{5}$$

where δ_{ni} is called the *Kronecker delta*.

In constructing the properties and identities of some special numbers, binomial coefficients are frequently involved. That is why the q-analogue of the binomial coefficients plays an important role in developing the theory of the q-analogue of these special numbers. From [3][6], the q-analogue of the binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad q \neq 1.$$

This q-analogue is also called the *q-binomial coefficient*, a *Gaussian coefficient*, or a *Gaussian polynomial*. Using the notation

$$[k]_q = \frac{q^k - 1}{q - 1}, \quad q \neq 1,$$

we can rewrite the q-binomial coefficient as $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q![n-k]_q!} = \dfrac{[n]_q}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ where $[k]_q!$ denotes the q-factorial which is given by $[k]_q! = [k]_q[k-1]_q \ldots [2]_q[1]_q$. It can easily be verified that, when $q \to 1$, the q-binomial coefficients turn into the usual binomial coefficients.

Unlike the usual binomial coefficients, the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ have only a limited number of properties and identities. The following are some of the properties and identities for $\begin{bmatrix} n \\ k \end{bmatrix}_q$ which are taken from [6]. We have the triangular recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \tag{6}$$

the identity

$$\sum_{k\ even} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{k\ odd} \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{7}$$

the inverse relation

$$f_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k, \tag{8}$$

and the orthogonality relation

$$\sum_{j=i}^{n} (-1)^{n-j} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q = \sum_{j=i}^{n} (-1)^{j-i} q^{\binom{j-i}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q = \delta_{ni}. \tag{9}$$

Analogous to the expansion in (1), we have

$$\prod_{r=0}^{n-1} (1 + xq^r) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \tag{10}$$

Moreover, the q-binomial coefficient $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ may be interpreted as a polynomial in q whose coefficient in q^k counts the number of distinct partitions of k elements which fit inside an $m \times n$ rectangle [1].

The p, q-analogue of the usual binomial coefficient which is also called the *p, q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i}, \quad p \neq q. \tag{11}$$

Using the notation $[k]_{pq} = \frac{p^k - q^k}{p-q}$ we can rewrite (11) as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \frac{[n]_{pq}!}{[k]_{pq}![n-k]_{pq}!} = \frac{[n]_{pq}}{[k]_{pq}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{pq}$$

where $[k]_{pq}! = [k]_{pq}[k-1]_{pq}\dots[2]_{pq}[1]_{pq}$. It may be worth noticing that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \prod_{i=1}^{k} \frac{q^{n-i+1} - p^{n-i+1}}{q^i - p^i} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \frac{[n]_{pq}!}{[n-k]_{pq}![n-(n-k)]_{pq}!}.$$

Hence,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \begin{bmatrix} n \\ k \end{bmatrix}_{qp} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{pq}. \tag{12}$$

Clearly, when $p = 1$, the p,q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{pq}$ reduces to the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{q}$.

In this paper, we develop the theory of p,q-analogue of the binomial coefficients by establishing some properties and identities parallel to those in (6) - (10). Some special cases will be disclosed which somehow give a clearer description of the p,q-binomial coefficients.

Recurrence Relations and Generating Function

For a quick computation of the first values of p,q-binomial coefficients, we have the following triangular recurrence relations which are of similar form with those in (2) and (6).

Theorem 1. *The p,q-binomial coefficients satisfy the following triangular recurrence relations*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} \tag{13}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} \tag{14}$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{pq} = 1$, $\begin{bmatrix} n \\ n \end{bmatrix}_{pq} = 1$, and $\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = 0$ for $k > n$.

Proof. Evaluating the right-hand side of (13), we get

$$\begin{aligned} p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} &= p^k \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i} + q^{n-i+1} \prod_{i=1}^{k-1} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i} \\ &= \frac{\left\{\prod_{i=1}^{k-1}(p^{n-i+1} - q^{n-i+1}\right\}\left\{p^k(p^{n-k+1} - q^{n-k+1} + q^{n-k+1}(p^k - q^k)\right\}}{\prod_{i=1}^{k}(p^i - q^i)} \\ &= \frac{\prod_{i=1}^{k}(p^{(n+1)-i+1} - q^{(n+1)-i+1})}{\prod_{i=1}^{k}(p^i - q^i)} = \begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq}. \end{aligned}$$

Note that we can rewrite (13) as $\begin{bmatrix} n+1 \\ k \end{bmatrix}_{qp} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{qp} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{qp}$. Using (12), we obtain (14). □

Taking $p = 1$, (14) yields $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ which is precisely equivalent to (6). Similarly, when $p = 1$, (13) gives $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ which is another form of the triangular recurrence relation for q-binomial coefficients.

Using Theorem 1, we can quickly generate the first values of $\begin{bmatrix} n \\ k \end{bmatrix}_{pq}$ as shown in the following table.

n/k	**1**	**2**	**3**	**4**	**5**
1	1				
2	$p+q$	1			
3	p^2+pq+q^2	p^2+pq+q^2	1		
4	$p^3+p^2q+pq^2$ $+q^3$	$(p^2+pq+q^2)(p^2+q^2)$	$p^3+p^2q+pq^2+q^3$	1	
5	$p^4+p^3q+p^2q^2$ $+pq^3+q^4$	$p^2(p^2+pq+q^2)(p^2+q^2)$ $+q^3(p^3+p^2q+pq^2+q^3)$	$p^3(p^3+p^2q+pq^2+q^3)$ $+q^2(p^2+pq+q^2)(p^2+q^2)$	$p^4+p^3q+p^2q^2$ $+pq^3+q^4$	1

Table 1. *Table of Values for* $\begin{bmatrix} n \\ k \end{bmatrix}_{pq}$

One may try to verify (12) for $n = 1, 2, 3, 4, 5$ using Table 1. In fact, we can easily see it if we write the entries of Table 1 in the following way:

$$1$$

$$1 \qquad 1$$

$$1 \qquad p+q \qquad 1$$

$$1 \qquad p^2+pq+q^2 \qquad p^2+pq+q^2 \qquad 1$$

$$1 \qquad p^3+p^2q+pq^2+q^3 \qquad (p^2+pq+q^2)(p^2+q^2) \qquad p^3+p^2q+pq^2+q^3 \qquad 1$$

$$1 \qquad \begin{matrix} p^4+p^3q+p^2q^2 \\ +pq^3+q^4 \end{matrix} \qquad \begin{matrix} p^2(p^2+pq+q^2)(p^2+q^2) \\ +q^3(p^3+p^2q+pq^2+q^3) \end{matrix} \qquad \begin{matrix} p^3(p^3+p^2q+pq^2+q^3) \\ +q^2(p^2+pq+q^2)(p^2+q^2) \end{matrix} \qquad \begin{matrix} p^4+p^3q+p^2q^2 \\ +pq^3+q^4 \end{matrix} \qquad 1$$

This figure is analogous to Pascal's Triangle of the usual binomial coefficients. As we can see, it possesses the same symmetry as Pascal's Triangle. However, the identities that we have for the p, q-binomial coefficients may not exactly be of the same structure as those in the usual binomial coefficients.

Note that if we apply (13) thrice to $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{pq}$, we obtain

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{pq} &= q^{n-(k+1)+1} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{pq} \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{k+1} \left(q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} + p^{k+1} \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_{pq} \right) \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \\
&\quad + p^{2(k+1)} \left(q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{pq} + p^{k+1} \begin{bmatrix} n-2 \\ k+1 \end{bmatrix}_{pq} \right) \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \\
&\quad + p^{2(k+1)} q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{pq} + p^{3(k+1)} \begin{bmatrix} n-2 \\ k+1 \end{bmatrix}_{pq}.
\end{aligned}$$

Continuing this process until the $(n-k)th$ application of (13), we get

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{pq} &= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{pq} \\
&\quad + p^{2(k+1)} q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{pq} + p^{3(k+1)} q^{n-k-3} \begin{bmatrix} n-3 \\ k \end{bmatrix}_{pq} \\
&\quad + \cdots + p^{(n-k)(k+1)} \begin{bmatrix} k \\ k \end{bmatrix}_{pq}.
\end{aligned}$$

This is a vertical recurrence relation for the p,q-binomial coefficients. On the other hand, if we rewrite (14) as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = p^{-(n-k)} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{pq} - p^{-(n-k)} q^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{pq}$$

and apply this recurrence relation to itself $(n-k)$ times, we obtain a horizontal recurrence relation for the p,q-binomial coefficients. Let us state these results in the following theorem.

Theorem 2. *The p,q-binomial coefficients satisfy the following vertical recurrence relation*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{pq} = \sum_{j=k}^{n} p^{(n-j)(k+1)} q^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{pq}$$

and horizontal recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \sum_{j=0}^{n-k} (-1)^j p^{-(j+1)(n-k)+\binom{j+1}{2}} q^{jk+\binom{j+1}{2}} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_{pq}.$$

For example, using the vertical recurrence relation in Theorem 2, we can compute $\begin{bmatrix}4\\3\end{bmatrix}_{pq}$ as follows:

$$\begin{aligned}\begin{bmatrix}4\\3\end{bmatrix}_{pq} &= p^{(3-2)(2+1)}q^{2-2}\begin{bmatrix}2\\2\end{bmatrix}_{pq} + p^{(3-3)(2+1)}q^{3-2}\begin{bmatrix}3\\2\end{bmatrix}_{pq}\\ &= p^3 + q(p^2+pq+q^2) = p^3+p^2q+pq^2+q^3.\end{aligned}$$

This is exactly the value of $\begin{bmatrix}4\\3\end{bmatrix}_{pq}$ that appears in Table 1. One may also try to compute $\begin{bmatrix}4\\3\end{bmatrix}_{pq}$ using the horizontal recurrence relation.

The vertical and horizontal recurrence relations in Theorem 2 may be considered p,q-analogues of the Hockey Stick identities, which are also known as Chu Shih-Chieh's identities. More precisely, if we let $p=1$, Theorem 2 will give the following vertical and horizontal recurrence relations for the q-binomial coefficients, respectively,

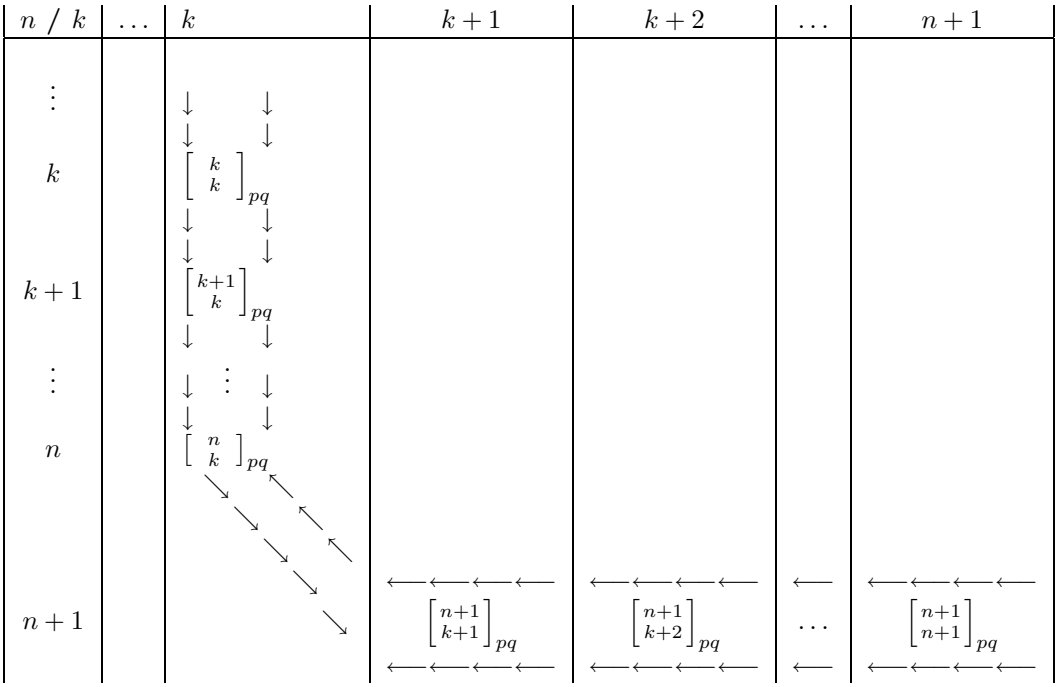
$$\begin{bmatrix}n+1\\k+1\end{bmatrix}_q = \sum_{j=k}^{n} q^{j-k}\begin{bmatrix}j\\k\end{bmatrix}_q \quad \text{and} \quad \begin{bmatrix}n\\k\end{bmatrix}_q = \sum_{j=0}^{n-k}(-1)^j q^{jk+\binom{j+1}{2}}\begin{bmatrix}n+1\\k+j+1\end{bmatrix}_q.$$

Furthermore, when $q\to 1$, the former will reduce to Chu Shih-Chieh's identity as given in the introduction and the latter will reduce to

$$\binom{n}{k} = \binom{n+1}{k+1} - \binom{n+1}{k+2} + \ldots + (-1)^{n-k}\binom{n+1}{n+1}$$

which is another recurrence relation for the usual binomial coefficients.

The vertical and horizontal recurrence relations in Theorem 2 can easily be remembered with the help of the pattern as shown in the table below.



The entries involved in the table to compute the value of $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{pq}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{pq}$ using the vertical and horizontal recurrence relations in Theorem 2 clearly form a hockey stick as illustrated by the arrows in the table.

We observe that

$$1+x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{pq} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{pq} x$$

$$\begin{aligned} (1+x)(p+qx) &= p + (p+q)x + qx^2 \\ &= p \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{pq} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{pq} x + q \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{pq} x^2 \end{aligned}$$

and

$$\begin{aligned} (1+x)(p+qx)(p^2+q^2x) &= p^3 + p(p^2+pq+q^2)x + q(p^2+pq+q^2)x^2 + q^3x^3 \\ &= p^3 \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{pq} + p \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{pq} x + q \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{pq} x^2 + q^3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{pq} x^3. \end{aligned}$$

This observation gives us an idea that we can also construct a horizontal generating function for the p,q-binomial coefficients which is analogous to those in (1) and (6). The next theorem contains the horizontal generating function mentioned above.

Theorem 3. *The horizontal generating function for the p,q-binomial coefficients is given by*

$$\prod_{r=0}^{n-1}(p^r + xq^r) = \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k.$$

Proof. We prove this by induction on n. For $n = 1, 2, 3$, the generating function is already verified above. Suppose it is true for some $n \geq 1$, that is,

$$\prod_{r=0}^{n-1}(p^r + xq^r) = \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k.$$

Then, using the inductive hypothesis, we have

$$\begin{aligned} \prod_{r=0}^{n}(p^r + xq^r) &= (p^n + xq^n) \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k \\ &= \sum_{k=0}^{n} p^{\binom{n-k}{2}+n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^k + \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}+n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} x^{k+1} \\ &= \sum_{k=0}^{n+1} \left\{ p^{\binom{n-k}{2}+n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + p^{\binom{n-k+1}{2}} q^{\binom{k-1}{2}+n} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} \right\} x^k \\ &= \sum_{k=0}^{n+1} p^{\binom{n-k+1}{2}} q^{\binom{k}{2}} \left\{ p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} \right\} x^k. \end{aligned}$$

Applying Theorem 1, we get $\prod_{r=0}^{n}(p^r+xq^r)=\sum_{k=0}^{n+1}p^{\binom{n-k+1}{2}}q^{\binom{k}{2}}\begin{bmatrix}n+1\\k\end{bmatrix}_{pq}x^k$. □

As a direct consequence of Theorem 3, we have the following corollary which contains an identity analogous to those in (3) and (7).

Corollary 1. *For $n\geq 1$, we have*

$$\sum_{k\ even}p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}=\sum_{k\ odd}p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}. \tag{15}$$

Proof. With $x=-1$, Theorem 3 gives

$$\prod_{r=0}^{n-1}(p^r-q^r)=\sum_{k=0}^{n}(-1)^k p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}.$$

Since $p^r-q^r=0$ when $r=0$, we have

$$\sum_{k=0}^{n}(-1)^k p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}=0.$$

This is precisely equivalent to (15). □

The next corollary can easily be deduced from Theorem 3 by taking $x=1$.

Corollary 2. *For $n\geq 1$, we have*

$$\sum_{k=0}^{n}\left\{\frac{p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}}{\prod_{r=0}^{n-1}(p^r+q^r)}\right\}=1. \tag{16}$$

Corollary 2 may be interpreted as follows. Consider an experiment with Ω as the sample space such that

$$|\Omega|=\prod_{r=0}^{n-1}(p^r+q^r),$$

and a random variable X that takes the value from 0 to n which determines the outcome of the experiment. Suppose that a given event $E_k\subset\Omega$ is associated with the value $X=k$ such that

$$|E_k|=p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}.$$

Then (16) shows that

$$Pr(X=k)=\frac{p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{pq}}{\prod_{r=0}^{n-1}(p^r+q^r)}$$

is a probability distribution of X.

The horizontal generating function in Theorem 3 may be a good tool in giving a combinatorial interpretation of the p,q-binomial coefficients. One may try to use the method of generating functions as discussed in [5],[16], and [19].

Orthogonality and Inverse Relations

Inverse relations are useful tools in deriving necessary formulas of some special numbers in combinatorics. The inverse relations in (4) and (8) were used many times in transforming some identities into a completely different and interesting form. For instance, the second form of the horizontal recurrence relation for the generalized Stirling numbers [7] was obtained by transforming the explicit formula of the generalized Stirling numbers using the inverse relation in (4), and the explicit formula of the q-analogue for the generalized Stirling numbers [8] was transformed into a horizontal recurrence relation using (8). Thus, for possible use in developing the theory of p,q-analogue of some special numbers, an inverse relation for the p,q-binomial coefficients must also be established.

The following theorem contains the orthogonality relation for the p,q-binomial coefficient which is necessary in establishing the inverse relation.

Theorem 4. *The orthogonality relation for the p,q-binomial coefficients is given by*

$$\sum_{j=i}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_{pq}\begin{bmatrix}j\\ i\end{bmatrix}_{pq}=\sum_{j=i}^{n}(-1)^{j-i}p^{\binom{j-i}{2}}q^{\binom{n-j}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_{pq}\begin{bmatrix}j\\ i\end{bmatrix}_{pq}=\delta_{ni}.$$

Proof. Clearly, it is true for $n=0,1$. Suppose that for some $n\geq 0$

$$H_n=\sum_{j=i}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_{pq}\begin{bmatrix}j\\ i\end{bmatrix}_{pq}=\delta_{ni}.$$

Then, by Theorem 1, we have

$$\begin{aligned}H_{n+1} &= \sum_{j=i}^{n+1}(-1)^{n+1-j}p^{\binom{n+1-j}{2}}q^{\binom{j-i}{2}}p^{j}\begin{bmatrix}n\\ j\end{bmatrix}_{pq}\begin{bmatrix}j\\ i\end{bmatrix}_{pq}\\ &+ \sum_{j=i}^{n+1}(-1)^{n+1-j}p^{\binom{n+1-j}{2}}q^{\binom{j-i}{2}}q^{n-j+1}\begin{bmatrix}n\\ j-1\end{bmatrix}_{pq}\begin{bmatrix}j\\ i\end{bmatrix}_{pq}.\end{aligned}$$

Using the facts that

$$\begin{bmatrix}n\\ n+1\end{bmatrix}_{pq}=0,\quad p^{\binom{n+1-j}{2}+j}=p^{\binom{n-j}{2}}p^{n},\ \text{ and }\ q^{\binom{j-i}{2}+(n-j+1)}=q^{\binom{j-1-i}{2}}q^{n-i},$$

we obtain

$$H_{n+1}=(-1)p^{n}\delta_{ni}+q^{n-i}\sum_{j=i-1}^{n}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_{pq}\begin{bmatrix}j+1\\ i\end{bmatrix}_{pq}.$$

Again, by applying Theorem 1, we get

$$H_{n+1} = -p^{n}\delta_{ni}+q^{n-i}p^{i}\delta_{ni}+q^{n-i}\sum_{j=i-1}^{n}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_{pq}q^{j-i+1}\begin{bmatrix}j\\ i-1\end{bmatrix}_{pq}$$

$$\begin{aligned} &= (-p^n + q^{n-i}p^i)\delta_{ni} + q^{n-i}q\sum_{j=i-1}^{n} p^{\binom{n-j}{2}}q^{\binom{j-(i-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \begin{bmatrix} j \\ i-1 \end{bmatrix}_{pq} \\ &= 0 + q^{n-i+1}\delta_{n,i-1} = \delta_{n+1,i}. \end{aligned}$$

This proves the first form of the orthogonality relation.

To prove the other form, we first interchange p and q to obtain

$$\sum_{j=i}^{n}(-1)^{j-i}p^{\binom{j-i}{2}}q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \begin{bmatrix} j \\ i \end{bmatrix}_{pq} = (-1)^{n-i}\sum_{j=i}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{qp} \begin{bmatrix} j \\ i \end{bmatrix}_{qp}.$$

Using (12) and the first form of the orthogonality relation, we have

$$\begin{aligned} \sum_{j=i}^{n}(-1)^{j-i}p^{\binom{j-i}{2}}q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \begin{bmatrix} j \\ i \end{bmatrix}_{pq} &= (-1)^{n-i}\sum_{j=i}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \begin{bmatrix} j \\ i \end{bmatrix}_{pq} \\ &= (-1)^{n-i}\delta_{ni} = \delta_{ni}. \end{aligned}$$ □

The first form of the orthogonality relation in Theorem 4 may be written as $\sum_{j=i}^{n} a_{nj}b_{ji} = \delta_{ni}$ where

$$a_{nj} = (-1)^{n-j}p^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \text{ and } b_{ji} = q^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_{pq}.$$

If we let M and $\widetilde{M}$ be two matrices such that $M = [a_{ij}]_{0\le i,j\le m}$ and $\widetilde{M} = [b_{ij}]_{0\le i,j\le m}$, then $M\widetilde{M} = [\delta_{ij}]_{0\le i,j\le m} = I_m$. That is, $\widetilde{M} = M^{-1}$ and $M = \widetilde{M}^{-1}$.

The next theorem is a kind of transformation from the identity in Theorem 3 that interchanges the role of x^k and $\prod_{r=0}^{n-1}(q^r + xp^r)$ in the formula.

Theorem 5. *The p,q-binomial coefficients satisfy the following relation*

$$p^{\binom{n}{2}}x^n = \sum_{j=0}^{n}(-1)^{n-j}p^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \prod_{r=0}^{j-1}(q^r + xp^r).$$

Proof. Clearly, $p^{\binom{n}{2}}x^n = \sum_{i=0}^{n}\delta_{ni}p^{\binom{i}{2}}x^i$. Then, using Theorem 4, we have

$$\begin{aligned} p^{\binom{n}{2}}x^n &= \sum_{i=0}^{n}\left\{\sum_{j=i}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}q^{\binom{j-i}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \begin{bmatrix} j \\ i \end{bmatrix}_{pq}\right\} p^{\binom{i}{2}}x^i \\ &= \sum_{j=0}^{n}(-1)^{n-j}p^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} \left\{\sum_{i=0}^{j} q^{\binom{j-i}{2}}p^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_{pq} x^i\right\}. \end{aligned}$$

Applying Theorem 3 completes the proof of the theorem. □

We notice that when $x = 0$, Theorem 5 gives $\sum_{j=0}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}q^{\binom{j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq} = 0$, which is precisely equivalent to Corollary 1. On the other hand, when $x = 1$, Theorem 5 yields

$$\sum_{j=0}^{n}(-1)^{n-j}p^{\binom{n-j}{2}-\binom{n}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq}\prod_{r=0}^{j-1}(q^r+p^r) = 1$$

which consequently gives

$$\sum_{j=1}^{n}(-1)^{n-j}p^{\binom{n-j}{2}-\binom{n}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq}\prod_{r=0}^{j-1}(q^r+p^r) = 0 \quad \text{if } n \text{ is even.}$$

That means

$$\sum_{j\ even(\neq 0)}p^{\binom{n-j}{2}-\binom{n}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq}\prod_{r=0}^{j-1}(q^r+p^r) = \sum_{j\ odd}p^{\binom{n-j}{2}-\binom{n}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq}\prod_{r=0}^{j-1}(q^r+p^r)$$

if n is even.

With the aid of Theorem 4, we can easily prove the inverse relation for the p, q-binomial coefficients which is given in the following theorem.

Theorem 6. *The inverse relation for the p, q-binomial coefficients is given by*

$$f_n = \sum_{j=0}^{n}(-1)^{n-j}p^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq} g_j \Longleftrightarrow g_n = \sum_{j=0}^{n}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq} f_j.$$

Proof. ($\Rightarrow$) Using the given hypothesis, we have

$$\begin{aligned}\sum_{j=0}^{n}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq} f_j &= \sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{j-i}p^{\binom{j-i}{2}}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq}\begin{bmatrix} j \\ i \end{bmatrix}_{pq} g_i \\ &= \sum_{i=0}^{n}\left\{\sum_{j=i}^{n}(-1)^{j-i}p^{\binom{j-i}{2}}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq}\begin{bmatrix} j \\ i \end{bmatrix}_{pq}\right\} g_i.\end{aligned}$$

Applying the second form of the orthogonality relation in Theorem 4, we obtain

$$\sum_{j=0}^{n}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq} f_j = \sum_{j=0}^{n}\delta_{ni}g_i = g_n.$$

($\Leftarrow$) This can easily be done by following the same argument as above and applying the first form of the orthogonality relation. □

If we set $f_n = p^{\binom{n}{2}}x^n$ and $g_j = \prod_{r=0}^{j-1}(q^r + xp^r)$, then, using Theorem 6, the identity in Theorem 5 becomes

$$\prod_{r=0}^{n-1}(q^r+xp^r) = \sum_{j=0}^{n}q^{\binom{n-j}{2}}p^{\binom{j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_{pq} x^j$$

which gives the expansion in Theorem 3 by interchanging p and q.

Some Remarks and Related Literature

There are still many properties and identities that we need to establish for the p,q-binomial coefficients which are parallel to those in the usual binomial coefficients. For instance, Vandermonde's identity, which is given by

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}, \tag{17}$$

can be obtained using the Binomial Theorem by multiplying the expansions $(x+1)^m$ and $(x+1)^n$ to get the coefficients of x^k which turns out to be the right-hand side of (17), which is then compared to the coefficients of x^k in the expansion of $(x+1)^{m+n}$.

The corresponding q-analogue of Vandermonde's identity is also known as the q-Vandermonde identity [10], which is given by

$$\begin{bmatrix} m+n \\ k \end{bmatrix}_q = \sum_{r=0}^{k} q^{r(m-k+r)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ k-r \end{bmatrix}_q,$$

and can be derived by paralleling what has been done to obtain (17), but this time using the identity in q-binomial theory

$$(A+B)^m(A+B)^n = (A+B)^{m+n}$$

with operators A and B that q-commute (i.e., $BA = qAB$). With these facts, one may try to construct a p,q-analogue of Vandermonde's identity by defining some operators analogous to those operators that q-commute.

The q-binomial coefficients were included in the classical families of numbers that arise as a particular case of U-Stirling numbers by Medicis and Leroux [9]. More precisely, the q-binomial coefficients can be expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\Phi \in T^A(k,n-k)} \prod_{c \in \Phi} q^{|c|} \tag{18}$$

where $T^A(k, n-k)$ denotes the set of A-tableaux (defined in [9]) with exactly $n-k$ columns whose lengths are in the set $\{0, 1, 2, \ldots, k\}$. This formula may give a combinatorial interpretation of the q-binomial coefficients in the context of 0-1 tableau as defined in [9]. Furthermore, identity (18) may be written as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{0 \le j_1 \le j_2 \le \ldots \le j_{n-k} \le k} \prod_{i=1}^{n-k} q^{j_i} = \sum_{0 \le j_1 \le j_2 \le \ldots \le j_{n-k} \le k} q^{j_1+j_2+\ldots+j_{n-k}}.$$

If we can express the p,q-binomial coefficients parallel to this expression, then the p,q-binomial coefficients may also be given a combinatorial interpretation in the context of 0-1 tableau.

In the study of (p,q)-hypergeometric series ${}_r\Phi_s$ [10][13] which is defined by

$$
\begin{aligned}
&{}_r\Phi_s((a_{1p},a_{1q}),\ldots,(a_{rp},a_{rq});(b_{1p},b_{1q}),\ldots,(b_{sp},b_{sq});(p,q),z)\\
&\quad=\sum_{n=0}^{\infty}\frac{((a_{1p},a_{1q}),\ldots,(a_{rp},a_{rq});(p,q))_n}{((b_{1p},b_{1q}),\ldots,(b_{sp},b_{sq});(p,q))_n((p,q);(p,q))_n}\left((-1)^n(q/p)^{n(n-1)/2}\right)^{1+s-r}z^n
\end{aligned}
$$

where $|q/p|<1$ and

$$((x_{1p},x_{1q}),\ldots,(x_{rp},x_{rq});(p,q))_n=\prod_{i=1}^{r}((x_{ip},x_{iq});(p,q))_n$$

with $((x_{ip},x_{iq});(p,q))_n=\prod_{m=0}^{n-1}(x_{ip}p^m-x_{iq}q^m)$, the p,q-binomial coefficients appeared in some of its identities. As mentioned in [13], when $r=1$ and $s=0$, the above (p,q)-hypergeometric series gives

$${}_1\Phi_0((p^n,q^n);-;(p,q),z)=\sum_{k=0}^{\infty}\begin{bmatrix}n-1+k\\ k\end{bmatrix}_{pq}=\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_{pq}(pq)^{k(k-1)/2}(-z)^k\right\}^{-1}\tag{18}$$

and this can further be written as

$${}_1\Phi_0((p^n,q^n);-;(p,q),z)=\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_{q^{-1}q}(-z)^k\right\}^{-1}$$

with

$$\begin{bmatrix}n\\ k\end{bmatrix}_{q^{-1}q}=\frac{((q^{-1},q);(q^{-1},q))_n}{((q^{-1},q);(q^{-1},q))_k((q^{-1},q);(q^{-1},q))_{n-k}}$$

which should be relevant in the context of quantum groups. Now, taking $a=p\zeta_p$ and $b=p^n\zeta_q$ with $z=\zeta_q/\zeta_p$, we can rewrite (18) as

$$((a,b);(p,q))_n=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_{pq}(-1)^kp^{\binom{n-k}{2}}q^{\binom{k}{2}}a^{n-k}b^k.\tag{19}$$

When $a=1$, (19) gives

$$\prod_{i=0}^{n-1}(p^i-bq^i)=\sum_{k=0}^{n}p^{\binom{n-k}{2}}q^{\binom{k}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_{pq}(-b)^k,$$

which can also be obtained from Theorem 3 by taking $x=-b$.

The (p,q)-hypergeometric series is a generalization of the basic hypergeometric series or the q-hypergeometric series which is given by

$${}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n(q;q)_n}\left((-1)^nq^{n(n-1)/2}\right)^{1+s-r}z^n,$$

such that when $a_{1p} = a_{2p} = \ldots = a_{rp} = b_{1p} = b_{2p} = \ldots = b_{sp} = 1, a_{1q} = a_1, a_{2q} = a_2, \ldots, a_{rq} = a_r, b_{1q} = b_1, b_{2q} = b_2, \ldots, b_{sq} = b_s$, and $p = 1$, ${}_r\Phi_s \to {}_r\phi_s$. Another form of a generalization of q-hypergeometric series is the bibasic hypergeometric series [10] with two bases q and q_1, which is given by

$$F(\bar{a}, \bar{c}; \bar{b}, \bar{d}; q, q_1, z) = \sum_{n=0}^{\infty} \frac{(\bar{a}; q)_n (\bar{c}; q_1)_n}{(\bar{a}; q)_n (\bar{c}; q_1)_n (q; q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} \left((-1)^n q_1^{n(n-1)/2}\right)^{s_1-r_1} z^n,$$

where

$$(\bar{a}; q)_n = \prod_{i=1}^{r} (a_i; q)_n, \qquad (a_i; q)_n = \frac{(a_i; q)_\infty}{(a_i q^n; q)_\infty}, \qquad (a_i; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

In this series, the two unconnected bases q and q_1 are regarded and assigned partially to different numerator and denominator parameters whereas in the (p, q)-hypergeometric series the parameters p and q are inseparable and assigned to all the numerator and denominator parameter doublets. As we can see in [13], it is very interesting and profitable to study the (p, q)-hypergeometric series as well as the p, q-binomial coefficients. It is worth mentioning that there is research related to physics which considers identities of p, q-binomial coefficients [12]. For instance, Katriel and Kibler [14] defined the p, q-binomial coefficients and derived a p, q-binomial theorem while discussing normal ordering for deformed boson operators obeying (p, q)-oscillator algebra introduced by R. Chakrabarti and R. Jagannathan in [4]. Smirnov and Wehrhahn [18] expressed the expansion of

$$\left(q^{J_0(1)} J_\pm(2) + J_\pm(1) p^{-J_0(2)}\right)^l$$

in terms of the p, q-binomial coefficients, where $\{J_0(1), J_\pm(1)\}$ and $\{J_0(2), J_\pm(2)\}$ are the generators of two commuting (p, q)-angular momentum algebras.

There are algorithmic approaches which can prove multi-basic hypergeometric series identities. Zeilberger's creative telescoping paradigm in the algebraic setting of difference fields [17] is one of these algorithms which computes a recurrence for the given sum $S(n) = \sum_{k=0}^{m} f(n, k)$ where m might depend linearly on n, by finding $c_0(n), \ldots, c_\delta(n)$ and $g(n, k)$ such that $g(n, k+1) - g(n, k) = c_0(n) f(n, k) + \ldots + c_\delta(n) f(n+\delta, k)$. Another is Gosper's summation algorithm which finds a hypergeometric closed form of an indefinite sum of hypergeometric terms, if such a closed form exists. This algorithm was extended by Bauer in [2] to the case when the terms are simultaneously hypergeometric and multibasic hypergeometric. Riese[15] also presented an algebraically motivated generalization of Gosper's algorithm to indefinite bibasic hypergeometric summation. Exploiting these algorithms might contribute to find and prove additional identities of the p, q-binomial coefficients.

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