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On meromorphic p-valent functions with positive coefficients

H. E. Darwish, M. K. Aouf and G. S. Sălăgean

Abstract

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z)=\frac{a_{p-1}}{z^p}+\sum_{n=1}^{\infty}a_{p+n-1}z^{p+n-1},\ \ a_{p-1}>0,\ \ a_{p+n-1}\geq 0,\ \ p\in\mathbb{N}=\{1,2,...\},$$

which are regular and p-valent in the punctured disc $U^*=\{z\ :\ 0<|z|<1\}$. Let $\Sigma_i(p)$ $(i\in\{0,1\})$ denote the subclasses of $\Sigma(p)$ satisfying $z_0^pf(z_0)=1$ and $-z_0^{p+1}f'(z_0)=p$, respectively. In this paper we obtain coefficient estimates, a distortion theorem, closure theorems and a radius of convexity of order δ $(0\leq\delta<p)$ for certain subclasses $\Sigma_iS(p,\alpha,z_0)$, $i\in\{0,1\}$ of $\Sigma_i(p)$, $i\in\{0,1\}$.

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1 Introduction

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z)=\frac{a_{p-1}}{z^p}+\sum_{n=1}^{\infty}a_{p+n-1}z^{p+n-1},\ \ a_{p-1}>0,\ \ a_{p+n-1}\geq 0,\ \ p\in\mathbb{N},\tag{1}$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. A function f in $\Sigma(p)$ is said to belong to $\Sigma^* S(p, \alpha)$, the class of meromorphically p-valent starlike of order α functions $(0 \leq \alpha < p)$ if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} < -\alpha \quad \text{for } z \in U^*. \tag{2}$$

A function f in $\Sigma(p)$ is said to be meromorphically p-valent convex of order α if

$$\mathrm{Re}\frac{zf'(z)}{f'(z)} + 1 < -\alpha \quad \text{for } z \in U^*. \tag{3}$$

For a given real number $z_0 \in U^*$ let $\Sigma_i(p)$ $(i \in \{0, 1\})$ be the subclass of $\Sigma(p)$ satisfying $z_0^p f(z_0) = 1$ and $-z_0^{p+1} f'(z_0) = p$, respectively. Let

$$\Sigma_i^* S(p, \alpha, z_0) = \Sigma^* S(p, \alpha) \cap \Sigma_i(p), \quad i \in \{0, 1\}.$$

We note that for $p = 1$ the classes $\Sigma_i^* S(1, \alpha, z_0) = Q_i^*(\alpha, z_0)$, $(i \in \{0, 1\})$ were studied by Uralegaddi and Ganigi [2].

In this paper we obtain coefficient estimates, a distortion theorem, closure theorems and a radius of convexity of order δ $(0 \leq \delta < p)$ for the classes $\Sigma_i^* S(p, \alpha, z_0)$ $(i \in \{0, 1\})$. The techniques used are similar to those of Silverman [1] and Uralegaddi and Ganigi [2].

2 Coefficient estimates

Theorem 1 *Let the function* f *be defined by (1). Then* $f(z)$ *is in the class* $\Sigma^*S(p,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(p+n-1+\alpha)a_{p+n-1} \leq (p-\alpha)a_{p-1}. \tag{4}$$

The result is sharp for the functions

$$f_{n+p-1}(z) = \frac{a_{p-1}}{z^p} + \frac{(p-\alpha)a_{p-1}}{p+n-1+\alpha}\, z^{p+n-1}, \quad n \in \mathbb{N}. \tag{5}$$

The proof follows exactly the lines of the proof of Theorem 1 in [2], hence it is omitted here.

Theorem 2 *Let the function* f *be defined by (1). Then* $f \in \Sigma_0^*S(p,\alpha,z_0)$ *if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{p+n-1+\alpha}{p-\alpha} + z_0^{2p+n-1}\right] a_{p+n-1} \leq 1.$$

Proof. Since $f(z) \in \Sigma_0^*S(p,\alpha,z_0)$ we have

$$z_0^p f(z_0) = 1 = a_{p-1} + \sum_{n=1}^{\infty} a_{p+n-1}\, z_0^{2p+n-1} \quad (a_{p-1} > 0, \;\; a_{p+n-1} \geq 0))$$

which gives

$$a_{p-1} = 1 - \sum_{n=1}^{\infty} a_{p+n-1}\, z_0^{2p+n-1}. \tag{6}$$

Substituting the value of a_{p-1} in Theorem 1 we complete the proof.

Corollary 1 *Let the function f defined by (1) be in the class $\Sigma_0^* S(p,\alpha,z_0)$. Then*

$$(7)\qquad a_{p+n-1} \leq \frac{p-\alpha}{p+n-1+\alpha+(p-\alpha)\, z_0^{2p+n-1}}, \quad n \in \mathbb{N}.$$

The equality in (7) is attained for the function $f = f_{p+n-1}$ given by

$$(8) f_{p+n-1}(z) = \frac{p+n-1+\alpha+(p-\alpha)\, z^{2p+n-1}}{z^p \left[p+n-1+\alpha+(p-\alpha)\, z_0^{2p+n-1}\right]}, \quad n \in \mathbb{N}.$$

Theorem 3 *Let the function f be defined by (1). Then $f \in \Sigma_1^* S(p,\alpha,z_0)$ if and only if*

$$\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha} - \frac{p+n-1}{p}\, z_0^{2p+n-1}\right] a_{p+n-1} \leq 1.$$

Proof. Since $-z_0^{p+1} f'(z_0) = p,$ we have

$$(9)\qquad a_{p-1} = 1 + \sum_{n=1}^{\infty} \frac{p+n-1}{p}\, a_{p+n-1}\, z_0^{2p+n-1}.$$

Eliminating a_{p-1} from (4) and (9) we get the required result.

Corollary 2 *Let the function f defined by (1) be in the class $\Sigma_1^* S(p,\alpha,z_0)$. Then*

$$(10)\quad a_{p+n-1} \leq \frac{p-\alpha}{p+n-1+\alpha-(p-\alpha)\dfrac{p+n-1}{p}\, z_0^{2p+n-1}},$$

$n \in \mathbb{N}$.

The equality in (10) is attained for the function $f = f_{p+n-1}$ *given by*

$$f_{n+p-1}(z) = \frac{p+n-1_{\alpha} + (p-\alpha)\, z^{2p+n-1}}{z^p \left[p+n-1+\alpha-(p-\alpha)\dfrac{p+n-1}{p}\, z_0^{2p+n-1} \right]} \tag{11}$$

$n \in \mathbb{N}$.

3 Distortion Theorem

Theorem 4 *Let the function* f *defined by (1) be in the class* $\Sigma_0^* S(p,\alpha,z_0)$. *Then*

$$|f(z)| \geq \frac{p+\alpha-(p-\alpha)\, r^{2p}}{r^p \left[p+\alpha+(p-\alpha)\, z_0^{2p} \right]}$$

for $0 < |z| = r < 1$. *The result is sharp.*

Proof. It follows from Theorem 2 that

$$\sum_{n=1}^{\infty} a_{p+n-1} \leq \frac{p-\alpha}{p+\alpha+(p-\alpha)\, z_0^{2p}} \tag{12}$$

and

$$a_{p-1} = 1 - \sum_{n=1}^{\infty} a_{p+n-1} z_0^{2p+n-1} \geq \frac{p+\alpha}{p+\alpha+(p-\alpha) z_0^{2p}}. \tag{13}$$

Hence we have

$$|f(z)| \geq \frac{a_{p-1}}{r^p} - r^p \sum_{n=1} a_{p+n-1} \geq \frac{p+\alpha-(p-\alpha)\, r^{2p}}{r^p \left[p+\alpha+(p-\alpha)\, z_0^{2p} \right]}$$

by using (12) and (13). Further the result is sharp for the function $f = f_p$ given by (see (8))

$$f_p(z) = \frac{p + \alpha + (p - \alpha)\, z^{2p}}{z^p \left[p + \alpha + (p - \alpha)\, z_0^{2p} \right]}.$$

4 Closure Theorems

Let the functions f_j be defined, for $j \in \{1, 2, ..., m\}$, by

$$f_j(z) = \frac{a_{p-1,j}}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,j}\, z^{p+n-1}, \tag{14}$$

$$a_{p-1,j} > 0, \quad a_{p+n-1,j} \geq 0 \text{ for } z \in U^*.$$

We shall prove the following results for the closure of functions in the classes $\Sigma_i^* S(p, \alpha, z_0)$.

Theorem 5 *Let the functions f_j, $j \in \{1, 2, ..., m\}$ defined by (14) be in the class $\Sigma_0^* S(p.\alpha, z_0)$. Then the function h defined by*

$$h(z) = \sum_{j=1}^{m} d_j\, f_j(z) \quad d_j \geq 0 \tag{15}$$

is also in the same class $\Sigma_0^* S(p, \alpha, z_0)$, where

$$\sum_{j=1}^{m} d_j = 1. \tag{16}$$

Proof. According to the definition of h we can write

$$h(z) = \frac{b_{p-1}}{z^p} + \sum_{n=1}^{\infty} b_{p+n-1}\, z^{p+n-1},$$

where

$$b_{p-1} = \sum_{j=1}^{m} d_j a_{p-1,j}$$

and

$$b_{p+n-1} = \sum_{j=1}^{m} d_j a_{p+n-1,j}, \quad j \in \{1, 2, ..., m\}.$$

Since $f_j(z)$ are in $\Sigma_0^* S(p, \alpha, z_0)$, $(j \in \{1, 2, ..., m\})$, by means of Theorem 2, we have

$$\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha} + z_0^{2p+n-1} \right] a_{p+n-1,j} \leq 1$$

for every $j \in \{1, 2, ..., m\}$. Therefore we have

$$\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha} + z_0^{2p+n-1} \right] \left(\sum_{j=1}^{m} d_j\, a_{p+n-1,j} \right) =$$

$$= \sum_{j=1}^{m} d_j \left(\sum_{n=1}^{\infty} \left[\frac{p+n-1+\alpha}{p-\alpha} + z_0^{2p+n-1} \right] a_{p+n-1,j} \right) \leq$$

$$\leq \sum_{j=1}^{m} d_j = 1$$

which shows that $h \in \Sigma_0^* S(p, \alpha, z_0)$. This completes the proof.

Theorem 6 *Let the functions* f_j *defined by (14) be in the class* $\Sigma_1^* S(p, \alpha, z_0)$ *for every* $j \in \{1, 2, ..., m\}$*. Then the function* g *defined by (15) is also in the same class* $\Sigma_1^* S(p, \alpha, z_0)$ *under the condition (16).*

The proof of the theorem is obtained by using the same technique as in the proof of Theorem 5.

Theorem 7 *The class* $\Sigma_0^* S(p, \alpha, z_0)$ *is closed under convex linear combinations.*

Proof. Let the functions f_j $j \in \{1, 2\}$ defined by (14) be in the class $\Sigma_0^* S(p, \alpha, z_0)$. It is sufficient to prove that the function g defined by

$$g(z) = \lambda f_1(z) + (1 - \lambda) f_2(z), \ \ 0 \leq \lambda \leq 1,$$

is also in the class $\Sigma_0^* S(p, \alpha, z_0)$.

Since

$$g(z) = \frac{\lambda a_{p-1,1} + (1 - \lambda) a_{p-1,2}}{z^p} +$$

$$+ \sum_{n+1}^{\infty} \{\lambda a_{p+n-1,1} + (1 - \lambda) a_{p+n-1,2}\} \ z^{p+n-1},$$

with the aid of Theorem 2 we have

$$\sum_{n=1}^{\infty} \left[\frac{p + n - 1 + \alpha}{p - \alpha} + z_0^{2p+n-1} \right] \{\lambda a_{p+n-1,1} + (1 - \lambda) a_{p+n-1,2}\} \leq 1$$

which implies that $g \in \Sigma_0^* S(p, \alpha, z_0)$.

In a similar manner, by using Theorem 3, we can prove the following theorem:

Theorem 8 *The class* $\Sigma_1^* S(p, \alpha, z_0)$ *is closed under convex linear combinations.*

Theorem 9 *Let define* $f_{p-1}(z) = z^{-p}$ *and let* $f_{p+n-1}, \;\; n \in \mathbb{N}$ *be defined by (8). Then* f *is in the class* $\Sigma_0^* S(p, \alpha, z_0)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1}\, f_{p+n-1}(z), \tag{17}$$

where $\lambda_{p+n-1} \geq 0 \;\; (n \in \mathbb{N} \cup \{0\})$ *and*

$$\sum_{n=0}^{\infty} \lambda_{p+n-1} = 1. \tag{18}$$

Proof. Assume that

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1}\, f_{p+n-1}(z) = \lambda_{p-1}\, f_{p-1}(z) + \sum_{n=1}^{\infty} \lambda_{p+n-1}\, f_{p+n-1}(z) =$$

$$= \frac{\lambda_{p-1}}{z^p} + \sum_{n=1}^{\infty} \frac{\left[p + n - 1 + \alpha + (p - \alpha)\, z^{2p+n-1}\right] \lambda_{p+n-1}}{z^p \left[p + n - 1 + \alpha + (p - \alpha)\, z_0^{2p+n-1}\right]} =$$

$$= \frac{1}{z^p} \left[\lambda_{p-1} + \sum_{n=1}^{\infty} \frac{(p + n - 1 + \alpha)\, \lambda_{p+n-1}}{p + n - 1 + \alpha + (p - \alpha)\, z_0^{2p+n-1}}\right] +$$

$$+\sum_{n=1}^{\infty} \frac{(p-\alpha)\,\lambda_{p+n-1}\, z^{p+n-1}}{p+n-1+\alpha+(p-\alpha)\, z_0^{2p+n-1}}.$$

Then it follows that

$$\sum_{n=1}^{\infty} \left(\frac{p+n-1+\alpha}{p-\alpha} + z_0^{2p+n-1}\right) \frac{(p-\alpha)\,\lambda_{p+n-1}}{p+n-1+\alpha+(p-\alpha)z_0^{2p+n-1}} =$$

(19)

$$= \sum_{n=1}^{\infty} \lambda_{p+n-1} = 1 - \lambda_{p-1} \leq 1.$$

Also by definition we have $z_0^p\, f_{p+n-1}(z_0) = 1$. Therefore

$$z_0^p\, f(z_0) = \sum_{n=0}^{\infty} \lambda_{p+n-1}\, z_0^p\, f_{p+n-1}(z_0) = \sum_{n=0}^{\infty} \lambda_{p+n-1} = 1.$$

This implies $f \in \Sigma_0(p)$. So, by (19) and by Theorem 2, $f \in \Sigma_0^* S(p, \alpha, z_0)$.

Conversely, assume that the function f defined by (1) belongs to the class $\Sigma_0^* S(p, \alpha, z_0)$. Then

$$a_{p+n-1} \leq \frac{p-\alpha}{p+n-1+\alpha+(p-\alpha)z_0^{2p+n-1}}, \quad n \in \mathbb{N}.$$

Setting

$$\lambda_{p+n-1} = \frac{p+n-1+\alpha+(p-\alpha)z_0^{2p+n-1}}{p-\alpha}\, a_{p+n-1}, \quad n \in \mathbb{N}$$

and

$$\lambda_{p-1} = 1 - \sum_{n=1}^{\infty} \lambda_{p+n-1},$$

we can see that f can be expressed in the form (17). This completes the proof.

In a similar manner we can prove the following theorem.

Theorem 10 *Let define* $f_{p-1}(z) = z^{-p}$ *and let* $f_{p+n-1}, \quad n \in \mathbb{N}$ *be defined by (11). Then* f *is in the class* $\Sigma_1^* S(p, \alpha, z_0)$ *if and only if it can be expressed in the form(17) where* $\lambda_{p+n-1} \geq 0 \ \ (n \in \mathbb{N} \cup \{0\})$ *and they satisfy (18).*

5 Radius of Convexity

In this section we determine the radius of convexity of order $\delta, \ \ 0 \leq \delta < p,$ for the classes $\Sigma_i^* S(p, \alpha, z_0), \quad i \in \mathbb{N}.$

Theorem 11 *Let the function* f *defined by (1) be in the class* $\Sigma_0^* S(p, \alpha, z_0)$ *or* $\Sigma_1^* S(p, \alpha, z_0)$; *then* f *is convex of order* $\delta, \ \ 0 \leq \delta < p,$ *in the punctured disc* $\{z : 0 < |z| < r^*(p, \alpha, \delta)\},$ *where*

$$r^*(p, \alpha, \delta) = \inf_n \left\{ \frac{p(p-\delta)(p+n-1+\alpha)}{(p-\alpha)(p+n-1)(3p+n-1-\delta)} \right\}^{\frac{1}{2p+n-1}}, \quad n \in \mathbb{N}.$$

The result is sharp.

Proof. It is sufficient to demonstrate that (see (3))

$$\left| 1 + p + \frac{zf'(z)}{f'(z)} \right| \leq p - \delta, \ \ 0 \leq \delta < p, \ \ \text{for} \ \ 0 < |z| < r^*(p, \alpha, \delta).$$

We have

$$\left| \frac{(zf'(z))' + p\, f'(z)}{f'(z)} \right| \leq$$

$$\leq \frac{\displaystyle\sum_{n=1}^{\infty}(p+n-1)(2p+n-1)\,a_{p+n-1}|z|^{2p+n-1}}{\displaystyle p\,a_{p-1}-\sum_{n=1}^{\infty}(p+n-1)\,a_{p+n-1}|z|^{2p+n-1}}.$$

Thus

$$\left|\frac{(zf'(z))'+p\,f'(z)}{f'(z)}\right| \leq p-\delta$$

if

$$\sum_{n=1}^{\infty}(p+n-1)(3p+n-1-\delta)\,a_{p+n-1}|z|^{2p+n-1} \leq p\,(p-\delta)\,a_{p-1} \tag{20}$$

When $f \in \Sigma_0^* S(p,\alpha,z_0)$, using (6) we find that the inequality (20) is equivalent to

$$\sum_{n=1}^{\infty}\left[(p+n-1)(3p+n-1-\delta)\,|z|^{2p+n-1}+p\,(p-\delta)\,z_0^{2p+n-1}\right]a_{p+n-1} \leq p\,(p-\delta). \tag{21}$$

But Theorem 2 ensures

$$\sum_{n=1}^{\infty}\left\{p(p-\delta)\left[\frac{p+n-1+\alpha}{p-\alpha}+z_0^{2p+n-1}\right]\right\}a_{p+n-1} \leq p(p-\delta).$$

Hence (21) holds if

$$\left[(p+n-1)(3p+n-1-\delta)\,|z|^{2p+n-1}+p\,(p-\delta)\,z_0^{2p+n-1}\right]a_{p+n-1} \leq$$

$$\leq p(p-\delta)\left(\frac{p+n-1+\alpha}{p-\alpha}+z_0^{2p+n-1}\right)a_{p+n-1}, \quad n \in \mathbb{N},$$

or if

$$|z| \leq \left\{\frac{p(p-\delta)(p+n-1+\alpha)}{(p-\alpha)(p+n-1)(3p+n-1-\delta)}\right\}^{\frac{1}{2p+n-1}}, \quad n \in \mathbb{N}.$$

Hence f is convex of order δ, $(0 \leq \delta < p)$ in $0 < |z| < r^*(p, \alpha, \delta)$.

In the case when $f \in \Sigma_1^* S(p, \alpha, z_0)$, by using (9) we find that the inequality (20) is equivalent to

$$\sum_{n=1}^{\infty}(p+n-1)\left[(3p+n-1-\delta)|z|^{2p+n-1}-(p-\delta)z_0^{2p+n-1}\right]a_{p+n-1} \leq p\,(p-\delta). \tag{22}$$

Therefore, in view of Theorem 3, the inequality (22) holds if

$$(p+n-1)\left[(3p+n-1-\delta)\,|z|^{2p+n-1}-(p-\delta)\,z_0^{2p+n-1}\right]a_{p+n-1} \leq$$

$$\leq p(p-\delta)\left(\frac{p+n-1+\alpha}{p-\alpha}-\frac{p+n-1}{p}z_0^{2p+n-1}\right)a_{p+n-1}, \quad n \in \mathbb{N},$$

or if

$$|z| \leq \left\{\frac{p(p-\delta)(p+n-1+\alpha)}{(p-\alpha)(p+n-1)(3p+n-1-\delta)}\right\}^{\frac{1}{2p+n-1}}, \quad n \in \mathbb{N}.$$

The sharpness for the class $\Sigma_0^* S(p, \alpha, z_0)$ follows by taking the function f given by (8), and for the class $\Sigma_1^* S(p, \alpha, z_0)$ the sharpness follows if we take the function f given by (11). This completes the proof of the theorem.

Remark. The conclusion of Theorem 11 is independent of the point z_0.

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H. E. Darwish
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura, Egypt
E-mail: SINFAC@MUM.MANS.EUN.EG

M. K. Aouf
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura, Egypt
E-mail: SINFAC@MUM.MANS.EUN.EG

G. S. Sălăgean
Babes-Bolyai University
Faculty of Mathematics and Computer Science
str.M. Kogalniceanu nr. 1
3400 Cluj-Napoca, Romania
E-mail: salagean@math.ubbcluj.ro