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On a subclass of starlike functions

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Abstract

In this paper we obtain a sufficient condition expressed by

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf'(z)}{f'(z)}+1\right),$$

such that f satisfies the condition $\left|\dfrac{zf'(z)}{f(z)}-1\right|<M$, where $0<M\le 1$.

For $M=1$ this condition was obtained in [1].

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1 Introduction and preliminaries

Let A_n denote the class of functions f of the form

$$f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\ldots,\quad z\in U,$$

which are analytic in the unit disc $U=\{z\in\mathbb{C};|z|<1\}$.

Let $A_1=A$ and let

$$S^*=\left\{f\in A,\ \mathrm{Re}\,\frac{zf'(z)}{f(z)}>0,\ z\in U\right\}$$

be the class of starlike functions in the unit disc U.

If f and g are analytic in U, then we say that f is subordonate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is the function w, analytic in U with $w(0) = 0$, $|w(z)| < 1$, for any $z \in U$ such that $f(z) = g(w(z))$, for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let

$$J(\alpha, f, z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f'(z)} + 1\right),$$

where $f \in A$ and α is a real number.

If $\operatorname{Re} J(\alpha, f, z) > 0$, $z \in U$ we say that the function f is α-convex. This notion was introduced by P.T. Mocanu in [4].

If we let

$$p(z) = \frac{zf'(z)}{f(z)},$$

then

$$J(\alpha, f, z) = p(z) + \alpha\frac{p'(z)}{p(z)}$$

and the condition for α-convexity can be written

$$\operatorname{Re}\left[p(z) + \alpha\frac{p'(z)}{p(z)}\right] > 0, \quad z \in U.$$

Let $\mathcal{H}[a, n]$ denote the class of analytic functions in U of the form

$$f(z) = a + a_n z^n + \dots, \quad z \in U.$$

The function $L(z,t)$, $z \in U$, $t \geq 0$ is called a subordination chain if

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots, \quad z \in U$$

is analytic and univalent in U for any $t \geq 0$ and if

$$L(z, t_1 \prec L(z, t_2), \text{ when } 0 \leq t_1 \leq t_2.$$

Lemma A. [6] *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are constants $r \in (0,1]$ and $M > 0$ such that:*

(i) $L(z,t)$ is analytic in $|z| \leq r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for every $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$;

(ii) there is a function $p(z,t)$ analytic in U for any $t \geq 0$ and measurable in $[0,\infty)$ for any $z \in U$ so that $\operatorname{Re} p(z,t) > 0$ for $z \in U$, $t \geq 0$ and

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z} p(z,t)$$

for $|z| < r$ and for almost any $t \geq 0$.

By using the method of differential subordinations developed by S.S. Miller and P.T. Mocanu (see [1], [2], [3]) it is easy to prove the following result.

Lemma B. *Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and let h be an univalent function on $\overline{U}$. Suppose that the differential equation*

$$\psi(q, nzq') = h$$

has a solution $q \in \mathcal{H}[a,n]$ which is univalent on $\overline{U}$ and $q'(z) \neq 0$, for $|z| = 1$.

Moreover let assume that the function

$$L(z,t) = \psi(q(z), (1+t)nzq'(z)), \quad z \in U,\ t \geq 0$$

is a subordination chain.

If $p \in \mathcal{H}[a,n]$ satisfies the differential subordination

$$\psi[p(z), zp'(z)] \prec h(z)$$

then $p(z) \prec q(z)$.

In the paper [5] the following result was obtained:

Let n be a positive integer and let $\alpha > 0$. If $f \in A_n$ and

$$J(\alpha, f, z) \prec 1 + z + \frac{n\alpha z}{1+z} \equiv h(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + z,$$

or

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

2 Main results

Theorem 1. *Let $\alpha > 0$, $0 < M \leq 1$, $p \in \mathcal{H}[a, n]$,*

$$h_M(z) = 1 + Mz + \frac{\alpha n M z}{1 + Mz}. \tag{1}$$

If

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec h_M(z)$$

then

$$p(z) \prec 1 + Mz.$$

Proof. Let $q(z) = 1 + Mz$, $0 < M \leq 1$ and let

$$h_M(z) = q(z) + \frac{\alpha n z q'(z)}{q(z)} = 1 + Mz + \frac{\alpha n M z}{1 + Mz}.$$

It is easy to show that h_M is univalent in U and $h_M \prec h_1 = h$.

In order to apply Lemma B it is sufficient to show that the function

$$L(z, t) = 1 + Mz + (1 + t)\frac{\alpha n M z}{1 + Mz}, \quad t \geq 0, \ z \in U$$

is a subordination chain.

A simple calculation yields

$$\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} = \frac{1}{\alpha n}(1+Mz) + (1+t)\frac{1}{1+Mz}$$

and we have

$$\operatorname{Re} \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} > 0,$$

for all $t \geq 0$ and $z \in U$.

Hence by Lemma A we deduce that $L(z,t)$ is a subordination chain and the conclusion of Theorem 1 follows from Lemma B. $\square$

For $M = 1$ Theorem 1 reduces to Theorem A.

If we let

$$p(z) = \frac{zf'(z)}{f(z)}$$

and

$$J(\alpha, f, z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right),$$

Theorem 1 can be rewritten in the equivalent form:

Theorem 2. *Let $\alpha > 0$, $0 < M \leq 1$ and h_M given by (1). If $f \in A_n$ and $J(\alpha, f, z) \prec h_M(z)$ then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

We next specialize Theorem 2 by considering the maximum disc centered at the point 1 which is contained in $h_M(U)$.

We obtain the following result.

Corollary. *Let* $\alpha > 0$, $0 < M \leq 1$.

If $f \in A_n$ *and*

$$|J(\alpha, f, z) - 1| < M\left(1 + \frac{\alpha n}{1+M}\right)$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

Proof. If we let $z = e^{i\theta}$, then

$$h_M(e^{i\theta}) = 1 + Me^{i\theta} + \frac{\alpha n e^{i\theta} M}{1 + Me^{i\theta}} = u + iv$$

where

$$\begin{cases} u = 1 + M\cos\theta + \alpha n M \dfrac{\cos\theta + M}{1 + M^2 + 2M\cos\theta} \\ v = M\left[1 + \dfrac{\alpha n}{1 + M^2 + 2M\cos\theta}\right]\sin\theta. \end{cases}$$

Letting $\cos\theta = t \in [-1, 1]$ the parametric equations of the boundary of $h_M(U)$ are given by

$$\begin{cases} u = 1 + Mt + \alpha n M \dfrac{t + M}{1 + M^2 + 2Mt} \\ v = M\left[1 + \dfrac{\alpha n}{1 + M^2 + Mt}\right]\sqrt{1 - t^2}. \end{cases}$$

We need to find the minimum distance from 1 to the boundary of $h_M(U)$.

If we let $\alpha n = a$, R this distance and we note $\mu = 1 + M^2 + a$, then we have

$$R^2(t) = (u-1)^2 + v^2 =$$

$$= \frac{M^2}{(1+M^2+2Mt)^2}\{[2Mt^2+\mu t + aM]^2 + [\mu + 2Mt]^2(1-t^2)\} =$$

$$= M^2 \frac{4M^2(1+a)t^2 + 2M\mu(a+2)t + \mu^2 + a^2M^2}{(1+M^2+2Mt)^2} \equiv \phi(t).$$

It is easy to show that

$$\frac{\partial \phi(t)}{\partial t} = \frac{K}{(1+M^2+2Mt)^2}(At+B), \quad t \in [-1,1],$$

where

$$K > 0$$

$$A = -2M(1-M^2+a)$$

$$B = -[(1+M^2+a)(1-M^2) + 2aM^2].$$

Since for $t \in [-1,1]$ we have

$$At + B \leq -(1-M^2)^2(1-M^2+a) < 0,$$

we deduce

$$\phi'(t) < 0.$$

Hence

$$\min_{t\in[-1,1]} \phi(t) = \phi(1) = R^2(1),$$

where

$$R(1) = u(1) - 1 = M\left(1 + \frac{a}{1+M}\right). \ \square$$

In [7] the following result was obtained.

Theorem B. If $f \in A_n$ satisfies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M,$$

where $M = M_n$ is the solution of the equation

$$\cos\frac{M}{n} = M, \tag{2}$$

then Re $f'(z) > 0$.

Combining our Corollary with Theorem B we easily deduce the following result which shows that certain subclasses of α-convex functions satisfy Re $f'(z) > 0$.

Theorem 3. *If $\alpha > 0$ and $f \in A_n$ satisfies*

$$|J(\alpha, f, z) - 1| < M_n\left(1 + \frac{\alpha n}{1+M_n}\right),$$

where M_n is the solution of the equation (2) then Re $f'(z) > 0$.

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