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ON THE DARBOUX TRANSFORMATION. I

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Abstract. Automorphisms of the family of all Sturm-Liouville equations $y'' = qy$ are investigated. The classical Darboux transformation arises as a particular case of a general result.

1. The Darboux theorem. Let nonvanishing functions $y = y(x)$, $z = z(x)$ be solutions of the Sturm-Liouville equations

$$y'' = p(x)y, \quad z'' = q(x)z, \tag{1}$$

where the potentials $p = p(x)$, $q = q(x)$ differ by a constant: $p - q = \lambda \in \mathbb{R}$. Then the function $\bar{y} = y' - yz'/z$ satisfies the Sturm-Liouville equation

$$\bar{y}'' = \bar{p}(x)\bar{y} \qquad (\bar{p} = p - 2(\frac{z'}{z})'). \tag{2}$$

For a direct proof, various modifications and numerous applications on the soliton theory we refer to [1]. For a very striking proof based on factorization of differential operators see [2]. Our aim is to propose a more systematic approach within the framework of Monge systems (underdetermined systems of ordinary differential equations).

2. Generalization. *Retaining the notation and all assumptions as above, let us introduce new variables*

$$u = \frac{y'}{y} + \frac{z'}{z}, \quad v = \frac{y'}{y} - \frac{z'}{z} \tag{3}$$

and choose quite arbitrary functions $f = f(\lambda)$, $g = g(\lambda, x, v) \neq 0$. Then the functions

$$\tilde{y} = e^{\frac{1}{2}\int(\tilde{u}+\tilde{v})dx}, \qquad \tilde{z} = e^{\frac{1}{2}\int(\tilde{u}-\tilde{v})dx}, \tag{4}$$

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where

$$\tilde{u} == \frac{1}{g}(f - \frac{\partial g}{\partial x} - v'\frac{\partial g}{\partial v}), \quad \tilde{v} = g, \tag{5}$$

satisfy certain Sturm-Liouville equations

$$\tilde{y}'' = \tilde{p}(x)\tilde{y}, \quad \tilde{z}'' = \tilde{q}(x)\tilde{z} \tag{6}$$

such that the difference of new potentials $\tilde{p} = \tilde{p}(x)$, $\tilde{q} = \tilde{q}(x)$ *is equal to the constant* $\tilde{p} - \tilde{q} = f \in \mathbb{R}$.

Proof. Clearly

$$\frac{y''}{y} - \frac{z''}{z} = p - q = \lambda = \text{ const.} \tag{7}$$

In terms of variables (3), this second-order relation turns into the first-order Monge system

$$v' + uv = \lambda, \quad \lambda' = 0. \tag{8}$$

We shall look for transformations of (8) into itself. Assuming that the independent variable is preserved, $\tilde{x} = x$, we shall quite arbitrarily choose the transformed $\tilde{\lambda} = f(\lambda)$ and $\tilde{v} = g(\lambda, x, v)$. Then clearly

$$(v')\tilde{} = \left(\frac{dv}{dx}\right)^{\tilde{}} = \frac{d\tilde{v}}{dx} = (\tilde{v})' = \frac{\partial g}{\partial x} + v'\frac{\partial g}{\partial v},$$

hence $\tilde{v}' + \tilde{u}\tilde{v} = \tilde{\lambda}$ (see (8)) gives (5):

$$\tilde{u}' = (\tilde{\lambda} - \tilde{v}')/\tilde{v} = \frac{1}{g}(f - \frac{\partial g}{\partial x} - v'\frac{\partial g}{\partial v}).$$

Finally, using the counterparts of (3) for the transformed variables, namely

$$\tilde{u} = \frac{\tilde{y}'}{\tilde{y}} + \frac{\tilde{z}'}{\tilde{z}}, \qquad \tilde{v} = \frac{\tilde{y}'}{\tilde{y}} - \frac{\tilde{z}'}{\tilde{z}},$$

the sought result (4) follows at once. According to the construction, the difference of new potentials is

$$\tilde{p} - \tilde{q} = \frac{\tilde{y}''}{\tilde{y}} - \frac{\tilde{z}''}{\tilde{z}} = \tilde{v}' + \tilde{u}\tilde{v} = \tilde{\lambda} = f(\lambda). \quad \square$$

3. Some particular cases. For the special choice $f = A\lambda$, $g = Bv$, where $A, B \in \mathbb{R}$ are constants, one can easily find

$$\tilde{u} = (\frac{A}{B} - 1)\frac{v'}{v} + \frac{A}{B}u,$$

whence

$$\tilde{y} = v^{\frac{1}{2}(\frac{A}{B}-1)} y^{\frac{1}{2}(\frac{A}{B}+B)} z^{\frac{1}{2}(\frac{A}{B}-B)},$$
$$\tilde{z} = v^{\frac{1}{2}(\frac{A}{B}-1)} y^{\frac{1}{2}(\frac{A}{B}-B)} z^{\frac{1}{2}(\frac{A}{B}+B)}.$$

These results simplify if, for example, $A = B$ or $A = B^2$. In the latter case, it follows in particular that the potentials of the couple $\tilde{y} = vy^3$, $\tilde{z} = vz^3$ (the choice $B = 3$) or the potentials of $\tilde{y} = 1/vy$, $\tilde{z} = 1/vz$ (the choice $B = -1$), differ by a constant.

Analogously, for the special choice $f = A\lambda$, $g = B\lambda/v$ where $A, B \in \mathbb{R}$ are constants, one can find $\tilde{u} = Av/B + v'/v$, whence

$$\tilde{y} = v^{\frac{1}{2}(1+B)} y^{\frac{1}{2}(\frac{A}{B}+B)} z^{-\frac{1}{2}(\frac{A}{B}-B)}, \tilde{z} = v^{\frac{1}{2}(1-B)} y^{\frac{1}{2}(\frac{A}{B}-B)} z^{-\frac{1}{2}(\frac{A}{B}+B)}.$$

This result simplifies if, for example, $B = \pm 1$, or $A = \pm B$. In particular, if $A = B = 1$ (hence $f = \lambda$ and $g = \lambda/v$) we obtain $\tilde{u} = v + v'/v$, hence $\tilde{y} = y' - yz'/z = \bar{y}$ (the Darboux transformation), $\tilde{z} = 1/z$. It follows that

$$\tilde{q} = \frac{\tilde{z}''}{\tilde{z}} = -q + 2(\frac{z'}{z})^2 = q - 2(\frac{z'}{z})'$$

hence $\tilde{p} = \tilde{q} + f = p - 2(z'/z)' = \bar{p}$. This is the Darboux theorem.

4. Some comments. We shall seek the true sense of the above elementary proof. First of all, it is clear that the main result may still be generalized if, for example, either the function q is allowed to depend on some higher order derivatives or the variable x is not preserved. In full generality, let us deal with the infinite-dimensional space $\mathbf{M}$ with coordinates $\lambda, x, v, v', v'', \ldots$ (owing to (8_1), u can be expressed in terms of them, and owing to (8_2) the derivatives of λ are not needed). Then the infinite system

$$d\lambda = dv - v'dx = dv' - v''dx = \cdots = 0 \tag{9}$$

clearly involves all given data. Every mapping of (9) into itself which turns the above coordinates into new ones $\tilde{\lambda}, \tilde{x}, \tilde{v}, (v')\tilde{}, (v'')\tilde{}, \ldots$ (and preserves the formula (8_1)) can be as for a generalization of the classical Darboux result.

In Section (2) above, we have apriori postulated $\tilde{x} = x$, $\tilde{\lambda} = f(\lambda)$, $\tilde{v} = g(\lambda, x, v)$, but this is not necessary. For instance, for the choice $\tilde{x} = x$, $\tilde{\lambda} = \lambda$, $\tilde{v} = v'/v$, one can easily find the transformed variable $\tilde{u} = \lambda v/v' - v''/v' - v'/v$, whence $\tilde{y} = v^\lambda/v'$, $\tilde{z} = v^\lambda/v'v^2$. We shall not continue in this direction here, however.

It is worth mentioning that if we are interested in automorphism of (9), i.e., in bijective transformations, then all truncated finite dimensional systems

$$d\lambda = 0, \quad d\lambda = dv - v'dx = 0, \quad d\lambda = dv - v'dx = \\ = dv' - v''dx = 0, \ldots, \tag{10}$$

are transformed into themselves (see [3]). It follows that such automorphisms are (prolonged) contact transformations depending on the parameter λ. If, moreover, the invariance $\tilde{x} = x$ is postulated, then necessarily $\tilde{v} = g(\lambda, x, v)$ is a certain function of λ, x, v (as follows from the invariance of the system (10_2)), and the invertibility is guaranteed if

$$f' \neq 0, \quad \partial g / \partial v \neq 0. \tag{11}$$

So we have the following result: Assuming (11), the generalization of Section 2 includes all transformations between the families of couples $(y, z) \to (\tilde{y}, \tilde{z})$ which are bijective.

Adopting the point of view of infinite-dimensional Pfaffian systems, various developments are at hand, but we briefly mention only the following one: if instead of (1) we deal with the equations

$$y''' = p(x)y, \quad z''' = q(x)z \tag{12}$$

of third order, then the problem leads to the Monge system

$$v'' + \frac{3}{2}(uv)' + \frac{1}{4}u(u^2 + 3v^2) = \lambda, \quad \lambda' = 0 \tag{13}$$

for the same functions (3) as above. Such a system cannot be investigated by elementary methods. We postpone these problems to a future publication.

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