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ON THE EXISTENCE OF CONTINUOUS MODIFICATIONS OF VECTOR-VALUED RANDOM FIELDS

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Abstract. We consider vector-valued random fields on a general index set in $\mathbb{R}^d$ and show the existence of continuous modifications if some Kolmogorov type condition is satisfied. Furthermore, we prove an extension result for Hölder continuous vector valued mappings.

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1. Introduction

The main purpose of this paper is to prove the existence of locally Hölder continuous modifications of vector-valued random fields satisfying Kolmogorov type conditions.

Suppose we are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a separable real Banach space $(E, \|.\|_E)$. By $\mathcal{B}(E)$ we denote the σ-algebra of Borel sets of E. Furthermore, let $\Theta \subset \mathbb{R}^d$ be a nonempty subset.

A *random field* $X = (X_t)_{t\in\Theta}$ indexed by Θ and taking values in E is a family of measurable mappings $X_t : \Omega \to E$, $t \in \Theta$. The random field X is said to be *continuous* (*Hölder continuous, locally Hölder continuous*) if for each $\omega \in \Omega$ the mapping $t \mapsto X_t(\omega)$ from Θ into E is continuous (Hölder continuous, locally Hölder continuous).

Let us recall that for given $A \subset \mathbb{R}^d$, $A \neq \varnothing$, and a Banach space $(F, \|.\|_F)$ a mapping $g : A \to F$ is said to be *α-Hölder continuous* ($\alpha > 0$) if there exists a constant $c > 0$ such that

$$\|g(x)\|_F \le c \quad \text{and} \quad \|g(x) - g(y)\| \le c \cdot |x - y|^\alpha$$

hold for all $x, y \in A$. The smallest constant with this property is denoted by $|g|_\alpha$. Furthermore, the mapping g is called *locally α-Hölder continuous* if for each compact subset $B \subset A$ the restriction of g to B is α-Hölder continuous.

In many applications, for instance, if one wants to show further measurability properties or the pathwise boundedness of X the continuity of the paths of the field is very helpful. Therefore sufficient conditions for the existence of continuous modifications are of high importance.

The paper is divided into three sections. The next section is devoted to the main result and its proof. In the third section we prove an extension result for Hölder continuous mappings taking values in a Banach space.

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2. Main Result

Before we formulate the main result of this paper let us recall that a subset $\mathcal{O} \subset \mathbb{R}^d$ is said to be a domain if it is open, nonempty and connected.

We shall prove the following assertion.

Theorem 2.1. *Let $X = (X_t)_{t\in\Theta}$ be an E-valued random field satisfying*

$$\mathbb{E}\,\|X_t\|_E^p < \infty, \quad t \in \Theta, \tag{1}$$

and

$$\mathbb{E}\,\|X_t - X_s\|_E^p \le c \cdot |t-s|^{d+\beta}, \quad s,t \in \Theta, \tag{2}$$

for some constants $c > 0$, $\beta > 0$ and $p \ge 1$. Furthermore, suppose that one of the following two conditions holds:

(i) $\dfrac{d+\beta}{p} \in (0,1]$;

(ii) Θ *is a domain in $\mathbb{R}^d$ or the closure of a domain in $\mathbb{R}^d$.*

Then there exists a continuous modification $\tilde{X}$ of X that is locally Hölder continuous with exponent γ for every $\gamma \in (0, \frac{\beta}{p})$.

Condition (2) is a special case of the well known Kolmogorov condition. In case $E = \mathbb{R}$, $\Theta = [0,1]^d$ one can find proofs of the corresponding results, for instance, in [5] or in [8]. The authors in [6] consider more general Banach spaces. More general index sets are considered in [1] in the case of a general separable Banach space and in [3] and [4] respectively in the case of $E = \mathbb{R}$. In the proof of the corresponding result in [1] the authors use Sobolev type embedding arguments. Therefore they have to assume that Θ satisfies some regularity condition like the cone condition. In our proof we do not need such an assumption. We use some of the ideas in [3, the proof of Theorem 7] and [4, pp. 493–496] in a modified form.

In order to show the existence of a continuous modification of the random field X we will use the following auxiliary results.

Theorem 2.2. *Let $A \subset \mathbb{R}^d$ be a nonempty subset, F be a real Banach space and $g : A \to F$ be a mapping that is Hölder continuous with exponent $\alpha \in (0,1]$. Then there exists an extension $\overline{g} : \mathbb{R}^d \to F$ of g that is Hölder continuous with the same exponent.*

For the proof of this result we refer to the following section. In the same way as in the real valued case one proves the next result.

Lemma 2.3. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a domain and F be a real Banach space. Assume that $g : \mathcal{O} \to F$ or $g : \overline{\mathcal{O}} \to F$ is a mapping that is locally Hölder continuous with exponent $\alpha > 1$. Then g is a constant function.*

Remark. The statement of Theorem 2.2 is not true if $\alpha > 1$ holds. A simple counterexample is given by the mapping $g : [0,1] \cup [2,3] \to \mathbb{R}$ with $g(x) = 0$, $x \in [0,1]$, and $g(x) = 1$, $x \in [2,3]$. This mapping is α-Hölder continuous for any

$\alpha > 1$. By Lemma 2.3 an extension $\overline{g} : \mathbb{R} \to \mathbb{R}$ of g that is Hölder continuous with an exponent greater than one has to be a constant function. This leads to a contradiction.

Now, we are able to prove our main result.

Proof of Theorem 2.1. For $\Theta = [0,1]^d$ the existence of a continuous modification $\tilde{X}$ of X that is locally Hölder continuous with exponent γ for every $\gamma \in (0, \beta/p)$ follows from [6, Theorem I.2.1]. From this we get the corresponding result in the the case of $\Theta = [-N, N]^d$, $N \in \mathbb{N}$, by means of a space transformation. Now, the existence of a locally γ-Hölder continuous modification in case of $\Theta = \mathbb{R}^d$ follows by standard arguments.

Condition (1) yields $X_t \in \mathcal{L}_p(\Omega; E) := \mathcal{L}_p((\Omega, \mathcal{F}, \mathbb{P}); (E, \mathcal{B}(E)))$, $t \in \Theta$. For each $t \in \Theta$ we identify X_t and its equivalence class and consider the mapping $g : \Theta \to L_p(\Omega; E)$ defined by $g(t) := X_t$, $t \in \Theta$. By (2) this mapping is locally Hölder continuous with exponent $\frac{d+\beta}{p}$. Let us assume now that the condition (ii) is satisfied and that $\frac{d+\beta}{p} > 1$ holds. Since $L_p(\Omega; E)$ is a Banach space by Lemma 2.3, there exists an extension $\overline{g} : \mathbb{R}^d \to L_p(\Omega; E)$ that is Hölder continuous with the same exponent. For every $t \in \mathbb{R}^d \setminus \Theta$ we fix a representant $\xi_t \in \overline{g}(t)$ and define

$$Y_t := \begin{cases} X_t\,; & t \in \Theta \\ \xi_t\,; & t \in \mathbb{R}^d \setminus \Theta. \end{cases}$$

The random field $Y = (Y_t)_{t \in \mathbb{R}^d}$ satisfies conditions (1) and (2). By the arguments at the beginning of the proof there exists a modification $\tilde{Y} = (\tilde{Y}_t)_{t \in \mathbb{R}^d}$ of Y which is locally Hölder continuous with exponent γ for every $\gamma \in (0, \beta/p)$. Now, the random field $\tilde{X}$ given by $\tilde{X}_t := \tilde{Y}_t$, $t \in \Theta$, has the desired properties.

For the remaining case ($\frac{d+\beta}{p} \leq 1$) let us assume for a while that Θ is bounded. Then the mapping $g : \Theta \to L_p(\Omega; E)$ is Hölder continuous with exponent $\frac{d+\beta}{p}$. By Theorem 2.2 there exists an extension $\overline{g} : \mathbb{R}^d \to L_p(\Omega; E)$ that is Hölder continuous with the same exponent, and we can proceed as in the previous case.

Finally, if Θ is unbounded and $0 < \frac{d+\beta}{p} \leq 1$ holds, let N_0 be the smallest $N \in \mathbb{N}$ such that $\Theta \cap [-N, N]^d \neq \varnothing$ and consider the fields $X^N := (X_t)_{t \in \Theta \cap [-N,N]^d}$, $N \geq N_0$. From the previous considerations there follows the existence of modifications $\tilde{X}^N$ of X^N which are locally Hölder continuous with exponent γ for every $\gamma \in (0, \beta/p)$. This yields the existence of the desired modification $\tilde{X}$ of X by standard arguments. □

3. Proof of the Extension Result

In this section we prove Theorem 2.2. In the case of $F = \mathbb{R}$ and of a closed subset $A \subset \mathbb{R}^d$ one can find a proof of the corresponding result in [7, Chapter IV]. Here the mapping $\overline{g} : \mathbb{R}^d \to \mathbb{R}$ is constructed by means of an extension of Whitney type. We will use some of the ideas from this proof in the case of general Banach spaces F in a modified form.

Proof of Theorem 2.2. Step 1. Let us suppose that A is closed and nonempty. Without loss of generality we additionally assume $A \neq \mathbb{R}^d$.

Like in [7] we define the following concepts. By a cube we mean a closed cube in $\mathbb{R}^d$ with the sides parallel to the axis. Two such cubes are said to be disjoint if their interiors are disjoint. For a cube Q we denote its diameter by $\operatorname{diam}(Q)$ and its distance from A by $\operatorname{dist}(Q, A)$.

Furthermore, we construct a partition of unity analogously to the approach of [7], namely, there exists a countable family $\mathcal{H} = \{Q_1, Q_2, \dots, Q_k, \dots\}$ of cubes satisfying the following conditions (see [7, Chapter VI, Theorem 1 and its proof]):

- $\bigcup_k Q_k = A^c$;
- The cubes Q_k, $k = 1, 2, \dots$, are mutually disjoint;
- $\operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, A) \leq 4 \operatorname{diam}(Q_k)$.

For fixed $\varepsilon \in (0, 1/4)$ we define cubes Q_k^* by

$$Q_k^* := (1+\varepsilon) \cdot (Q_k - x^k) + x^k \quad k \in \mathbb{N}.$$

Here x^k denotes the center of Q_k. For the family $\mathcal{H}$ there exists $N \in \mathbb{N}$ such that each point $x \in A^c$ is contained in a neighborhood intersecting at most N cubes Q_k^*. (cf. [7, VI.1, proof of Proposition 3]) Moreover, there exist continuously differentiable mappings $\varphi_k^* : \mathbb{R}^d \to [0, 1]$, $k \in \mathbb{N}$, having the following three properties:

- $\varphi_k^*(x) = 1$, $x \in Q_k$, and $\varphi_k^*(x) = 0$, $x \notin Q_k^*$,
- $\sum_k \varphi_k^*(x) = 1$, $x \in A^c$,
- There exists a constant $c_1 > 0$ such that
$$\left| \frac{\partial}{\partial x_i} \varphi_k^*(x) \right| \leq c_1 \cdot (\operatorname{diam}(Q_k))^{-1}, \quad i = 1, \dots, d,$$
holds for all $x \in A^c$ and all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$ we fix a point $p_k \in A$ with

$$\operatorname{dist}(Q_k, A) = \operatorname{dist}(Q_k, p_k).$$

Since A is closed one can easily verify that such a point exists. Moreover, there exists a suitable constant $c_2 > 0$ with

$$|y - p_k| \leq c_2 \cdot |y - x|$$

for all $k \in \mathbb{N}$, each $x \in Q_k^*$ and each $y \in A$ (see [7, VI.2(11)]). We define the mapping $\overline{g} : \mathbb{R}^d \to F$ by

$$\overline{g}(x) := \begin{cases} g(x) & \text{if} \quad x \in A \\ \sum_k \varphi_k^*(x) \cdot g(p_k) & \text{if} \quad x \in A^c. \end{cases}$$

Since every point $x \in A^c$ is contained in a neighborhood that intersects at most N cubes Q_k^*, only at most N mappings φ_k^* do not vanish in this neighborhood. Hence $\overline{g}(x)$ is well defined for $x \in A^c$. Moreover, we have

$$\|\overline{g}(x)\|_F \leq \sum_k \varphi_k^*(x) \cdot \|g(p_k)\|_F \leq |g|_\alpha \cdot \sum_k \varphi_k^*(x) = |g|_\alpha, \quad x \in A^c.$$

For the proof of the Hölder continuity of $\overline{g}$ we need an estimate for the Fréchet derivative of $\overline{g}$. For this purpose let $\delta(x) := \operatorname{dist}(x, A)$, $x \in A^c$. Then

$$c_3 \cdot \operatorname{diam}(Q_k) \le \delta(x) \le c_4 \cdot \operatorname{diam}(Q_k), \quad x \in Q_k^*,$$

holds for all $k \in \mathbb{N}$ and two suitable constants $c_3, c_4 > 0$. Since φ_k^* are continuously differentiable and since at most N mappings φ_k^* are different from zero in a neighborhood of each point $x \in A^c$, we see that $\overline{g}$ is continuously Fréchet differentiable in A^c. In the following we will show the existence of a constant $c_5 > 0$ with

$$\|\overline{g}'(x)\| \le c_5 \cdot \delta(x)^{\alpha-1}, \quad x \in A^c. \tag{3}$$

For this let $x \in A^c$ be arbitrary but fixed. Then there exist $M \in \{1, \dots, N\}$, $k_1, \dots, k_M \in \mathbb{N}$ and $\rho > 0$, such that the ball $B_\rho(x) := \{y \in \mathbb{R}^d : |x - y| < \rho\}$ intersects only the cubes $Q_{k_1}^*, \dots, Q_{k_M}^*$. Hence only the mappings $\varphi_{k_1}^*, \dots, \varphi_{k_M}^*$ are different from zero in $B_\rho(x)$. We can choose M, $k_1, \dots, k_M$ and ρ such that $x \in Q_{k_i}^*$, $i = 1, \dots, M$, holds for all $i = 1, \dots, M$. (Otherwise there exists a neighborhood of x that is not contained in $Q_{k_i}^*$.) For each $z \in B_\rho(x)$ we get

$$\overline{g}(z) = \sum_{i=1}^{M} \varphi_{k_i}^*(z) \cdot g(p_{k_i}),$$

and thus

$$\overline{g}'(z)h = \sum_{i=1}^{M} \langle \nabla \varphi_{k_i}^*(z), h \rangle \cdot g(p_{k_i}), \quad h \in \mathbb{R}^d.$$

From $1 = \sum_k \varphi_k^*(z) = \sum_{i=1}^M \varphi_{k_i}^*(z)$, $z \in B_\rho(x)$, follows $\sum_{i=1}^M \nabla \varphi_{k_i}^* = 0$ on $B_\rho(x)$. Consequently,

$$\overline{g}'(z)h = \sum_{i=1}^{M} \langle \nabla \varphi_{k_i}^*(z), h \rangle \cdot (g(p_{k_i}) - g(y))$$

holds for all $y \in A$ and all $h \in \mathbb{R}^d$. Choosing $y \in A$ with $|y - x| = \delta(x)$ we get

$$\begin{aligned}
\|\overline{g}'(x)\| &= \sup_{|h|=1} \|\overline{g}'(x)h\|_F \\
&\le \sup_{|h|=1} \sum_{i=1}^{M} \|(g(p_{k_i}) - g(y))\|_F \cdot |\nabla \varphi_{k_i}^*(x)| \cdot |h| \\
&\le \sum_{i=1}^{M} c_g \cdot |p_{k_i} - y|^\alpha \cdot \sqrt{d} \cdot c_1 \cdot \operatorname{diam}(Q_{k_i})^{-1} \\
&\le \sum_{i=1}^{M} c_g \cdot c_2^\alpha \cdot |x - y|^\alpha \cdot \sqrt{d} \cdot c_1 \cdot c_4^{-1} \cdot \delta(x)^{-1} \\
&\le N \cdot c_g \cdot c_2^\alpha \cdot \delta(x)^\alpha \sqrt{d} \cdot c_1 \cdot c_4^{-1} \cdot \delta(x)^{-1} \\
&= c_5 \cdot \delta(x)^{\alpha-1}
\end{aligned}$$

for a suitable constant c_5. Thus (3) is shown.

Now we will show the Hölder continuity of $\overline{g}$. Let $y \in A$ and $x \in A^c$ be arbitrary but fixed, and $k_1, \ldots, k_M$ such that $x \in Q^*_{k_i}$, $i = 1, \ldots, M$, and $x \notin Q_k$, $k \notin \{k_1, \ldots, k_M\}$ holds. Then

$$\begin{aligned}
\|\overline{g}(x) - \overline{g}(y)\|_F &= \Big\| \sum_{i=1}^{M} \varphi^*_{k_i}(x) \cdot g(p_{k_i}) - g(y) \Big\|_F \\
&= \Big\| \sum_{i=1}^{M} \varphi^*_{k_i}(x) \cdot (g(p_{k_i}) - g(y)) \Big\|_F \\
&\leq \sum_{i=1}^{M} \varphi^*_{k_i}(x) \cdot \|g(p_{k_i}) - g(y)\|_F \\
&\leq |g|_\alpha \cdot c_2^\alpha \cdot |y - x|^\alpha \sum_{i=1}^{M} \varphi^*_{k_i}(x) \\
&= |g|_\alpha \cdot c_2^\alpha \cdot |y - x|^\alpha .
\end{aligned}$$

Now, let $x, y \in A^c$ be arbitrary but fixed. First we assume that the line segment $\overline{xy}$ connecting x and y satisfies $\operatorname{dist}(\overline{xy}, A) > |x - y|$. In this case there exists an open neighborhood of $\overline{xy}$ that is contained in A^c. From the mean value theorem for the Fréchet derivative ([2, Theorem 175.3]) there follows

$$\begin{aligned}
\|\overline{g}(x) - \overline{g}(y)\|_F \;\leq\; |x - y| \cdot \sup_{z \in \overline{xy}} \|\overline{g}'(z)\| &\leq c_5 \cdot |x - y| \cdot \sup_{z \in \overline{xy}} \delta(z)^{\alpha - 1} \\
&\leq c_5 \cdot |x - y|^\alpha .
\end{aligned}$$

In the case of $\operatorname{dist}(\overline{xy}, A) \leq |x - y|$ there exist $x' \in \overline{xy}$ and $y' \in A$ with $|x' - y'| \leq |x - y|$. Since $|x - y'| \leq 2|x - y|$ and $|y - y'| \leq 2|x - y|$ holds, from the above considerations we get

$$\begin{aligned}
\|\overline{g}(x) - \overline{g}(y)\|_F &\leq \|\overline{g}(x) - \overline{g}(y')\|_F \;+\; \|\overline{g}(y') - \overline{g}(y)\|_F \\
&\leq |g|_\alpha \cdot c_2^\alpha \cdot (|x - y'|^\alpha + |y' - y|^\alpha) \\
&\leq 2^{\alpha+1} \cdot |g|_\alpha \cdot c_2^\alpha \cdot |x - y|^\alpha .
\end{aligned}$$

Step 2. Now, let $A \subset \mathbb{R}^d$ be an arbitrary nonempty subset.
Since g is uniformly continuous on A, by standard arguments one can construct a continuous extension $\tilde{g} : \overline{A} \to F$, where $\tilde{g}$ is even Hölder continuous with the same exponent α. From the above considerations we get an α-Hölder continuous extension $\overline{g} : \mathbb{R}^d \to F$ of $\tilde{g}$. Obviously, $\overline{g}$ is an extension of g, too. □

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