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FRACTIONAL INTEGRODIFFERENTIATION IN HÖLDER CLASSES OF ARBITRARY ORDER

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ABSTRACT. Hölder classes of variable order $\mu(x)$ are introduced and it is shown that the fractional integral I^{α}_{0+} has Hölder order $\mu(x)+\alpha$ $(0<\alpha,\ \mu_+,\ \alpha+\mu_+<1,\ \mu_+=\sup\mu(x))$.

The connection between the smoothness of image and the smoothness of density is an important problem in the theory of integral operators. In particular, the same problem for fractional integral operators $I^{\alpha}_{0+}\varphi$ in the interval $[a,b]$ was thoroughly studied by G. H. Hardy and J. E. Littlewood (see [1], p. 56). They showed that the operator $I^{\alpha}_{0+}\varphi$ acts from $H^{\alpha}_0[a,b]$ into $H^{\alpha+\mu}_0[a,b]$ where $0<\mu,\alpha,\mu+\alpha<1$, and $\varphi\in H^{\mu}_0$ means that $\varphi\in H^{\mu}$ and $\varphi(a)=0$, i.e., ane improvement in smoothness takes place. In fact, the following stronger statement by the same authors is likewise true: $I^{\alpha}_{0+}\varphi$ implements an isomorphism between these classes.

The above results were subsequently generalized on weighted classes, generalized Hölder classes, and Besov spaces [1–3].

It is wellknown [1] that the connection between the fractional integrals I^{α}_{+} and I^{α}_{-} by means of a singular integral operator is important for the study of isomorphism in weighted classes. Here we consider classes of functions having order μ everywhere but the Hölder order being $\mu_0>\mu$ at the point $x_0\in[a,b]$ (see the definition and the examples below). It should be noted (see [4]) that the singular integral does not preserve such a class. The question arises whether the fractional integral preserves the local Hölder order. To this we give a positive answer.

Here we introduce Hölder classes of variable order $\mu(x)$ and show that I^{α}_{0+} has Hölder order $\mu(x)+\alpha$ $(0<\mu,\mu_+,\alpha+\mu_+<1,\ \mu_+=\sup\mu(x))$.

If, in addition, we assume $\mu(x)$ to have smoothness of the form

$$|\mu(x+h)-\mu(x)|<c|\ln h|^{-1}$$

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(c denotes all constants used in this paper) we will be able to study the action of a fractional derivative and to solve the problem of an isomorphism. Moreover, we shall extend the results to the case of power weights.

1. Introduction of Spaces

We will introduce a class of functions $f(x)$ on the segment $[a,b]$ satisfying the Hölder condition with a variable index.

Definition 1.1. We denote by $H^{\mu(x)}$ a class of continuous functions on the segment $[a,b]$ with the norm

$$\|f\|_{\mu} = \|f\|_{C[a,b]} + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\mu(x)}} = \|f\|_{C} + A_f, \tag{1.1}$$

where $\mu(x)$ is defined for all $x \in [a,b]$, $0<\mu\leq 1$. We introduce the notation $\mu_- = \inf_{x\in[a,b]} \mu(x)$, $\mu_+ = \sup_{x\in[a,b]} \mu(x)$, assuming throughout the paper that $\mu_- > 0$.

Clearly, $H^{\mu(x)}$ is a linear space and also $H^{\gamma(x)} \subset H^{\mu(x)}$ if $\mu(x)\leq\gamma(x)$. For $H^{\mu(x)}[a,b]$ the following ordinary Hölder properties are satisfied: 1) the space $H^{\mu(x)}$ is complete; 2) if $f(x)\in H^{\mu(x)}[a,c]$ and $f(x)\in H^{\nu(x)}[c,b]$, and $f(x)$ is continuous for $x=c$, then $f(x)\in H^{\gamma(x)}[a,b]$, where $\gamma(x)=\mu(x)$ for $a\leq x<c$, $\gamma(x)=\nu(x)$ if $c<x\leq b$, and $\gamma(x)=\min(\mu(c),\nu(c))$; 3) if $f(x)\in H^{\mu(x)}[a,b]$, $g(x)\in H^{\nu(x)}[a,b]$, then $f[g(x)]\in H^{\gamma(x)}[a,b]$, where $\gamma(x)=\mu(x)\nu(x)$.

We adduce some examples of functions belonging to this space.

Example 1. $f(x) = \prod\limits_{k=1}^{n} |x-x_k|^{\delta_k}$, $0<\delta_k\leq 1$, $x_k\in[a,b]$, where $\mu(x)=\delta_k$ for $|x-x_k|\leq\varepsilon$, $\varepsilon=\frac{1}{4}\inf_{i\neq j}|x_i-x_j|$, and $\mu(x)=1$ for the rest of x.

Example 2. Let $[a,b]=[0,1]$, $x_1=1/2$, $x_k=1/2+1/k$, and $f(x)=|x-x_1|+\sum_{k=2}^{\infty}|x-x_k|^{1/2}2^{-k}$. It is evident that $f(x)\in H^{1/2}[0,1]$, but for $x=1/2$ the Hölder order is higher: $\mu(1/2)=1$. Namely,

$$\big|f(x)-f(1/2)\big| \leq A\big|x-1/2\big|,$$

where

$$A = 1+2\sum_{k=1}^{\infty} k^{1/2}2^{-k}.$$

This fact can be proved using the inequality

$$\big|(1+t)^{\mu}-1\big| \leq c|t|, \quad t\geq -1, \quad 0<\mu\leq 1. \tag{1.2}$$

The next example is a generalization of the previous one.

Example 3. Let $Q = \{x_k\}_{k=1}^{\infty}$ be an ordered sequence of rational numbers from $[0,1]$. We represent it in the form $Q = \cup Q_l$, where $\{x_{k_l}\}_{l=1}^{\infty} \in Q_l$ and $\frac{1}{k+1} < |x_{k_l} - \frac{1}{2}| \le \frac{1}{k}$. We introduce the notation

$$f(x) = |x - x_0| + \sum_{k=2}^{\infty} 2^{-k} \sum_{l=1}^{\infty} |x_{k_l} - x|^{\delta_{k_l}} 2^{-l},$$

$x_0 = \frac{1}{2}$, $\delta_{k_l} \in (0,1)$ (for example, $\delta_{k_l} = \frac{1}{2}$), and $\delta_0 = \inf_{k,l} \delta_{k_l} > 0$. Then $f(x) \in H^{\mu(x)}[0,1]$, where $\mu(x) = \frac{1}{2}$ if $x \neq \frac{1}{2}$, and $\mu(\frac{1}{2}) = 1$.

We also introduce a class of functions $h^{\mu(x)}[a,b]$ satisfying a stronger condition than (1.1).

Definition 1.2. Let $h^{\mu(x)}$ be a set of functions from $H^{\mu(x)}[a,b]$ which have the same norm and for which

$$\lim_{\delta \to 0} A_f(\delta) = \lim_{\delta \to 0} \sup_{|x-y|<\delta} \frac{|f(x) - f(y)|}{|x - y|^{\mu(x)}} = 0. \tag{1.3}$$

It is evident that $h^{\mu(x)}[a,b] \subset H^{\mu(x)}[a,b]$.

If $f(a) = 0$, the classes H and h will be denoted by $H_0^{\mu(x)}[a,b]$ and $h_0^{\mu(x)}[a,b]$, respectively. Note that in $H_0^{\mu(x)}[a,b]$ the norm is defined by one term A_f from (1.1).

For $h^{\mu(x)}[a,b]$ we have

Theorem 1.1. *The functions from $C^{\infty}[a,b]$, $-\infty < a < b < \infty$, are dense in $h^{\mu(x)}$ if $\mu(x)$ is a function satisfying the inequality*

$$|\mu(x+h) - \mu(x)| \le c|\ln h|^{-1} \tag{1.4}$$

and $0 < \mu(x) < 1$.

Proof. (It can be assumed for simplicity that $[a,b] = [0,1]$.) Let $\varphi(x) \in h^{\mu(x)}[0,1]$. Then

$$\psi(x) = \begin{cases} \varphi(x) - \varphi(0)(1-x) - \varphi(1)x, & 0 \le x \le 1, \\ 0, & x \notin [0,1] \end{cases}$$

belongs to $C_0(R^1)$. It is clear that $\psi(x) \in H^{\widetilde{\mu}(x)}$, where $\widetilde{\mu}(x)$ is equal to $\mu(x)$ if $0 \le x < 1$, to $\mu(0)$ if $x \le 0$, and to $\mu(1)$ if $x \ge 0$.

Throughout the proof of this theorem it will be assumed that $\varphi(x)$ has the above-indicated properties for $\psi(x)$ and $\widetilde{\mu}(x)$ will be replaced by $\mu(x)$.

Let

$$\varphi_\varepsilon(x) = \int_0^{\varepsilon} k_\varepsilon(t) \varphi(x - t)\, dt, \tag{1.5}$$

where $k(t) \in C_0^\infty(R_1)$, $k(t) \geq 0$, $\operatorname{supp} k(t) \subset (0,1)$, $k_\varepsilon(t) = \varepsilon^{-1} k(\frac{t}{\varepsilon})$, and $\int_{R_1} k(t)dt = 1$. Then $\varphi_\varepsilon(t)$ is the average for $\varphi(x)$ and $\|\varphi_\varepsilon - \varphi\|_{C(R_1)} \to 0$ for $\varepsilon \to 0$.

Our aim is to estimate $\|\varphi_\varepsilon - \varphi\|_\mu$. We have

$$\varphi_\varepsilon(x) - \varphi(x) = \int_0^\infty k_\varepsilon(t)[\varphi(x-t) - \varphi(x)]\, dt$$

and, introducing the notation

$$(\Delta_h \psi)(x) = \psi(x+h) - \psi(x), \tag{1.6}$$

we can write

$$\Delta_h(\varphi_\varepsilon - \varphi)(x) = \Big(\int_0^\delta + \int_\delta^\infty\Big) k_\varepsilon(t)\big[(\Delta_h\varphi)(x-t) - (\Delta_h\varphi)(x)\big]\, dt = I_1 + I_2.$$

Further, $|(\Delta_h\varphi)(x)| \leq A_\varphi(t)h^{\mu(x)}$ when $h < \tau$; here $A_\varphi(\tau)$ is defined by (1.3), and $A_\varphi(\tau) \to 0$ when $\tau \to 0$.

For $(\Delta_h\varphi)(x-t)$ we have the inequality

$$\big|(\Delta_h\varphi)(x-t)\big| \leq A_\varphi(\tau)|h|^{\mu(x-t)} \leq A_\varphi(\tau)|h|^{\mu(x)},$$

where $\mu(x)$ satisfies (1.4).

Therefore for I_1 we have for $h < \tau$ that

$$|I_1| \leq 2A_\varphi(\tau)|h|^{\mu(x)} \int_0^\delta k_\varepsilon(t)\, dt \leq 2A_\varphi(\tau)|h|^{\mu(x)}.$$

This expression can be made small at the expense of an appropriate choice of τ with respect to $\varphi(x)$.

Next, we fix τ and estimate I_1 for $h > \tau$ by the equality

$$I_1 = \int_0^\delta k_\varepsilon(t)\big[(\Delta_{-t}\varphi)(x)(x+h) - (\Delta_{-t}\varphi)(x)\big]\, dt.$$

We obtain

$$\big|(\Delta_{-t}\varphi)(x)\big| \leq c|t|^{\mu(x)}; \quad \big|(\Delta_{-t}\varphi)(x+h)\big| \leq c|t|^{\mu(x+h)} \leq c|t|^{\mu(x)}.$$

Now for $h > \tau$ we can write

$$|I_1| \leq 2c\int_0^\delta k_\varepsilon(t)|t|^{\mu(x)}dt \leq 2c\delta^{\mu(x)}\int_0^1 k_\varepsilon(t)y^{\mu(x)}dy \leq$$
$$\leq 2c\tau^{-\mu(x)}h^{\mu(x)}\delta^{\mu(x)}\int_0^1 k_\varepsilon(y)y^{\mu(x)}dy.$$

This expression is $o(h^{\mu(x)})$ at the expense of choosing a small δ. Now we fix δ and estimate I_2. We have

$$\|I_2\| \le 2c|h|^{\mu(x)} \int_{\delta}^{\infty} k_{\varepsilon}(t)\,dt = 2c|h|^{\mu(x)} \int_{\delta/\varepsilon}^{\infty} k(y)\,dy \to 0$$

when $\varepsilon \to 0$. $\square$

2. Fractional Integrals and Derivatives in $H_0^{\mu(x)}[0,1]$

We consider the action of fractional integral operators in $H_0^{\mu(x)}[0,1]$. For this we should give some definitions.

Definition 2.1. Let $\varphi(x)$ be an arbitrary function from $L_1(0,1)$. The operators below will be called the Riemann–Liouville fractional integrals:

$$\left(I_{0+}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\varphi(t)\,dt, \tag{2.1}$$

$$\left(I_{1-}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1}\varphi(t)\,dt \tag{2.2}$$

for $x \in (0,1)$, $0 < \alpha < 1$. Integrals (2.1) and (2.2) will be called the left-hand and the right-hand one, respectively.

Definition 2.2. Let $f(x)$ be defined on $[0,1]$. The expression

$$\left(\mathbb{D}_{0+}^{\alpha} f\right)(x) = \frac{f(x)}{\Gamma(1-\alpha)x^{\alpha}} + \\ + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-(1+\alpha)}\big(f(x)-f(t)\big)\,dt \tag{2.3}$$

is called the Marchaud derivative (it is defined for sufficiently good functions; for details see [1]).

The next theorem giving the sufficient condition for the function $f(x)$ to be represented by the fractional integral of a function from L_p (see [1, Theorem 13.5]) will be necessary hereafter to demonstrate the theorem on isomorphism.

Theorem 2.1. *If $f(x) \in L_p(a,b)$ and*

$$\int_0^{b-a} t^{1-\alpha}\omega_p(f,t)\,dt < \infty,$$

then $f(x)$ can be represented by the fractional integral of a function from $L_p(a,b)$, $1 < p < \frac{1}{\alpha}$, where

$$\omega_p(f,h) = \sup_{|t|<h} \left\{ \int_0^1 |f(x)-f(x-t)|^p dx \right\}^{1/p}. \tag{2.4}$$

Now we can formulate the main result of this paper.

Theorem 2.2 (on isomorphism). *Let $\mu(x)$ satisfy* (1.4) *and* $0 < \mu(x) < 1$, $\mu_+ + \alpha < 1$, $0 < \alpha < 1$. *Then the fractional integral operator* I^α_{0+} *establishes an isomorphism between the spaces* $H_0^{\mu(x)}$ *and* $H_0^{\mu(x)+\alpha}$.

The proof of Theorem 2.2 is based on the theorems on the action of the fractional operator $I^\alpha_{0+}\varphi$ and the fractional derivative $\mathbb{D}^\alpha_{0+} f$ in $H_0^{\mu(x)}$. These theorems and their proofs are given below.

Theorem 2.3. *If* $\varphi(x) \in H_0^{\mu(x)}[a, b]$ *and* $\mu_+ + \alpha < 1$, $0 < \alpha < 1$, *then* $f \in H_0^{\mu(x)+\alpha}$, *where* $f(x) = I^\alpha_{0+}\varphi$.

Theorem 2.4. *If* $f(x) \in H_0^{\mu(x)+\alpha}[0, 1]$, $\mu_+ + \alpha < 1$, *and* $\mu(x)$ *satisfies* (1.4), *then* $(\mathbb{D}^\alpha_{0+} f)(x) \in H_0^{\mu(x)}[0, 1]$.

Proof of Theorem 2.3. We will estimate $(\Delta_h I^\alpha_{0+}\varphi)(x)$, taking into account (1.5). There are two separate proofs for $h > 0$ and $h < 0$, since we cannot interchange the points x and $x + h$ considered in $H_0^{\mu(x)}[0, 1]$.

Let $h > 0$. We have

$$c\big(\Delta_h I^\alpha_{0+}\varphi\big)(x) = \int_0^{x+h} t^{\alpha-1}\varphi(x+h-t)dt - \int_0^x t^{\alpha-1}\varphi(x-t)dt = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_0^x \big[(t+h)^{\alpha-1} - t^{\alpha-1}\big]\big[\varphi(x-t) - \varphi(x)\big]\,dt,\\ I_2 &= \int_0^h \big[\varphi(x+h-t) - \varphi(x)\big]\,dt,\\ I_3 &= \int_x^{x+h} t^{\alpha-1}\,dt. \end{aligned}$$

Taking into account that $|(\Delta_t\varphi)(x)| \le ct^{\mu(x)}$ and estimating each term separately, we obtain

$$\begin{aligned} |I_1| &\le c\int_0^x t^{\mu(x)}\big|(t+h)^{\alpha-1} - t^{\alpha-1}\big|\,dt \le \\ &\le ch^{\mu(x)+\alpha}\int_0^\infty y^{\mu(x)}\big|(1+y)^{\alpha-1} - y^{\alpha-1}\big|\,dy. \end{aligned}$$

It is obvious that the obtained integral is finite $(\mu_+ + \alpha < 1)$ and we have the required estimate

$$|I_1| \le ch^{\mu(x)+\alpha}$$

for I_1. The integrals I_2 and I_3 are estimated quite easily. We get

$$|I_2| \le c \int_0^h t^{\alpha-1}(h-t)^{\mu(x)}\, dt = ch^{\mu(x)+\alpha},$$
$$|I_3| \le cx^{\mu(x)} \int_x^{x+h} t^{\alpha-1}\, dt \le c \int_x^{x+h} t^{\mu(x)+\alpha-1}\, dt \le ch^{\mu(x)+\alpha}.$$

The proof for the case $h > 0$ is completed. The case $h < 0$ is not difficult and we leave it out. □

Remark 1. Estimates as above are correct for convolutions. For $K\varphi$ we have the estimate

$$|(\Delta_h K\varphi)(x)| \le c\Big\{ \int_0^x t^{\mu(x)} |(\Delta_h k)(t)|\, dt +$$
$$+ \int_0^h |k(t)|(h-t)^{\mu(x)} dt + x^{\mu(x)} \Big| \int_x^{x+h} k(t)\, dt \Big| \Big\}.$$

It is obvious that for $k(t) = t^{\alpha-1}$ we obtain Theorem 2.3.

Remark 2. Using the well-known representation $I^\alpha_{1-}\varphi = VI^\alpha_{0+}V\varphi$, where $V\varphi = \varphi(1-x)$, for $x \in [0,1]$, we find that Theorem 2.3 holds for $I^\alpha_{1-}\varphi$ provided that $\varphi(1) = 0$.

Proof of Theorem 2.4. We introduce the notation (see (2.3))

$$\psi(x) = \int_0^x \frac{f(x) - f(x-t)}{t^{1+\alpha}}\, dt. \tag{2.5}$$

First we will show that $\psi(x) \in H_0^{\mu(x)}[0,1]$ and then consider the first term of (2.3). In common with the proof of Theorem 2.3, here we will also treat the cases $h > 0$ and $h < 0$ separately.

Let $h > 0$. We have

$$(\Delta_h \psi)(x) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^x \big[f(x) - f(x-t)\big]\big[(t+h)^{-\alpha-1} - t^{-\alpha-1}\big] dt,$$
$$I_2 = \int_{-h}^0 \frac{f(x+h) - f(x-h)}{(t+h)^{1+\alpha}}\, dt,$$
$$I_3 = \int_0^x \frac{f(x+h) - f(x)}{(t+h)^{1+\alpha}}\, dt.$$

We will estimate each term separately. It is easy to verify that for I_1 and I_3

$$
\begin{aligned}
|I_1| &\le c\int_0^x t^{\mu(x)+\alpha}\big[(t+h)^{-\alpha-1}-t^{-\alpha-1}\big]\,dt \le \\
&\le ch^{\mu(x)}\int_0^\infty y^{\mu(x)+\alpha}\big[(y+1)^{-\alpha-1}-y^{-\alpha-1}\big]\,dy \le ch^{\mu(x)}, \\
|I_3| &\le ch^{\mu(x)+\alpha}\int_0^x (t+h)^{-\alpha-1}\,dt \le ch^{\mu(x)}\int_0^\infty (y+1)^{-\alpha-1}\,dy = ch^{\mu(x)}.
\end{aligned}
$$

The term I_2 is estimated by applying (2.4) for $\mu(x)$:

$$|I_2| \le c\int_{-h}^0 (t+h)^{\mu(x+h)-1}\,dt = ch^{\mu(x+h)}.$$

We obtain the final estimate for I_2 if $\mu(x+h)\ge\mu(x)$, i.e., $|I_2|\le ch^{\mu(x)}$. For $\mu(x+h)\le\mu(x)$ we can write

$$h^{\mu(x+h)} = e^{-(\mu(x)-\mu(x+h))\ln h}\cdot h^{\mu(x)} \le ch^{\mu(x)}.$$

Thus we finally show that

$$|I_2| \le ch^{\mu(x)}.$$

To finish the case $h>0$ we have to show that $|\Delta_h(\frac{f(x)}{x^\alpha})| \le ch^{\mu(x)}$. We leave out the proof of the latter estimate, since it is similar to the case of constant Hölder order (see [5]).

Now let $h<0$. For convenience, instead of $x+h$ for $h<0$ we will consider $x-h$ with $h>0$. Then we will decompose the terms rather than otherwise (see [2]):

$$
\begin{gathered}
(\Delta_{-n}\mathbb{D}^\alpha_{0+}f)(x) = (x-h)^{-\alpha}f(x-h) - x^{-\alpha}f(x) + \\
+\alpha\int_0^{x-h}\frac{f(x-h)-f(x-h-t)}{t^{1+\alpha}}\,dt - \alpha\int_0^x\frac{f(x)-f(x-t)}{t^{1+\alpha}}\,dt = \\
= I_1+I_2+I_3+I_4,
\end{gathered}
$$

where

$$
\begin{aligned}
I_1 &= \int_{-h}^0 \frac{f(x)-f(x-h-t)}{(t+h)^{1+\alpha}}\,dt, \\
I_2 &= \int_0^{x-h}\big[f(x-h-t)-f(x-h)\big]\Big[\frac{1}{t^{1+\alpha}}-\frac{1}{(t+h)^{1+\alpha}}\Big]\,dt, \\
I_3 &= f(x-h)[x^{-\alpha}-(x-h)^{-\alpha}], \\
I_4 &= h^{-\alpha}[f(x)-f(x-h)].
\end{aligned}
$$

We will estimate each term separately. For I_1 and I_4 we easily obtain

$$|I_1| \leq c\int_{-h}^{0}(t+h)^{\mu(x)-1}dt = ch^{\mu(x)},$$
$$|I_4| \leq h^{-\alpha}h^{\mu(x)+\alpha}dt \leq ch^{\mu(x)}.$$

To estimate the term I_2 we have to consider two cases, $x-h \leq h$ and $x-h > h$.

For $x-h \leq h$ we have

$$|I_2| \leq c\int_0^h t^{\mu(x-h)+\alpha-1-\alpha}dt = ch^{\mu(x-h)}.$$

By virtue of (2.4) we have $|I_2| \leq ch^{\mu(x)}$.

For $x-h > h$ we have

$$|I_2| \leq c\Big|\Big(\int_0^h + \int_h^{x-h}\Big)t^{\mu(x-h)+\alpha}\Big[\frac{1}{t^{1+\alpha}} - \frac{1}{(t+h)^{1+\alpha}}\Big]dt\Big| \leq$$
$$\leq c\Big|\int_0^h t^{\mu(x-h)-1}dt\Big| + ch^{\mu(x-h)}\int_1^\infty y^{\mu(x-h)+\alpha}y^{-2-\alpha}dy \leq ch^{\mu(x)}.$$

Now we will estimate I_3. Keeping in mind that $f(x-h) \leq (x-h)^{\mu(x-h)+\alpha}$, for $x-h \leq h$ we obtain

$$|I_3| \leq c(x-h)^{\mu(x-h)} \leq ch^{\mu(x)}.$$

When $x-h > h$ we have

$$|I_3| \leq c(x-h)^{\mu(x-h)+\alpha}(x-h)^{-\alpha-1}h \leq ch^{\mu(x)}.$$

This completes the proof of Theorem 2.4. □

Thus it remains for us to prove the theorem on isomorphism.

Proof of Theorem 2.2. It is necessary to show that any function from $H_0^{\mu(x)+\alpha}[0,1]$ is representable in the form $I_{0+}^\alpha\varphi$, where $\varphi \in H_0^{\mu(x)}[0,1]$. To this end we use Theorem 2.1. We make sure that for $1 < p < 1/\alpha$

$$\int_0^1 t^{-\alpha-1}\omega_p(f,t)\,dt < \infty.$$

Clearly,

$$\omega_p(f,h) \leq \sup_{|t|<h}\Big\{\int_0^1 t^{\mu(x)+\alpha}dx\Big\}^{1/p} \leq ch^{\mu_-+\alpha}$$

and we obtain $f(x) = I_{0+}^\alpha\varphi$, where $\varphi(x) \in L_p(0,1)$. Using the condition $f(x) \in H_0^{\mu(x)}[0.1]$ and Theorem 2.4, we find that $\varphi = \mathbb{D}_{0+}^\alpha f \in H_0^{\mu(x)}[0,1]$. □

It should be noted that the theorem on isomorphism in the Hölder space with constant order is wellknown (see [1]). Here we will give only the formulations of the theorems on the action of the operators I_{0+}^{α} and $\mathbb{D}_{0+}^{\alpha}$ in h_0^{μ}, since their proofs are rather awkward.

Theorem 2.5. *Let* $0 < \mu, \alpha, \mu + \alpha < 1$. *Then* $I_{0+}^{\alpha}\varphi$ *acts from* $h_0^{\mu}[0,1]$ *into* $h_0^{\mu+\alpha}[0,1]$.

Theorem 2.6. *If* $f \in h_0^{\mu+\alpha}[0,1]$, *then* $\mathbb{D}_{0+}^{\alpha} f \in h_0^{\mu}[0,1]$, *where* $0 < \mu, \alpha$, $\mu + \alpha < 1$.

Finally, we note that the same results are true in the weight case as well. Namely, we have

Theorem 2.7. *If* $\varphi(x) \in H_0^{\mu(x)}(x^{\nu})$ *and* $\mu_+ + \alpha < 1$, $0 < \alpha < 1$, *then* $f(x) \in H_0^{\mu(x)}(x^{\nu})$, *where* $f(x) = I_{0+}^{\alpha}\varphi$ *and* $\nu = 1 + \mu_-$.

Theorem 2.8. *Let* $f(x) \in H_0^{\mu(x)+\alpha}(x^{\nu})$, $\mu_+ + \alpha < 1$, μ *satisfy* (1.4) *and* $\nu \leq \mu_- + 1$. *Then* $(\mathbb{D}_{0+}^{\alpha} f)(x) \in H_0^{\mu(x)}(x^{\nu})$.

Theorem 2.9. *Let* μ *satisfy* (1.4) *and* $0 < \mu(x) < 1$, $0 < \alpha < 1$, $\mu_+ + \alpha < 1$, $\nu \leq \mu_- + 1$. *Then the fractional integral operator* I_{0+}^{α} *establishes an isomorphism between the spaces* $H_0^{\mu(x)}(x^{\nu})$ *and* $H_0^{\mu(x)+\alpha}(x^{\nu})$.

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