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A condition for starlikeness

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Abstract

In this paper we obtain a sufficient condition for starlikeness, which are expressed by the argument of $(1-\lambda)\dfrac{f(z)}{z}+\lambda f'(z)$.

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1 Introduction and preliminaries

$\lim\limits_{n\to\infty}$ Let A_n denote the class of functions of the form:

$$f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\ldots,$$

which are analytic in the unit disc $U=\{z|\ |z|<1\}$.

If $n=1$, $A_1=A$.

We let $\mathcal{H}[1,n]$ denote the class of analytic functions in the unit disc of the form:

$$f(z)=1+a_nz^n+a_{n+1}z^{n+1}+\ldots,\quad z\in U$$

and let

$$S^* = \left\{ f \in A,\ \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$$

be the class of starlike functions in the unit disc U.

The function $f \in A$, is called strongly-starlike of order γ, $0 < \gamma \leq 1$, if:

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \gamma \frac{\pi}{2}.$$

Let denote by $S^*[\gamma]$ the class of all such functions.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ for any $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

In this paper we obtain a condition for strongly-starlikeness by using the argument of $(1-\lambda)\dfrac{f(z)}{z} + \lambda f'(z)$.

For this aim we need to apply the following lemma.

Lemma. [1] *Let $\lambda > 0$, let n be a positive integer and let $\beta_0 = \beta_0(\lambda, n)$ be the root of the equation:*

$$\beta\pi = \frac{3\pi}{2} - \arctan(n\lambda\beta).$$

Let

$$(1) \qquad \alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi} \arctan(n\lambda\beta), \quad 0 < \beta \le \beta_0.$$

If $P \in \mathcal{H}[1, n]$ *then*

$$P(z) + \lambda z P'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \text{ implies } P(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

2 Main results

Theorem 1. *If* $f \in A_n$, $\lambda \ge 1$, $0 < \gamma \le 1$ *and*

$$(2) \qquad (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

where $\alpha = \alpha_n(\lambda, \gamma)$ *satisfies the equation*

$$(3) \qquad 2 \arctan[\lambda n(1-\alpha)] + \pi(\gamma - 2\alpha) = 0,$$

then $f \in S^*[\gamma]$.

Proof. If we let $P(z) = \dfrac{f(z)}{z}$ and $Q(z) = (1-\lambda)\dfrac{f(z)}{z} + \lambda f'(z)$. condition (2) can be written:

$$Q(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

which is equivalent to

$$P(z) + \lambda z P'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

By applying lemma we deduce

$$P(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{4}$$

where β satisfies condition (1).

If we write (3) as

$$2\arctan[\lambda n(\gamma-\alpha)] + \pi(\gamma-\alpha-\alpha) = 0$$

and we let $\gamma - \alpha = \beta$ then (3) becomes

$$2\arctan \lambda n\beta + \pi(\beta-\alpha) = 0$$

and by letting $\alpha\pi = \beta\pi + 2\arctan(\lambda n\beta)$ we obtain $\alpha = \beta + \frac{2}{\pi}\arctan(\lambda n\beta)$ which is exactly (1).

Hence, from (1) and (3) we deduce:

$$\alpha + \beta = \gamma. \tag{5}$$

Since $\lambda \geq 1$ and

$$f'(z) = \frac{1}{\lambda}Q(z) + \frac{\lambda-1}{\lambda}P(z) = \frac{1}{\lambda}Q(z) + \left(1-\frac{1}{\lambda}\right)P(z),$$

we deduce that $f'(z)$ is a convex combination of P and Q.

On the other hand, from (1) we deduce that $\alpha > \beta$ hence

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{6}$$

If we let

$$p(z) = \frac{zf'(z)}{f(z)},$$

then we have $p \in \mathcal{H}[1, n]$ and

$$p(z) = \frac{zf'(z)}{P(z)}.$$

From (4) we have $|\arg P(z)| < \dfrac{\beta\pi}{2}$, from (6) we have $|\arg f'(z)| < \dfrac{\alpha\pi}{2}$ and from (5) we have $\alpha + \beta = \gamma$. Hence

$$|\arg p(z)| \leq |\arg f'(z)| + |\arg P(z)| < \alpha\frac{\pi}{2} + \beta\frac{\pi}{2} = (\alpha+\beta)\frac{\pi}{2} = \gamma\frac{\pi}{2},$$

which shows that $f \in S^*[\gamma]$.

Theorem 1 can be written in the following equivalent form.

Theorem 2. *If* $0 < \mu \leq 1$ *and* $g \in \mathcal{H}[1, n]$ *satisfies the condition*

$$|\arg g(z)| < \theta_n(\mu) = \alpha_n\left(\frac{1}{\mu}\right)\frac{\pi}{2},$$

where $\alpha_n(\lambda)$ *is given by (3), then the function*

$$f(z) = \mu z\int_0^1 t^{\mu-1}g(tz)dt \tag{7}$$

is strongly-starlike.

Proof. If we let $\lambda = \frac{1}{\mu}$ and $g \in \mathcal{H}[1, n]$ the equation

$$(\mu - 1)\frac{f(z)}{z} + f'(z) = \mu g(z)$$

has the solution $f \in A_n$ given by (7).

Since $g(z) = (1 - \lambda)\dfrac{f(z)}{z} + \lambda f'(z)$ we have

$$(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

and from Theorem 1 we deduce that f is strongly-starlike.

In the case $\gamma = 1$ this result was obtained in [4].

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