General Mathematics Vol. 7 /1999

The fine spectra of discrete Cesàro operator on bv_0

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Abstract

In a series of papers, the authors has previously investigated the spectra and fine spectra for discrete Cesàro operators, considered as bounded operators over various sequence spaces. In this paper I examined the fine spectra of discrete Cesàro operators as operator over bv_0, the spaces of null sequences of bounded variation:

$$\sigma_r(C_1, bv_0) = \{\lambda : \lambda I - C_1 \text{ injective and } \overline{(\lambda I - C_1)(bv_0)} \neq bv_0\},$$

$$\sigma_c(C_1, bv_0) = \{\lambda : \lambda I - C_1 \text{ injective, nesurjective and } \overline{(\lambda I - C_1)(bv_0)} = bv_0\}.$$

In [3] the author investigated the spectrum and punctual spectrum for $C_1 : bv_0 \longrightarrow bv_0$.

1991 Mathematical Subject Classification ....

If $(x_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers, C_1x is the sequence of arithmetic means of x:

$$(C_1x)_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k \quad , \quad n = 0, 1, 2, \ldots$$

C_1 is called the discrete Cesàro operator.

The space bv_0 is the space of null sequences of bounded variation. Then $bv_0 = bv \cap c_0$, where

$$bv = \left\{ x \mid \sum_k |x_k - x_{k-1}| < \infty \right\} .$$

From Okutoyi [[3], Lemma 2], $C_1 \in B(bv_0)$. Let $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots \right\}$.

Theorem 1 *Let $C_1 : bv_0 \longrightarrow bv_0$. Then this spectrum is*

$$\sigma(C_1, bv_0) = \left\{ \lambda - \frac{1}{2} \leq \frac{1}{2} \right\} .$$

Proof [[3], Theorem 2.2].

Theorem 2 *Let C_1 defined on bv_0. If λ satisfies $\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$ and $\lambda \notin S$ then $\lambda \in \sigma_r(C_1, bv_0)$.*

Proof. First of all $\lambda I - C_1$ is a triangle, therefore $(\lambda I - C_1)^{-1}$ exists.

Consider the adjoint matrix $\lambda I - C_1^*$. $C_1^* \in B(bv_0^*) = B(bs)$ with entries $c_{00}^* = 1$, $c_{n0}^* = c_{0n}^* = 0$ for $n > 0$ and $c_{nk}^* = c_{k-1,n-1}$ for $n, k > 0$. bv_0^* is isomorphic with $bs = \left\{ x \mid \sup_n \left| \sum_{k=0}^{n} x_k \right| < \infty \right\}$ ([3]).

Suppose $(\lambda I - C_1^*)x = 0$. Then

$$(\lambda - 1)x_0 = 0$$

and

$$(1)\qquad \left(\lambda - \frac{1}{n}\right) x_n - \sum_{k=n+1}^{\infty} c^*_{nx} x_k = 0 \qquad \text{for } n > 0 \ .$$

Thus $x_0 = 0$ and, from (1), x_1 is arbitrary and,

$$(2)\ \ x_n = \frac{x_1}{\lambda^{n-1}} \prod_{j=0}^{n-2} \left(\lambda - \frac{1}{j+1}\right) = x_1 \prod_{j=0}^{n-2} \left(1 - \frac{1}{\lambda(j+1)}\right) =$$

$$\left(1 - \frac{1}{\lambda}\right) \frac{x_1}{n-1} \prod_{j=1}^{n-2} \left(1 + \left(1 - \frac{1}{\lambda}\right) \frac{1}{j}\right) \ .$$

Now $\left|1 + \left(1 - \frac{1}{\lambda}\right) \frac{1}{j}\right| < 1$ for all j sufficiently large if and only if

$$\left(1 + (1+\alpha) \frac{1}{j}\right)^2 + \left(\beta \frac{1}{j}\right)^2 < 1,$$

where $-\dfrac{1}{\lambda} = \alpha + i\beta$. The above inequality is equivalent to

$$2(1+\alpha) + ((1+\alpha)^2 + \beta^2) \frac{1}{j} < 0$$

for all j sufficiently large. The above inequality will be true for all j sufficiently large if $2(1+\alpha) + ((1+\alpha)^2 + \beta^2) < 0$, which is equivalent to

$$\left|\lambda - \frac{1}{2}\right| < \frac{1}{2} \ .$$

Let

$$z_n = \prod_{j=1}^{n-2} \left(1 + \left(1 - \frac{1}{\lambda}\right) \frac{1}{j}\right) \ .$$

Then

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|1+\left(1-\frac{1}{\lambda}\right)\frac{1}{n-1}\right| .$$

From the hypothesis on λ, and the discussion in the preceding formula,

$$\left|1+\left(1-\frac{1}{\lambda}\right)\frac{1}{n-1}\right| \leq \varepsilon < 1 \tag{3}$$

for all n sufficiently large, and $\sum |z_n|$ is convergent.

Since

$$\left|\left(1-\frac{1}{\lambda}\right)\frac{x_1}{n-1}\right|$$

is bounded, it follows that $\sum |x_n|$ is convergent, so that $(\lambda I - C_1^*)x = 0$ has nonzero solutions.

By [2, Theorem II 3.7], $\lambda I - C_1$ does not have dense range. Therefore to verify that $\lambda \in \sigma_r(C_1, bv_0)$ it is sufficient, from [1] to show that $\lambda I - C_1^*$ is surjective.

Suppose $y = (\lambda I - C_1^*)x$, $x, y \in bv_0 = bs$. Then $(\lambda - 1)x_0 = y_0$ and

$$\left(\lambda - \frac{1}{n}\right)x_n - \sum_{k=n+1}^{\infty}\frac{x_k}{k} = y_n \quad , \quad n > 0 . \tag{4}$$

Choose $x_1 = 0$ and solve for x in terms of y to get

$$-\sum_{k=2}^{\infty}\frac{x_k}{k} = y_1 \tag{5}$$

$$\left(\lambda - \frac{1}{n}\right)x_n = y_n + \sum_{k=n+1}^{\infty}\frac{x_k}{k} . \tag{6}$$

Continuing this process, if B is a lover triangular matrix defined by $B_y = x$, then B has entries

$$b_{00} = \frac{1}{\lambda - 1}\,, \qquad b_{nn} = \frac{1}{\lambda}\,, \qquad n > 1$$

$$b_{21} = -\frac{1}{\lambda}\,, \qquad b_{n,n-1} = \frac{1}{(n-1)\lambda^2}\,, \quad n > 2$$

$$b_{n1} = -\frac{1}{\lambda}\prod_{j=1}^{n-2}\left(1 - \frac{1}{\lambda j}\right), \qquad n > 2$$

$$b_{nk} = -\frac{1}{\lambda^2 k}\prod_{j=k}^{n-2}\left(1 - \frac{1}{\lambda j}\right), \qquad 1 < k < n-1$$

and $b_{nk} = 0$ otherwise.

To show that $B \in B(bs)$ it is sufficient to show that $\sum_n |b_{nk}|$ is finite, independent of k

$$\sum_n |bn_0| = \frac{1}{|\lambda - 1|}\,.$$

We may write

$$1 - \frac{1}{\lambda(j+1)} = \frac{j}{j+1}\left(1 + \left(1 - \frac{1}{\lambda}\right)\frac{1}{j}\right)\,.$$

Also

$$\sup_{n>1}\left|\frac{1}{n-1}\right| \le 1\,.$$

Therefore

$$\sum_n |bn_1| \le \frac{1}{|\lambda|}\left(1 + \sum_{n=3}^{\infty}\prod_{j=1}^{n-2}\left|1 + \left(1 - \frac{1}{\lambda}\right)\frac{1}{j}\right|\right)\,.$$

and, for $k > 1$,

$$\sum_n |b_{nk}| \leq \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2} + \frac{1}{|\lambda|^2} \sum_{n=k+2}^{\infty} \prod_{j=k}^{n-2} \left| 1 + \left(1 - \frac{1}{\lambda}\right) \frac{1}{j} \right| .$$

Since $k > 1$, the series in the second inequality is dominated by the series in the first inequality which, from (3), is absolutely convergent. Therefore $\|B\| < \infty$.

Since $(\lambda I - C_1)^{-1}$ is bounded, it is continuous, and $\lambda \in \sigma_r(C_1, bv_0)$.

Theorem 3 *Let $C_1 : bv_0 \longrightarrow bv_0$. If λ satisfies $\left|\lambda - \frac{1}{2}\right|$, $\lambda \neq 1, \frac{1}{2}$, then $\lambda \in \sigma_c(C_1, bv_0)$.*

Proof. Suppose $\lambda \neq 1, \frac{1}{2}$, and satisfying $\left|\lambda - \frac{1}{2}\right| = \frac{1}{2}$. Since $\lambda I - C_1$ is a triangle, $(\lambda I - C_1)^{-1}$ exists.

Consider $(\lambda I - C_1^*)x = 0$. As in Theorem 2, $x_0 = 0$, x_1 is arbitrary and (x_n) satisfies (2) for all $n > 0$.

The condition on λ, implies that

$$\left| 1 + \left(1 - \frac{1}{\lambda}\right) \frac{1}{n} \geq 1 \right|$$

for $n \geq N$, $N \in \mathbb{N}$.

Thus $|x_n| \geq c \frac{1}{n-1}$ for $n \geq N$, where c is a constant independent of n. The series $\sum \frac{1}{n-1}$ is divergent, so $(x_n) \in bs$ implies $x_1 = 0$, hence $x = 0$ and $(\lambda I - C_1^*)^{-1}$ exists. Since, from [2], $\lambda \in \sigma(C_1, bv_0)$ and $\lambda \neq 1, \frac{1}{2}$, $\lambda \in \sigma_c(C_1, bv_0)$.

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