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# Iteration of the Pompeiu integral operator and complex higher order equations

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**Abstract**

The paper contains a brief presentation of the Pompeiu integral operator and applications to the Dirichlet problem in the unit disc and to complex elliptic second order equations.

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## 1 Pompeiu integral operator

The complex form of the Gauß theorem

$$\frac{1}{2i}\int\limits_{\partial D} w(z)\,dz = \int\limits_{D} w_{\overline{z}}(z)\,dx\,dy\,,$$

$$z = x + iy\,,\ \ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\,,$$

for bounded regular domains $D$ in the complex plane $\mathbb{C}$ and $w \in C^1(D;\mathbb{R}) \Cap C^0(\overline{D};\mathbb{R})$ lead to a representation formula for these functions

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \, \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{D} w_{\overline{\zeta}}(\zeta) \, \frac{d\xi \, d\eta}{\zeta - z} , \ \zeta = \xi + i\eta , \ z \in D ,$$

generalizing the Cauchy formula for analytic functions. It is called the Cauchy–Pompeiu formula.

Treating this formula in the case of the unit disc $D = \mathbb{D}$ in the same way as the Poisson formula is deduced from the Cauchy formula leads to the Cauchy–Schwarz–Poisson–Pompeiu formula

$$\begin{aligned} w(z) \ = \ & \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \mathrm{Re}\, w(\zeta) \, \frac{\zeta + z}{\zeta - z} \, \frac{d\zeta}{\zeta} + \frac{1}{2\pi} \int\limits_{\partial D} \mathrm{Im}\, w(z) \, \frac{d\zeta}{\zeta} \\ & - \frac{1}{\pi} \int\limits_{\partial \mathbb{D}} \left( \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} + \frac{z \, \overline{w_{\overline{\zeta}}(\zeta)}}{1 - z\overline{\zeta}} \right) d\xi \, d\eta , \quad z \in \mathbb{D} . \end{aligned}$$

This formula becomes symmetric if $i$ Im $w(0)$ is subtracted,

$$w(z) - i \, \mathrm{Im}\, w(0) = \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} \mathrm{Re}\, w(\zeta) \, \frac{\zeta + z}{\zeta - z} \, \frac{d\zeta}{\zeta} -$$

$$- \frac{1}{2\pi} \int\limits_{\mathbb{D}} \left\{ \frac{w_{\overline{\zeta}}(\zeta)}{\zeta} \, \frac{\zeta + z}{\zeta - z} + \frac{\overline{w_{\overline{\zeta}}(\zeta)}}{\overline{\zeta}} \, \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} \right\} d\xi \, d\eta , \quad z \in \mathbb{D} .$$

Similarly,

$$w(z) - \mathrm{Re}\, w(0) = \frac{1}{2\pi} \int\limits_{\partial \mathbb{D}} \mathrm{Im}\, w(z) \, \frac{\zeta + z}{\zeta - z} \, \frac{d\zeta}{\zeta}$$

$$- \frac{1}{2\pi} \int\limits_{\mathbb{D}} \left\{ \frac{w_{\overline{\zeta}}(\zeta)}{\zeta} \, \frac{\zeta + z}{\zeta - z} - \frac{\overline{w_{\overline{\zeta}}(\zeta)}}{\overline{\zeta}} \, \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}} \right\} d\xi \, d\eta , \quad z \in \mathbb{D} ,$$

holds.

The area integral in the Cauchy–Pompeiu formula is called the Pompeiu operator

$$T\rho(z) := -\frac{1}{\pi}\int\limits_D \rho(\zeta)\,\frac{d\xi\,d\eta}{\zeta - z}\,, \quad z \in \mathbb{C}\,,$$

acting on spaces $L_p(\overline{D};\mathbb{C})$ and $L_{p,2}(D;\mathbb{C})$ for $1 < p$, see [9]. In case of $D = \mathbb{D}$

$$\widehat{T}\rho(z) := -\frac{1}{\pi}\int\limits_{\mathbb{D}} \left(\frac{\rho(\zeta)}{\zeta - z} + \frac{z\overline{\rho(\zeta)}}{1 - z\overline{\zeta}}\right) d\xi\,d\eta\,, \quad z \in \mathbb{D}\,,$$

or

$$\check{T}\rho(z) := -\frac{1}{2\pi}\int\limits_{\mathbb{D}} \left\{\frac{\rho(\zeta)}{\zeta}\,\frac{\zeta + z}{\zeta - z} + \frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\,\frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\right\} d\xi\,d\eta\,, \quad z \in \mathbb{D}\,,$$

have similar properties. They are important for treating complex first order equations as those of generalized analytic functions and, more general, the generalized Beltrami equation, see [9, 1, 5, 7, 8, 10].

For higher order equations iterations of the $T$–operator in proper combination with the $\overline{T}$–operator

$$\overline{T}\rho(z) := -\frac{1}{\pi}\int\limits_D \rho(\zeta)\,\frac{d\xi\,d\eta}{\overline{\zeta - z}}\,, \quad z \in D\,,$$

are adequate. For $D = \mathbb{D}$ the $\check{T}$– and $\overline{\check{T}}$–operators can be used in the same way.

## 2 Complex elliptic second order equations

Iterations of these operators of any order are studied in full detail and extensively in [6]. Here we will just consider $T^2$ and $T\overline{T}$ and see how they serve to treat complex second order equations with main part

$$Lw := aw_{\overline{z}\,\overline{z}} + bw_{\overline{z}z} + cw_{zz}\,.$$

Assuming $L$ to be elliptic, i.e.

$$a\zeta^2 + b\zeta\overline{\zeta} + c\overline{\zeta}^2 \neq 0 \quad \text{for } \zeta \in \mathbb{C}\,,\ \ \zeta \neq 0\,,$$

there are essentially two different cases. By a complete homeomorphism of a proper Beltrami equation $L$ can be transformed into a multiple of either $\partial^2/\partial\overline{z}^2 + q\,\partial^2/\partial\overline{z}\,\partial z$ or $\partial^2/\partial\overline{z}\,\partial z + q\,\partial^2/\partial\overline{z}^2$ with some $q$ satisfying $|q| < 1$, see [4]. In case $\partial^2/\partial\overline{z}^2$ is the leading term $T^2$ is needed while in the case where $\partial^2/\partial\overline{z}\,\partial z$ is dominant $T\overline{T}$ is useful. We have for a bounded regular domain $D$ and $\rho \in L_p(\overline{D};\mathbb{C})$

$$T^2\rho(z) = \frac{1}{\pi^2}\int\limits_D\int\limits_D \rho(\widetilde{\zeta})\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta}-\zeta}\,\frac{d\xi\,d\eta}{\zeta - z}\,,$$

$$T\overline{T}\rho(z) = \frac{1}{\pi^2}\int\limits_D\int\limits_D \rho(\widetilde{\zeta})\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\overline{\widetilde{\zeta}-\zeta}}\,\frac{d\xi\,d\eta}{\zeta - z}\,,$$

$$-\frac{1}{\pi}\int\limits_D \frac{d\xi\,d\eta}{(\zeta-\widetilde{\zeta})\,(\zeta - z)} =$$

$$= -\frac{1}{(\widetilde{\zeta} - z)\pi} \int\limits_D \left( \frac{1}{\zeta - \widetilde{\zeta}} - \frac{1}{\zeta - z} \right) d\xi \, d\eta =$$

$$= \frac{1}{\widetilde{\zeta} - z} \left[ \overline{\widetilde{\zeta} - z} - \frac{1}{2\pi i} \int\limits_{\partial D} \left[ \frac{1}{\zeta - \widetilde{\zeta}} - \frac{1}{\zeta - z} \right] \overline{\zeta} \, d\zeta \right] ,$$

$$-\frac{1}{\pi} \int\limits_D \frac{d\xi \, d\eta}{\overline{(\zeta - \widetilde{\zeta})} \, (\zeta - z)} = \log |z - \widetilde{\zeta}|^2 - \frac{1}{2\pi i} \int\limits_{\partial D} \log |\zeta - \widetilde{\zeta}|^2 \frac{d\zeta}{\zeta - z} \, .$$

Hence,

$$T^2 \rho(z) \ = \frac{1}{\pi} \int\limits_D \rho(\zeta) \, \frac{\overline{\zeta - z}}{\zeta - z} \, d\xi \, d\eta - \frac{1}{2\pi i} \int\limits_{\partial D} T \rho(\zeta) \, \frac{\overline{\zeta} \, d\zeta}{\zeta - z} \, ,$$

$$T\overline{T} \rho(z) \ = \frac{2}{\pi} \int\limits_D \rho(\zeta) \, \log |\zeta - z| \, d\xi \, d\eta -$$

$$- \frac{1}{2\pi i} \int\limits_{\partial D} \frac{2}{\pi} \int\limits_D \rho(\widetilde{\zeta}) \, \log |\widetilde{\zeta} - \zeta| \, d\widetilde{\xi} \, d\widetilde{\eta} \, \frac{d\zeta}{\zeta - z} \, .$$

While the area integrals in these representations for $T^2$ and $T\overline{T}$ are the main contributions the boundary integrals are proper analytic functions. In a similar way or by conjugating the last formula

$$\overline{T} T \rho(z) \ = \frac{2}{\pi} \int\limits_D \rho(\zeta) \, \log |\zeta - z| \, d\xi \, d\eta +$$

$$+ \frac{1}{2\pi i} \int\limits_{\partial D} \frac{2}{\pi} \int\limits_D \rho(\widetilde{\zeta}) \, \log |\widetilde{\zeta} - \zeta| \, d\widetilde{\xi} \, d\widetilde{\eta} \, \overline{\frac{d\overline{\zeta}}{\zeta - z}}$$

is found where an antianalytic function is involved. Replacing the area integral in the last two formulas by

$$S\rho(z) := -\frac{2}{\pi}\int_D \rho(\zeta)\, g(z,\zeta)\, d\xi\, d\eta\,, \quad z \in D\,,$$

where $g(z,\zeta) = -\log|z-\zeta| + h(z,\zeta)$ is the Green function of $D$ for the Laplacian and $h$ is harmonic in $z \in D$ and in $\zeta \in D$ both $T\overline{T}\rho$ and $\overline{T}T\rho$ are seen to differ from this integral by a sum of an analytic and an antianalytic function. While this sum is the general solution to the homogeneous Laplace equation $\partial^2/\partial\overline{z}\,\partial z\, w = 0$ the integral $S\rho$ provides a particular solution to the Poisson equation $\partial^2/\partial\overline{z}\,\partial z\, w = \rho$. Similarly,

$$T_1\rho(z) := \frac{1}{\pi}\int_D \rho(\zeta)\, \frac{\overline{\zeta - z}}{\zeta - z}\, d\xi\, d\eta$$

is a particular solution to $\partial^2/\partial\overline{z}^2\, w = \rho$ while $\psi(z) + \overline{z}\phi(z)$ is the general solution to the related homogeneous equation, where $\phi$ and $\psi$ are arbitrary analytic functions.

Boundary conditions which distinguish solutions within this class of general solutions turn out as boundary value problems for analytic functions and thus are essentially of Riemann–Hilbert or of Riemann type.

Natural conditions in both cases are

$$\mathrm{Re}\,\{\overline{\lambda(z)}\, w_{\overline{z}}(z)\} = 0\,,\ \mathrm{Re}\,\{\lambda(z)\, z'(s)\, w(z)\} = 0\,,$$
$$z = z(s) \in \partial D\,,$$

where $s$ is the arc length parameter of $\partial D$. The two boundary conditions involved here are adjoint to one another, [9], p. 228 and were suggested by A. Dzhuraev, see [8]. The particular case $\lambda \equiv 1$ is extensively treated in [4, 2], see also [3] especially for the unit disc. Here the method is just indicated.

**Theorem** *Let $D \subset \mathbb{C}$ be a smooth multiply connected bounded domain and $q_1, q_2$ be measurable functions in $\overline{D}$ satisfying*

$$|q_1(z)| + |q_2(z)| \leq q_0 < 1\,, \quad z \in \overline{D}\,, \quad q_0 \|\widehat{\Pi}\|_{L_p} < 1\,,$$

*where $\widehat{\Pi}$ is a proper singular integral operator on $L_p(\overline{D};\mathbb{C})$ defined below,*
$a_1, a_2, b_1, b_2, c_1, c_2 \in C^\alpha(\overline{D}),\ 0 < \alpha < 1,\ d \in L_p(\overline{D};\mathbb{C}),\ 2 < p.$

*Then the boundary value problem*

$$\mathrm{Re}\, w_{\overline{z}}(z) = 0\,, \quad \mathrm{Re}\,\{z'(s)\, w(z)\} = 0 \ \text{ on } \ \partial D$$

*for the equation*

$$\mathcal{L}\, w + a_1 w_z + a_2 \overline{w_z} + b_1 w_{\overline{z}} + b_2 \overline{w_{\overline{z}}} + c_1 w + c_2 \overline{w} = d$$

*has the Fredholm property. Here $\mathcal{L}$ is either $\mathcal{L}_1$ or $\mathcal{L}_2$, both being second order operators given by*

$$\mathcal{L}_1 w := w_{\overline{z}\,\overline{z}} + q_1 w_{\overline{z}z} + q_2 \overline{w_{\overline{z}z}},$$

$$\mathcal{L}_2 w := w_{\overline{z}z} + q_1 w_{\overline{z}\,\overline{z}} + q_2 \overline{w_{\overline{z}\,\overline{z}}}.$$

**Proof** Let $w_k$ be a solution to the differential equation for $\mathcal{L} = \mathcal{L}_k$, $k = 1, 2$, and denote $\rho_1 := w_{1\overline{z}\,\overline{z}}$ and $\rho_2 := \overline{w_{2\overline{z}z}}$. Then with proper analytic functions $\phi_k, \psi_k, k = 1, 2$,

$$w_1 = \psi_1 + \overline{z}\phi_1 + T_1\rho_1\,, \ \ T_1\rho_1(z) := \frac{1}{\pi}\int\limits_D \rho_1(\zeta)\,\frac{\overline{\zeta - z}}{\zeta - z}\,d\xi\,d\eta\,,$$

$$w_{1z} = \psi_1' + \overline{z}\phi_1' + \frac{\partial}{\partial z}T_1\rho_1\,, \ \ \frac{\partial}{\partial z}T_1\rho_1(z) = \frac{1}{\pi}\int\limits_D \rho_1(\zeta)\,\frac{\overline{\zeta - z}}{\zeta - z}\,\frac{d\xi\,d\eta}{\zeta - z}\,,$$

$$w_{1\overline{z}} = \phi_1 + T\rho_1\,, \ \ T\rho_1(z) := -\frac{1}{\pi}\int\limits_D \rho_1(\zeta)\,\frac{d\xi\,d\eta}{\zeta - z}\,,$$

$$w_{1\overline{z}z} = \phi_1' + \Pi\rho_1\,, \ \ \Pi\rho_1(z) := -\frac{1}{\pi}\int\limits_D \rho_1(\zeta)\,\frac{d\xi\,d\eta}{(\zeta - z)^2}\,;$$

$$w_2 = \psi_2 + \overline{\phi_2} + S\overline{\rho_2}\,, \ \ S\overline{\rho_2}(z) := -\frac{2}{\pi}\int\limits_D \overline{\rho_2(\zeta)}\,g(z,\zeta)\,d\xi\,d\eta\,,$$

$$w_{2z} = \psi_2' - \frac{2}{\pi}\int\limits_D \overline{\rho_2(\zeta)}\,h_z(z,\zeta)\,d\xi\,d\eta + T\overline{\rho_2} =: \widetilde{\psi}_2 + T\overline{\rho_2}\,,$$

$$w_{2\overline{z}} = \overline{\phi_2'} - \frac{2}{\pi}\int\limits_D \overline{\rho_2(\zeta)}\,h_{\overline{z}}(z,\zeta)\,d\xi\,d\eta + \overline{T\rho_2} =: \overline{\widetilde{\phi}_2} + \overline{T\rho_2}\,,$$

$$w_{2\overline{z}\,\overline{z}} = \overline{\phi_2''} - \frac{2}{\pi}\int\limits_D \overline{\rho_2(\zeta)}\,h_{\overline{z}\,\overline{z}}(z,\zeta)\,d\xi\,d\eta + \overline{\Pi\rho_2} = \overline{\widetilde{\phi}_2'} + \overline{\Pi\rho_2}\,.$$

The analytic functions $\phi_k, \psi_k$ and $\rho_k \in L_p(\overline{D};\mathbb{C})$, $2 < p$, have to be determined such that the boundary conditions together with the respective second order equation are satisfied.

The first boundary condition leads to

$$\mathrm{Re}\,\phi_1 = -\mathrm{Re}\,T\rho_1 =: h_1\,; \quad \mathrm{Re}\,\widetilde{\phi}_2 = -\mathrm{Re}\,T\rho_2 =: h_2 \ \text{ on } \ \partial D\,.$$

These problems are solvable if and only if ($k = 1$ or $k = 2$)

$$\int\limits_{\partial D} h_k \psi_\mu d\zeta = 0\,, \quad 1 \le \mu \le m\,,$$

where $\{\psi_1, \dots, \psi_m\}$ is a basis for the solution space to the adjoint boundary value problem

$$\mathrm{Re}\,\{z'(s)\,\psi(z)\} = 0 \ \text{ on } \ \partial D$$

for analytic functions. If these conditions are satisfied then, see [2],

$$\begin{aligned}\phi_k + T\rho_k \ &= -\int\limits_D \{J_L(z,\zeta)\rho_k(\zeta) + J_K(z,\overline{\zeta})\overline{\rho_k(\zeta)}\}d\xi d\eta + \\ &\quad + T\rho_k(z_0) + ic_{0k}\,,\end{aligned}$$

where $c_{0k}$ is an arbitrary real constant and with fixed $z_0 \in \overline{D}$

$$J_L(z,\zeta) := \int\limits_{z_0}^{z} L(t,\zeta)\,dt\,, \quad J_K(z,\overline{\zeta}) := \int\limits_{z_0}^{z} K(t,\overline{\zeta})\,dt\,,$$

$$L(z,\zeta) := -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\zeta}\,, \quad K(z,\overline{\zeta}) := -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\overline{\zeta}}\,.$$

The function $K(z,\overline{\zeta})$ is the Bergman kernel function, see [7].

The second boundary condition gives on $\partial D$

$$\begin{aligned}\mathrm{Re}\,\{z'(s)\psi_1\} \ &= \ -\mathrm{Re}\,\{z'(s)\,(\overline{z}\phi_1 + T_1\rho_1)\} \ =: \ h^1\,, \\ \mathrm{Re}\,\{z'(s)\psi_2\} \ &= \ -\mathrm{Re}\,\{z'(s)\,(\phi_2 + S\rho_2)\} \quad\ =: \ h^2\,.\end{aligned}$$

These problems are solvable if and only if ($k=1$ or $k=2$)

$$\int_{\partial D} h^k(\zeta)\, ds_\zeta = 0\,.$$

This just one condition serves to determine the not yet fixed real constant $c_{0k}$. It becomes

$$c_{0k} =$$

$$\mathrm{Im}\, \frac{1}{|D|} \iint_{D\,D} \{J_L(z,\zeta)\, \rho_k(\zeta) + J_K(z,\overline{\zeta})\, \overline{\rho_k(\zeta)}\}\, d\xi\, d\eta\, dx\, dy - \mathrm{Im}\, T\rho_k(z_0)$$

when $|D|$ denotes the area of $D$. The solution to the boundary value problem for $\psi_k$ then is

$$\psi_k = \psi_{0k} + \sum_{\mu=1}^{m} \gamma_{\mu k}\psi_\mu\,, \quad \gamma_{\mu k} \in \mathbb{R}\,, \quad 1 \le \mu \le m\,, \;\; k = 1, 2\,,$$

where

$$\psi_{0k}(z) := \frac{1}{\pi i} \int_{\partial D} (\boldsymbol{\ell}(z,\zeta) + \frac{1}{\zeta - z})\, h^k(\zeta)\, ds_\zeta$$

is a particular solution to the inhomogeneous problem.

Here

$$\boldsymbol{\ell}(z,\zeta) := \pi \int_{z_0}^{\zeta} \ell(z,t)\, dt\,,$$

$$\ell(z,\zeta) := \frac{2}{\pi} \frac{\partial^2 h(z,\zeta)}{\partial z\, \partial \zeta} = \frac{1}{\pi(\zeta - z)^2} - L(z,\zeta)\,,$$

see [7], is a multi–valued function in the multiply connected domain $D$. Nevertheless $\psi_{0k}$ is single–valued in $D$ because of the above condition determining $c_{0k}$. An application of the Green formula shows

$$
\begin{aligned}
w_{01}(z) &:= \psi_{01}(z) + \overline{z}\phi_1(z) + T_1\rho_1(z) = T(\phi_1 + T\rho_1)\,(z) - \\
&\qquad -\frac{1}{\pi}\int\limits_D \{\boldsymbol{\ell}(z,\zeta)\,[\phi_1(\zeta) + T\rho_1(\zeta)] + \\
&\qquad\qquad +[2h_z(z,\zeta) - \boldsymbol{\ell}(z,\zeta)][\overline{\phi_1(\zeta)} + \overline{T\rho_1(\zeta)}]\}d\xi\, d\eta\,,
\end{aligned}
$$

$$
\begin{aligned}
w_{02}(z) &:= \psi_{02}(z) + \overline{\phi_2(z)} + S\overline{\rho_2}(z) = T(\overline{\widetilde{\phi}_2 + T\rho_2})\,(z) - \\
&\qquad -\frac{1}{\pi}\int\limits_D \{\boldsymbol{\ell}(z,\zeta)\,[\overline{\widetilde{\phi}_2(\zeta) + T\rho_2(\zeta)}] + \\
&\qquad\qquad +[2h_z(z,\zeta) - \boldsymbol{\ell}(z,\zeta)][\widetilde{\phi}_2(\zeta) + T\rho_2(\zeta)]\}d\xi\, d\eta\,,
\end{aligned}
$$

which will together with $\sum_{\mu=1}^{m} \gamma_{\mu k}\psi_\mu$ form a solution to the boundary value problem for the second order equation as soon as $\rho_k$ is determined properly.

In order to find $\rho_k$ through the differential equation for $w_k$ the first order derivatives of $\phi_1, \psi_1, \widetilde{\phi}_2$ have to be determined. In fact, what is needed is just

$$
\begin{aligned}
w_{01z}(z) &= \Pi(\phi_1 + T\rho_1)\,(z) - \frac{1}{\pi}\int\limits_D \{\boldsymbol{\ell}_z(z,\zeta)\,[\phi_1(\zeta) + T\rho_1(\zeta)] \\
&\qquad +[2h_{zz}(z,\zeta) - \boldsymbol{\ell}_z(z,\zeta)]\,[\overline{\phi_1(\zeta) + T\rho_1(\zeta)}]\}d\xi\, d\eta\,;
\end{aligned}
$$

$$
\begin{aligned}
w_{02z}(z) \;=\; & \Pi(\overline{\widetilde{\phi}_2 + T\rho_2})\,(z) - \frac{1}{\pi}\int\limits_D \{\boldsymbol{\ell}_z(z,\zeta)\,[\overline{\phi_2(\zeta) + T\rho_2(\zeta)}] \\
& +[2h_{zz}(z,\zeta) - \boldsymbol{\ell}_z(z,\zeta)]\,[\widetilde{\phi}_2(\zeta) + T\rho_2(\zeta)]\}d\xi\, d\eta\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
w_{1\overline{z}z}(z) \; & = \phi_1'(z) + \Pi\rho_1(z) = \\
& = -\int\limits_D \{L(z,\zeta)\,\rho_1(\zeta) + K(z,\zeta)\,\overline{\rho_1(\zeta)}\}d\xi\, d\eta =: \\
& =: \widehat{\Pi}\rho_1(z)\,; \\
\overline{w_{2\overline{z}\,\overline{z}}} \; & = \widetilde{\phi}_2' + \Pi\rho_2 = \widehat{\Pi}\rho_2\,.
\end{aligned}
$$

Inserting the respective expressions into the differential equation leads to a singular integral equation of the same kind. We get

$$
\begin{aligned}
& \rho_1 + q_1\widehat{\Pi}\rho_1 + q_2\overline{\widehat{\Pi}\rho_1} + a_1 w_{01z} + a_2\overline{w_{01z}} + b_1(\phi_1 + T\rho_1) + b_2(\overline{\phi_1 + T\rho_1}) \\
& + c_1 w_{01} + c_2\overline{w_{01}} = d - \sum_{\mu=1}^{m}\gamma_{\mu 1}[a_1\psi_\mu' + a_2\overline{\psi_\mu'} + c_1\psi_\mu + c_2\overline{\psi_\mu}] =: d_1\,; \\
& \overline{\rho_2} + q_1\overline{\widehat{\Pi}\rho_2} + q_2\overline{\widehat{\Pi}\rho_2} + a_1 w_{02z} + a_2\overline{w_{02z}} + b_1(\overline{\widetilde{\phi}_2 + T\rho_2}) + b_2(\widetilde{\phi}_2 + T\rho_2) \\
& + c_1 w_{02} + c_2\overline{w_{02}} = d - \sum_{\mu=1}^{m}\gamma_{\mu 2}[a_1\psi_\mu' + a_2\overline{\psi_\mu'} + c_1\psi_\mu + c_2\overline{\psi_\mu}] =: d_2\,.
\end{aligned}
$$

Taking the complex conjugate of the last equation and observing that $\widetilde{\phi}_2$ is defined by $\rho_2$ in the same way as $\phi_1$ is by $\rho_1$ both $\rho_1$ and $\rho_2$ turn out to solve the same kind of integral equation. The singular integral operator $\widehat{\Pi}$ was found by A. Dzhuraev [7] and its $L_2$–norm shown to be 1, see [7, 2].

Hence, because of the continuity of the $L_p$–norm of $\widehat{\Pi}$ as a function of $p$ the operator $q_1\widehat{\Pi}\rho + q_2\overline{\widehat{\Pi}\rho}$ turns out to be contractive in $L_p(\overline{D};\mathbb{C})$ as long as $q_0\|\widehat{\Pi}\|_{Lp} < 1$ which holds for $p$ close enough to 2. Thus, the operator $\rho + q_1\widehat{\Pi}\rho + q_2\overline{\widehat{\Pi}\rho}$ is invertible in $L_p(\overline{D};\mathbb{C})$. Because the remaining operator expressed in terms of $\phi_1 + T\rho_1$ and $\widetilde{\phi}_2 + T\rho_2$ respectively turns out to be compact of $L_p(\overline{D};\mathbb{C})$ the Fredholm alternative applies to this singular integral equation.

It has to be argued about that this also is true for the boundary value problem for our second order equations. We just do this for the first case $k = 1$. Let the homogeneous integral equation for $\rho$, i.e. when $d_1 = 0$, have $N \geq 0$ nontrivial over $\mathbb{R}$ linearly independent solutions $\rho_{1\nu}$, $1 \leq \nu \leq N$, and the inhomogeneous equation $d_1 \neq 0$ be solvable if and only if the $N$ orthogonality conditions

$$\mathrm{Re} \int_D d_1(z)\, \sigma_\gamma(z)\, dx\, dy = 0\,, \quad 1 \leq \nu \leq N\,,$$

are satisfied, where $\{\sigma_1, \ldots, \sigma_N\}$ is a basis for the null space of the adjoint integral equation.

These conditions lead to an algebraic system of $N$ linear equations for the real coefficients $\gamma_{\mu 1}$, $1 \leq \mu \leq m$, in the general solution $\psi_1$. Let the rank of the coefficients matrix be $r \geq 0$. Then $r$ of the $\gamma_{\mu 1}$'s can be expressed by the remaining $m-r$ ones leaving $N-r$ equations as solvability conditions on $d$ and the remaining $\gamma_{\mu 1}$'s. Together with the above $m$ solvability conditions

$$\int\limits_{\partial D} \mathrm{Re}\, T\rho_1\psi_\mu d\zeta = 0\,, \quad 1 \leq \mu \leq m\,,$$

for the boundary value problem for $\phi_1$ in which $\rho_1$ has to be replaced by $\rho_{10}+\sum_{\nu=1}^{N} c_\nu \rho_{1\nu}$ we have altogether $N-r+m$ solvability conditions for the inhomogeneous problem. Here $\rho_{10}$ is a particular solution to the inhomogeneous integral equation and $c_\nu \in \mathbb{R}$, $1 \leq \nu \leq N$, are arbitrary.

The boundary problem has the Fredholm property if we can show that the homogeneous problem, i.e. if $d = 0$, has $N - r + m$ over $\mathbb{R}$ linearly independent solutions. To show this we consider the integral equation with $d = 0$ and choose $\gamma_{\mu 1} = \delta_{\mu\nu}$ where $\nu$ varies over all those indices of the remaining $m - r$ coefficients $\gamma_{\mu 1}$'s. Because this integral equation is solvable for the solvability conditions are satisfied we get a solution for any such $\nu$ and those for different $\nu$'s are obviously linearly independent. These are then $m-r$ ones which together with the $N$ linear independent solutions of the homogeneous integral equation, i.e. with $d$ and all the $\gamma_{\mu 1}$ vanishing, form $N + m - r$ linearly independent solutions to the homogeneous boundary value problem. Any other solution to the homogeneous problem is a linear combination of these.

## 3 Dirichlet problem for higher order equations in the unit disc

In the same way as the $T$–operator the $\widehat{T}$–operator can be iterated. But it is more convenient to treat the $\check{T}$–operator instead differing from $\widehat{T}$ just by an additive constant. Iteration of this operator demands the calculation of the following integrals for $z, t \in \mathbb{D}$, see [11], which can be found by applying the Green and the Cauchy–Pompeiu formula, in particular $\overline{z} = -\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{d\xi\, d\eta}{\zeta - z}$,

$1 = \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{d\xi\, d\eta}{1 - \overline{z}\zeta}$, $z \in \mathbb{D}$.

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} d\xi\, d\eta = 1\,, \quad \frac{1}{\pi}\int\limits_{\mathbb{D}} \zeta\, d\xi\, d\eta = 0\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\zeta - z}}{\zeta - z}\, d\xi\, d\eta = \frac{1}{2}\,\overline{z}^2\,, \quad \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\zeta - z}{\zeta - t}\, d\xi\, d\eta = 1 - |t|^2 + z\overline{t}\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\zeta - z}}{\zeta - t}\, d\xi\, d\eta = \overline{z}\overline{t} - \frac{1}{2}\,\overline{t}^2\,, \quad \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\zeta(\overline{\zeta - z})}{\zeta - z}\, d\xi\, d\eta = \frac{1}{2}\,|z|^2\overline{z} - \overline{z}\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{(\overline{\zeta - z})^2}{\zeta - z}\, d\xi\, d\eta = -\frac{1}{3}\,\overline{z}^3\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{(\zeta-z)^2}{\zeta-t}\,d\xi\,d\eta = (t-2z)\,(1-|t|^2) - z^2\overline{t}\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{(\overline{\zeta-z})^2}{\zeta-t}\,d\xi\,d\eta = \overline{z}\overline{t}^2 - \overline{z}^2\overline{t} - \frac{1}{3}\overline{t}^3\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{|\zeta-z|^2}{\zeta-t}\,d\xi\,d\eta = \frac{1}{2}\,(z-t)\,(\overline{t}^2 - 2\overline{z}\overline{t}) - \overline{z}\,;$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\zeta-z}{1-t\overline{\zeta}}\,d\xi\,d\eta = \frac{1}{2}\,t - z\,, \quad \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\overline{\zeta-z}}{1-t\overline{\zeta}}\,d\xi\,d\eta = -\overline{z}$$

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$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\zeta\overline{t}(\zeta-z)}{1-\zeta\overline{t}}\,d\xi\,d\eta = 0\,, \quad \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\zeta\overline{t}(\overline{\zeta-z})}{1-\zeta\overline{t}}\,d\xi\,d\eta = \frac{1}{2}\,\overline{t}\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{(\zeta-z)^2}{1-t\overline{\zeta}}\,d\xi\,d\eta = z^2 - zt + \frac{1}{3}\,t^2\,,$$

$$\tfrac{1}{\pi}\int\limits_{\mathbb{D}} \tfrac{(\overline{\zeta-z})^2}{1-t\overline{\zeta}}\,d\xi\,d\eta = \overline{z}^2\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{|\zeta-z|^2}{1-t\overline{\zeta}}\,d\xi\,d\eta = |z^2| - \frac{1}{2}\,t\overline{z} + \frac{1}{2}\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{(\zeta-z)^2}{1-z\overline{\zeta}}\,d\xi\,d\eta = \frac{1}{3}\,z^2\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{(\overline{\zeta-z})^2}{1-z\overline{\zeta}}\,d\xi\,d\eta = \overline{z}^2\,,$$

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{|\zeta-z|^2}{1-z\overline{\zeta}}\,d\xi\,d\eta = \frac{1}{2}\,(1+|z|^2)\,,\cdots\,.$$

It thus

$$\check{T}^2\rho(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} \mathrm{Re}\,(\zeta - z)\left[\frac{\rho(\zeta)}{\zeta}\,\frac{\zeta + z}{\zeta - z} + \frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\,\frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\right] d\xi\, d\eta$$

can be shown. Continuing this way inductively

$$\check{T}^k\rho(z) =$$

$$= \frac{(-1)^k}{2\pi(k-1)!}\int\limits_{\mathbb{D}} (2\,\mathrm{Re}\,(\zeta - z))^{k-1}\left[\frac{\rho(\zeta)}{\zeta}\,\frac{\zeta + z}{\zeta - z} + \frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\,\frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\right] d\xi\, d\eta$$

can be attained. Instead of iterating one easily can show this relation by exploiting the properties of the integral operator on the right hand side of the last equality, see [6]. Denote this operator for a moment by $\check{T}_k$. In order to show $\check{T}_k = \check{T}^k$ we observe $\mathrm{Re}\,\check{T}_k\rho = 0$ on $\partial\mathbb{D}$, $\mathrm{Im}\,\check{T}_k\rho(0) = 0$, $\frac{\partial}{\partial\overline{z}}\check{T}_k\rho = \check{T}_{k-1}\rho$ for $k > 1$ in $\mathbb{D}$ and $\check{T}_1 = \check{T}$. Hence, $\check{T}_k = \check{T}\check{T}_{k-1}$ and thus by the properties of the $\check{T}$–operator $\check{T}_k = \check{T}^k$ is proved inductively.

As the $\check{T}_k$–operators their $z$–derivatives exist as weakly singular integrals as long as the order of the derivative is less than $k$,

$$\frac{\partial^\ell}{\partial z^\ell}\check{T}_k\rho(z) = \sum_{\lambda=0}^{\ell}\binom{\ell}{\lambda}\frac{(-1)^{k-\lambda}(\ell-\lambda)!}{(k-\lambda-1)!}\,\frac{1}{\pi}\int\limits_{\mathbb{D}}(2\,\mathrm{Re}\,(\zeta - z))^{k-\lambda-1}$$

$$\left[\frac{\rho(\zeta)}{(\zeta - z)^{\ell-\lambda+1}} + \frac{\overline{\zeta}^{\ell-\lambda-1}\overline{\rho(\zeta)}}{(1 - z\overline{\zeta})^{\ell-\lambda+1}}\right] d\xi\, d\eta +$$

$$+\frac{(-1)^{k-\ell-1}}{(k-\ell-1)!}\,\frac{1}{2\pi}\int\limits_{\mathbb{D}}(2\,\mathrm{Re}\,(\zeta - z))^{k-\ell-1}\left[\frac{\rho(\zeta)}{\zeta} + \frac{\overline{\rho(\zeta)}}{\overline{\zeta}}\right] d\xi\, d\eta\,,$$

$$0 \le \ell \le k-1\,,$$

while the integrals with strong singularity

$$\frac{\partial^k}{\partial z^k}\check{T}_k\rho(z) = \frac{(-1)^k k}{\pi}\int\limits_{\mathbb{D}}\Bigg\{\Bigg(\frac{\zeta - z + \overline{\zeta - z}}{\zeta - z} - 1\Bigg)^{k-1}\frac{\rho(\zeta)}{(\zeta - z)^2}$$
$$+\Bigg(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\overline{\zeta}}\,\overline{\zeta} - 1\Bigg)^{k-1}\frac{\overline{\rho(\zeta)}}{(1 - z\overline{\zeta})^2}\Bigg\}\,d\xi\,d\eta$$

only exist in the Cauchy principal value sense.

**Theorem** *The Dirichlet problem with side conditions*

$$\operatorname{Re}\frac{\partial^\ell w}{\partial\overline{z}^\ell} = 0 \ \ on\ \partial D\,,\ \ \operatorname{Im}\frac{\partial^\ell w}{\partial\overline{z}^\ell}\Big|_{z=0} = 0\,,\ \ 0 \le \ell \le k-1\,,$$

*for the $k$–th order differential equation*

$$\frac{\partial^k w}{\partial\overline{z}^k}\ +q_1\frac{\partial^k w}{\partial\overline{z}^{k-1}\partial z} + q_2\frac{\partial^k w}{\partial\overline{z}^{k-1}\partial z}+$$
$$= \sum_{\ell=0}^{k-1}\sum_{\lambda=0}^{\ell}\left[a_{\lambda\ell}\frac{\partial^\ell w}{\partial\overline{z}^{\ell-\lambda}\partial z^\lambda} + b_{\lambda\ell}\overline{\frac{\partial^\ell w}{\partial\overline{z}^{\ell-\lambda}\partial z^\lambda}}\right] = f$$

*is solvable for given* $a_{\lambda\ell}, b_{\lambda\ell} \in C^\alpha(\overline{\mathbb{D}};\mathbb{C})$, $0 < \alpha < 1$, $f \in L_p(\overline{D};\mathbb{C})$, $q_1, q_2$ *measurable satisfying* $|q_1(z)| + |q_2(z)| \le q_0 < 1$, $q_0\|\check{\Pi}\|_{L_p} < 1$, $\check{\Pi} = \dfrac{\partial}{\partial z}\check{T}$.

*Any solution is given in the form* $w = \check{T}^k\rho$ *where* $\rho \in L_p(\overline{\mathbb{D}};\mathbb{C})$ *is a solution to the singular integral equation*

$$\rho + q_1\check{\Pi}\rho + q_2\overline{\check{\Pi}\rho} + \sum_{\ell=0}^{k-1}\sum_{\lambda=0}^{\ell}\left[a_{\lambda\ell}\frac{\partial^\lambda}{\partial z^\lambda}\check{T}_{k-\ell+\lambda}\rho + b_{\lambda\ell}\overline{\frac{\partial^\lambda}{\partial z^\lambda}\check{T}_{k-\ell+\lambda}\rho}\right] = f\,.$$

**Proof** From [7], p. 210 the operator

$$\check{\Pi}\rho(z) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} \left[\frac{\rho(\zeta)}{(\zeta - z)^2} + \frac{\overline{\rho(\zeta)}}{(1 - z\overline{\zeta})^2}\right] d\xi\, d\eta$$

is known to be unitary in $L_2(\overline{D};\mathbb{C})$ so that $q_0\|\check{\Pi}\|_{L_p} < 1$ for $p$ close enough to 2. Since the remaining integral operator in the singular integral equation for $\rho$ originating from the lower order derivatives of $w$ is compact in $L_p(\overline{\mathbb{D}};\mathbb{C})$ this integral equation has the Fredholm property. If $\rho$ is a solution then $w_p = \check{T}_k\rho$ is a particular solution to the differential equation. Its general solution then is

$$w = \check{T}_k\rho + \sum_{\ell=0}^{k-1}\frac{1}{\ell!}\,\phi_\ell(z)\,\overline{z}^\ell$$

where $\phi_\ell$ are analytic functions in $\mathbb{D}$. From

$$\frac{\partial^\ell w}{\partial\overline{z}^\ell} = \check{T}_{k-\ell}\rho + \sum_{\lambda=\ell}^{k-1}\frac{1}{(\lambda-\ell)!}\,\phi_\lambda(z)\,\overline{z}^{\lambda-\ell}\,, \quad 0 \le \ell \le k-1\,,$$

we see via the boundary and side conditions

$$\mathrm{Re}\,\phi_{k-1} = 0 \ \text{ on } \ \partial\mathbb{D}\,, \quad \mathrm{Im}\,\phi_{k-1}(0) = 0$$

i.e. $\phi_{k-1} = 0$. In the same manner inductively $\phi_\ell = 0$ for $0 \le \ell \le k-1$. Therefore the solution $w$ is just of the form $w = \check{T}_k\rho$.

More general $k$–th order differential equations could be handled if the properties of the singular operators $\dfrac{\partial^\ell \check{T}_\ell}{\partial z^\ell}$ are known in detail.

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