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Some inequalities with integral means

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Abstract
We generalize some inequalities of Seiffert, Pearce, Pecarić and Dragomir.

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1 Introduction

For $0 < a < b$, let us denote by

$$G(a,b) = (ab)^{\frac{1}{2}} \quad , \quad A(a,b) = \frac{a+b}{2}$$

and

$$H(a,b) = \frac{2ab}{a+b} \ ,$$

the geometric mean, the arithmetic mean respectively the harmonic mean of a and b.

In [5] H.J. Seiffert proved the following result.

Theorem A. *Let $f : [a,b] \to \mathcal{R}$ be a Riemann–integrable positive function and $g : [G(a,b), A(a,b)] \to \mathcal{R}$ a strictly positive increasing function. Then the inequality*

$$g(G(a,b)) < \frac{\int_a^b f(t)g(h(t))dt}{\int_a^b f(t)dt} < g(A(a,b))$$

holds, where $h(t) = G(t, a+b-t)$.

This result was generalized in [3] where, in the definition of h, the geometric mean was replaced by other concrete means (power means, extended logarithmic means or integral power means). We go further with this generalization considering some abstract means and also replacing the integral by a functional. So the results remember those from [6].

We deal also with some results of other type. In [1] some inequalities related to Hermite–Hadamard's inequality are proved.

Theorem B. *Let f be a differentiable convex function on $[a, b]$.*

i) If $f'(G(a,b)) \geq 0$, then

$$\frac{1}{b-a}\int_a^b f(x)dx \geq f(G(a,b)) \ .$$

ii) If $f'(H(a,b)) \geq 0$ then

$$\frac{1}{b-a}\int_a^b f(x)dx \geq f(H(a,b)) \ .$$

In what follows we generalize also these results in more directions.

2 A property of some means

Let $p : [a, b] \to \mathcal{R}$ be a positive Riemann–integrable function. As it is known, the expression

$$m_p(a,b) = \frac{1}{b-a}\int_a^b p(t)dt$$

represents the integral arithmetic mean of p on $[a, b]$. If p is strictly increasing, then

$$M_p(a, b) = p^{-1}(m_p(a, b))$$

defines a mean of a and b. For example, taking $p(t) = t^r$ we get the extended logarithmic mean.

We want to study the functions:

$$h_p(x) = m_p(x, a + b - x)$$

and

$$H_p(x) = M_p(x, a + b - x)\ .$$

They are symmetric, so that we study them only on $[a, A(a, b)]$.

Theorem 1 . *If p is a convex function on $[a, b]$ then*

$$p(A(a, b)) \le h_p(x) \le m_p(a, b) \quad , \quad a \le x \le b\ .$$

Proof. We have

$$h_p'(x) =$$

$$= \frac{2}{a+b-2x}\left[\frac{1}{a+b-2x}\int\limits_x^{a+b-x} p(t)dt - \frac{p(a+b-x)+p(x)}{2}\right]\ .$$

It follows from Hermite–Hadamard's inequality (see [6]) that h' is negative on $[a, A(a, b)]$ for p convex. Thus h_p is decreasing on $[a, A(a, b)]$ which gives the desired inequalities.

Corollary 1 . *If p is a strictly increasing convex function then*

$$A(a, b) \le H_p(x) \le M_p(a, b) \quad , \ \textit{for all } x\ .$$

Let us also considere some means defined otherwise. Let q be a positive function on $[a, b]$. The expression

$$n_q(a, b) = \frac{q(a) + q(b)}{2}$$

represents the discrete arithmetic mean of q on [a,b]. If a is strictly increasing, then

$$N_q(a,b) = q^{-1}(n_q(a,b))$$

defines a quasi–arithmetic mean of the numbers a and b. We study also the functions

$$k_q(x) = n_q(x, a+b-x)$$

and

$$K_q(x) = N_q(x, a+b-x) \ .$$

Theorem 2 . *If q is a convex function on $[a,b]$, then*

$$q(A(a,b)) \leq k_q(x) \leq n_q(a,b) \quad , \quad a \leq x \leq b \ .$$

Proof. For the first inequality we have

$$q(A(a,b)) = q\left(\frac{x+a+b-x}{2}\right) \leq$$

$$\frac{q(x)+q(a+b-x)}{2} = n_q(x, a+b-x) \ .$$

The second inequality follows from

$$q(x) \leq \frac{b-x}{b-a}q(a) + \frac{x-a}{b-a}q(b)$$

and

$$q(a+b-x) \leq \frac{x-a}{b-a}q(a) + \frac{b-x}{b-a}q(b) \ .$$

Corollary 2 *If q is a strictly increasing convex function, then*

$$A(a,b) \leq K_q(x) \leq N_q(a,b) \quad , \quad a \leq x \leq b \ .$$

3 The result of Seiffert

We generalize the result of Theorem A for an increasing functional, i.e. a functional $L : C[a,b] \to R$ with the property

$$L(f) \leq L(g) \quad \text{if} \quad f(x) \leq g(x) \quad , \quad a \leq x \leq b.$$

Common examples of such functionals are the isotonic linear functionals (see the next paragraph) but here we have other examples.

The monotony of L implies that if

$$m \leq f(x) \leq M \quad , \quad \text{for } \ a \leq x \leq b$$

then

$$L(m) \leq L(f) \leq L(M) \ .$$

So we have the following properties.

Corollary 3 *If L is an increasing functional on $C[a,b]$ and p a convex function on $[a,b]$, then*

$$L(p(A(a,b))) \leq L(h_p) \leq L(m_p(a,b)) \ .$$

Corollary 4 *If L is an increasing functional on $C[a,b]$ and q a convex function on $[a,b]$, then*

$$L(q(A(a,b))) \leq L(k_q) \leq L(n_q(a,b)) \ .$$

Corollary 5 *If L is an increasing functional on $C[a,b]$ and p a strictly increasing convex function on $[a,b]$, then*

$$L(A(a,b)) \leq L(H_p) \leq L(M_p(a,b)) \ .$$

Corollary 6 *If L is an increasing functional on $C[a,b]$ and q is a strictly increasing convex function on $[a,b]$, then*

$$L(A(a,b)) \leq L(K_q) \leq L(N_q(a,b)) \ .$$

Example. Let f be a Riemann integrable positive function on $[a,b]$ and g be a strictly increasing continuous function on $[c,d]$. If for continuous functions $h:[a,b]\to[c,d]$ we define

$$L_g(h)=\frac{\int_a^b f(t)g(h(t))dt}{\int_a^b f(t)dt}$$

we get an increasing functional L_g. So Corollary 3 gives the following result. If p is a convex function on $[a,b]$, then

$$g(p(A(a,b)))\le L_g(h_p)\le g(m_p(a,b))\ . \tag{1}$$

If moreover p is supposed to be strictly increasing, replacing g by $g(p^{-1})$, we get the inequality

$$g(A(a,b))\le L_g(H_p)\le g(M_p(a,b))\ .$$

For $p(x)=x^r$ with $r>1$ we get the Theorem 2 from [3]. If we take in (1) $p(x)=w^r(x)$ and replace $g(x)$ by $g(x^{\frac{1}{r}})$, we have Theorem 3 from [3]. Analogously, from Corollary 4, we deduce that if q is a convex function on $[a,b]$, then

$$g(q(A(a,b)))\le L_g(k_q)\le g(n_q(a,b))\ .$$

If q is also strictly increasing, replacing g by $g(q^{-1})$, we have

$$g(A(a,b))\le L_g(K_q)\le g(N_q(a,b))\ ,$$

which for $q(x)=x^r$, $r>1$, gives an improvement of Theorem 1 from [3] because we have renonced at the assumption of differentialbility of g.

Remark 1 *If the convexity and/or the increasing monotony is repleaced by concavity respectively by decreasing monotony, we get the same or the reversed inequalities.*

4 The result of Dragomir

First of all let us generalize the result from [1] for an isotonic linear functional L. That is, let L be a functional defined on $C[a,b]$ with the properties

$$L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)\ ,\ \forall\alpha,\beta\in\mathcal{R},\beta\in\mathcal{R},\forall f,g\in C[a,b]$$

and

$$L(f)\geq 0\quad ,\quad \forall f\in C[a,b]\ ,\ f\geq 0\ .$$

We make also the unessential assumption $L(1)=1$, where the first 1 is the constant function with the values 1. We have the following improvement of of Jessenś inequality (see [2]).

Theorem 3 *Il L is an isotonic linear functional on $C[a,b]$ and f is a diferentiable convex function on $[a,b]$, then*

$$L(f)\geq \sup\{f(t):[L(e)-t]f'(t)\}\geq f(L(e))$$

where $e(x)=x$, for x in $[a,b]$.

Proof. We use the same basic inequality as in [1]. The function f being differentiable convex, then

$$f(x)\geq f(t)+(x-t)f'(t)\quad ,\quad \forall x,t\in[a,b].$$

Of course L is increasing and if we apply it for the functions of variable x we get

$$L(f)\geq f(t)+[L(e)-t]f'(t)$$

which gives the desired result.

Corollary 7 *If L and f satisfy the above conditions and $t\leq L(e)$, $f'(t)\geq 0$ then $L(f)\geq f(t)$.*

We remark that for

$$L(f) = A(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

we have $L(e) = A(a,b)$. So if we denote by P_r the power mean defined by

$$P_r = P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}$$

for $r \neq 0$ and $P_0 = G$, it is known that for $r < 1$, $P_r < A$. Thus

$$f'(P_r(a,b)) \geq 0 \quad \Rightarrow \quad A(f) \geq f(P_r(a,b)).$$

For $r = 0$ and $r = -1$ we get Theorem B.

For this last case of $A(f)$, we give also another generalization of this theorem. It is based on the following result from [4].

Theorem C. *Let f be a differentiable convex function on $[a,b]$ and*

$$c = \inf\{x \in [a,b] : f'(x) \geq 0\} \ .$$

Then

$$A(f) = \frac{1}{b-a}\int_a^b f(x)dx \geq \max\{f(x) : x \in I_f\}$$

where

$$I_f = \left[\frac{a+b}{2} - \frac{(b-c)^2}{2(b-a)} \ , \ \frac{a+b}{2} + \frac{(c-a)^2}{2(b-a)}\right] \ .$$

We deduce the following consequence.

Theorem 4 *Let f be a differentiable convex function on $[a,b]$. If for $r < 1$,*

$$f'(b - \sqrt{2(b-a)(A-P_r)}) \geq 0$$

or for $r > 1$,

$$f'(a + \sqrt{2(b-a)(P_r - A)}) \geq 0$$

then

$$A(f) \geq f(P_r(a,b)) \ .$$

Proof. The conditions assure that $P_r \in I_f$ and so the result follows from Theorem C.

Corollary 8 *Let* f *be a differentiable convex function on* $[a,b]$. *Then*

$$f'(b - \sqrt{b-a}(\sqrt{b} - \sqrt{a})) \geq 0 \quad \Rightarrow \quad A(f) \geq f(G(a,b)).$$

and

$$f'\left(b - (b-a)\sqrt{\frac{b-a}{b+a}}\right) \geq 0 \quad \Rightarrow \quad A(f) \geq f(H(a,b)) \ .$$

Remark 2 *As*

$$b - \sqrt{b-a}(\sqrt{b} - \sqrt{a}) \geq \sqrt{ab}$$

and

$$b - (b-a)\sqrt{\frac{b-a}{b+a}} \geq \frac{2ab}{(a+b)},$$

the corollary 8 improves Theorem B.

References

[1] S.S. Dragomir, *On Hadamard's inequalities for convex functions*, Mathem. Balkan. 6(1992), 3, 215–222.

[2] A. Lupaş, *A generalization of Hadamard inequalities for convex functions* , Publ. Elektrotehn. Fak., 544–576(1976), 115–121.

[3] C.E.M. Pearce; J. Pecarić, *Interpolation between functions of means*, (to appear).

[4] I. Raşa, *Inequalities for a nonlinear functional*, General Mathem. 2(1994), 1, 33–34.

[5] H.J. Seiffert, *Werte zwischen der geometrischen und den arithmetischen Mittel zweier Zahlen*, Elem. Math. 42(1987), 105–107.

[6] Gh. Toader; S.S. Dragomir, *Refinements of Jessen's inequality*, Demonstration Mathem. 28(1995), 2, 329–334.

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