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The construction of some means

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Abstract

Some methods of construction of means are considered and certain relations between means are established.

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1 Introduction

A mean of two positive real numbers is defined as a function $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, which has the property:

$$(1) \qquad \min(a,b) \le M(a,b) \le \max(a,b) \quad , \quad \forall a,b > 0 \ .$$

As $M(a,a) = a$, we will suppose in what follows that $0 < a < b$.

The following "exponential mean" was defined in [12]:

$$E = E(a,b) = \frac{a \cdot e^a - b \cdot e^b}{e^a - e^b} - 1$$

This mean was further studied and generalized in [10] and [11]. In [2] appears also the mean:

$$(2) \qquad S = S(a,b) = (a^a \cdot b^b)^{\frac{1}{a+b}}.$$

which is called symmetric geometric mean. It was also studied in [5], [6], [7] and [9].

Another exponential mean which we have in view in the present paper is:

$$F = F(a,b) = \frac{a \cdot e^a + b \cdot e^b}{e^a + b^b}\,. \tag{3}$$

In fact, some methods of construction of similar means will be considered and certain relations among means will be established.

2 Some families of means

In [4] the following family of means is defined. Let $f, g, h : \mathbb{R}_+ \to \mathbb{R}_+$ be three positive functions, h being moreover strictly monotone. Then:

$$F_{f,g,h} = F_{f,g,h}(a,b) = h^{-1}\left(\frac{h(a)f(a) + h(b)g(b)}{f(a) + g(b)}\right)$$

defines a mean. If $f = g$ and $h(x) = x$, we shall denote it by F_f. In the special case when $f = exp$, we get the mean defined by (3). For $h = log$, $F_{f,g,h}$ becomes:

$$S_{f,g} = S_{f,g}(a,b) = [a^{f(a)} \cdot b^{g(b)}]^{\frac{1}{f(a)+g(b)}}.$$

Again $S_f = S_{f,f}$ and in the special case when $f(x) = x$, we get the mean S given by (2).

On the other hand, S remembers the "identic mean" I defined by:

$$I = I(a,b) = \frac{1}{e} \cdot \left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}.$$

We can look for a generalization of the form:

$$I_f = I_f(a,b) = k \cdot \left[\frac{a^{f(a)}}{b^{f(b)}}\right]^{\frac{1}{f(a)-f(b)}} \tag{4}$$

The relation (1) impose the conditions

$$(5) \qquad \frac{1}{k} \leq \left(\frac{a}{b}\right)^{\frac{f(b)}{f(a)-f(b)}} \quad \text{and} \quad \left(\frac{a}{b}\right)^{\frac{f(b)}{f(a)-f(b)}} \leq \frac{1}{k}\ .$$

Let us now suppose that f is a differentiable function on $\mathbb{R}_+$. When b tends to a, from the inequalities (5) we get:

$$exp\left(\frac{f(a)}{af'(a)}\right) = \frac{1}{k}.$$

This is a simple differential equation, having the general solution

$$(6) \qquad f(a) = c \cdot a^n$$

where $c > 0$ and $n = \dfrac{-1}{log\, k}$, i.e. $k = e^{-\frac{1}{n}}$.

Conversely, is f is given by (6) and $n > 0$, then the expression I_f from (4) will represent a mean for $k = e^{-\frac{1}{n}}$.

Indeed, the first inequality of (1) gives

$$e^{\frac{1}{n}} \leq \left(\frac{a}{b}\right)^{\frac{b^n}{a^n-b^n}}\ .$$

Denoting $\dfrac{a}{b} = x \in (0,1)$, after taking logarithms, this becomes

$$\frac{1}{x^n-1}\log\, x \geq \frac{1}{n}\ ,$$

which for $y = x^n \in (0,1)$ is a well-known inequality. Analogously, the second inequality from (1) may be written as

$$\frac{x^n}{x^n-1}\log\, x \leq \frac{1}{n}\ ,$$

which, with the notation $y = x^n$, also becomes a standard inequality.

Therefore, we have obtained the general identic mean

$$I_n = I_n(a,b) = \frac{1}{e^{\frac{1}{n}}} \left(\frac{a^{a^n}}{b^{b^n}} \right)^{\frac{1}{a^n - b^n}} .$$

Turning now to the mean E, we also look for functions f such that

$$E_f = E_f(a,b) = \frac{af(a) - bf(b)}{f(a) - f(b)} - k \tag{7}$$

define a mean (for certain well chosen k). Again from (1) we have:

$$\frac{(a-b)f(b)}{f(a)-f(b)} \geq k \quad \text{and} \quad \frac{(a-b)f(a)}{f(a)-f(b)} \leq k \ ,$$

so, if we suppose that f has a derivative, for b tending to a we deduce the equation

$$\frac{f(a)}{f'(a)} = k \ ,$$

giving

$$f(a) = c \cdot q^a (q > 0) \ . \tag{8}$$

Conversely, if E_f is given by (7) and f by (8), we get a mean E_q taking $k = \dfrac{1}{\log\, q}$:

$$E_q = E_q(a,b) = \frac{a \cdot q^a - b \cdot q^b}{q^a - q^b} - \frac{1}{\log\, q} .$$

This can be verified directly, or by using a method developed in [13]: If f is continuous and monotonic, and g is positive, then the expression:

$$V_{f,g} = V_{f,g}(a,b) = f^{-1} \left(\frac{\int\limits_a^b f(x)g(x)dx}{\int\limits_a^b g(x)dx} \right)$$

defines a mean. For $f(x) = x$ and $g(x) = q^x$, we reobtain the mean E_q.

3 Some relations between means

The mean S can be also defined otherwise. For a given function $p : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$ let us consider its pair q, given by $q(a,b) = 1 - p(a,b)$. It is easy to see that

$$G_p = G_p(a,b) = a^{p(a,b)} b^{q(a,b)}$$

and

$$A_p = A_p(a,b) = a \cdot p(a,b) + b \cdot q(a,b)$$

define means, of geometric respectively arithmetic type. So

$$G_p \leq A_p \ .$$

For $p(a,b) = \dfrac{a}{a+b}$ we get $G_p = S$ and

$$A_p = C = C(a,b) = \frac{a^2 + b^2}{a+b}$$

the contraharmonic mean. And we have $S \leq C$, which was proved in [5] and [7]. Also for $p(x) = \dfrac{e^a}{e^a + e^b}$ we get $A_p = F$ and so we can obtain a similar inequality for F.

But we can deduce also some relations between F and S. As $f(x) = \dfrac{e^x}{x}$ is increasing on $[1, \infty)$, for $1 \leq a < b$ we have

$$\frac{e^b}{b} > \frac{e^a}{a}$$

which gives

$$C(a,b) < F(a,b) \quad , \text{ for } \ 1 \leq a < b$$

and also

$$S(a,b) < F(a,b) \quad , \text{ for } 1 \le a < b.$$

Similarly, f is decreasing on $(0,1)$, so for $0 < a < b < 1$ we have the previous inequalities reversed.

In [1] the so–called Gini means are used:

$$P_r(a,b) = exp\left(\frac{a^r \log a + b^r \log b}{a^r + b^r}\right).$$

They are related to the mean F by:

$$\log P_r(a,b) = \frac{1}{r} F(r \cdot \log a, r \cdot \log b).$$

Here we use also another sort of relations. In [13] it is remarked that if M is a mean and f is a bijection, we can define another mean by:

$$M(f) = M(f)(a,b) = f^{-1}(M(f(a), f(b))).$$

With this notation, we have

$$I_n(\exp) = E_{\exp(n)} \quad \text{or} \quad E_q = I_{\log q}(\exp)$$

which, for $n = 1$, or $q = e$, become

$$E = I(\exp) \tag{9}$$

that was given in [10]. It is also easy to see that

$$F = S(\exp). \tag{10}$$

Let us introduce two new means on the same way. Denote as usual (see [7]) by A, G and L the arithmetic, geometric respectively logarithmic mean (of the numbers a, b). Then define the means Δ and Λ by:

$$\Delta = A(\exp)\ , \tag{11}$$

respectively

$$\Lambda = L(\exp). \tag{12}$$

We have the following results:

Theorem 1 *One has the inequalities:*

$$3\Lambda > 2A + \Delta \tag{13}$$

and

$$3E > 2\Delta + A. \tag{14}$$

Proof. In [8] it is proved that $I^3 > A^2G$ while from [3] it is known that $L^3 > G^2A$. Replacing a by e^a and b by e^b, after logarithmating we get (13) and (14).

Theorem 2 *The following identity is valid:*

$$F - E = 1 - \exp[2A - (\Delta + \Lambda)] \ . \tag{15}$$

Proof. In [5] it is proved that

$$\log \frac{S}{I} = 1 - \frac{G^2}{A \cdot L} \ .$$

Replacing a by e^a, b by e^b, and using the relations (9) and (10), we get (15).

Remark 1 *In [10] the following inequalities are proved:*

$$A + \Delta - \Lambda < E < 2\Lambda - A \ .$$

In a similar way, we can prove:

Theorem 3 *The following inequalities hold true:*

$$2\Delta - E < F < 4\Delta - 3E \ . \tag{16}$$

Proof. *It is known (see [6]) that:*

$$\frac{A^2}{I} < S < \frac{A^4}{I^3} \ .$$

Taking into account (10) and (11), we get (16).

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