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Remarks on the gamma and beta distributions

Ioan Pienaru

Abstract

The following problem may be formulated: *by using only the frequency and the mean to find mean deviation.* In the last years many authors have obtained the exact mean deviation of some discrete distributions. A good source of references is A.R. Kamat's report (1963). Here we solve the above problem in the case of the following continuous distributions: Gamma, χ^2 and Beta. As an application, the exact probability of a certain event is given in a convenient form.

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1 Introduction

The notation and symbols used in this paper are as follows:

$$(a)_0 \equiv 1 \quad , \quad (a)_n \equiv a(a+1)\dots(a+n-1) \quad , \quad n \geq 1,$$

$F(a,b;c;z) \equiv_2 F_1(a,b;c;z)$ is the hypergeometric series of variable z with parameters a, b, c, that is, if n is a non–negative integer

$$F(a,b;c;z) =$$

$$= \begin{cases} \sum\limits_{k=0}^{\infty} \dfrac{(a)_k (b)_k}{(c)_k} \cdot \dfrac{z^k}{k!} & , if \quad c \neq -n \\ z^{n+1} \sum\limits_{k=0}^{\infty} \dfrac{(a+n+1)_k (b+n+1)_k}{(n+2)_k} \cdot \dfrac{z^k}{k!} & , if \quad c = -n, \end{cases}$$

the series being absolutely convergent for $|z| < 1$.

Further let

$$\Phi(a,c;z) =$$
$$= \begin{cases} \dfrac{1}{\Gamma(c)} \sum\limits_{k=0}^{\infty} \dfrac{(a)_k}{(c)_k} \cdot \dfrac{z^k}{k!} & , \; if \;\; c \neq -n \\ \dfrac{(a)_{n+1}}{(n+1)!} z^{n+1} \sum\limits_{k=0}^{\infty} \dfrac{(a+n+1)_k}{(n+2)_k} \cdot \dfrac{z^k}{k!} & , \; if \;\; c = -n. \end{cases}$$

We note that the function $\Phi(a,c;z)$ is an entire function of both parameters as well as of the variable z.

A one–dimensional continuous random variable X is said to have a gamma distributions with the parameter p, $p > 0$, if its frequency function is

$$g_p(x) = \begin{cases} 0 & \text{for } \; x \in (-\infty, 0] \\ \dfrac{1}{\Gamma(p)} x^{p-1} e^{-x} & \text{for } \; x \in (0, +\infty) \; . \end{cases}$$

Likewise X has a beta distribution, in the interval $(0,1)$, with the positive parameters p, q if the frequency is defined by

$$b_{p,q}(x) = \begin{cases} 0 & \text{for } \; x \in (-\infty, 0] \\ \dfrac{x^{p-1}(1-x)^{q-1}}{B(p,q)} & \text{for } \; x \in (0,1) \\ 0 & \text{for } \; x \in [1, +\infty) \; . \end{cases}$$

The following lemmas are proved in the first volume of the "Bateman manuscript project" (E.M.O.T.– 1953) edited by Erdélyi & Magnus & Oberhettinger & Tricomi.

Lemma 1 *If* $\alpha \in (0, +\infty)$, $\beta \in (0, 1]$, *then*

$$\int_0^{\alpha} g_p(x)dx = \alpha^p \Phi(p, p+1; -\alpha) \tag{1}$$

$$\int_0^\beta b_{p,q}(x)dx = \frac{\beta^p}{pB(p,q)} F(p, 1-q; p+1; \beta) \tag{2}$$

Lemma 2 *For $z \in (-\infty, +\infty)$ and $x \in (-1, 1)$*

$$\Phi(a, c; z) = \Phi(a-1, c; z) + z\Phi(a, c+1; z) \tag{3}$$

$$\Phi(a, c; z) = e^z \Phi(c-a, c; -z) \tag{4}$$

$$xF(a, b-1; c; x) = cF(a-1, b; c; x) - (a-b)xF(a, b; c+1; x) \tag{5}$$

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x) \tag{6}$$

For the proof of the formulas (1)–(6) see E.M.O.T.– 1953, p.266, p.87, p.253, p.58 and the page 64.

2 The mean deviation

Theorem 1 *Let X be a continuous random variable which has a gamma distribution with the parameter $p > 0$. If X has the mean x_0, then*

$$E(|X - x_0|) = 2x_0 g_p(x_0)$$

Proof. Because $x_0 = p$, we have

$$\begin{aligned} E(|X - x_0|) &= \int_0^p (p-x) g_p(x) dx + \int_p^{+\infty} (x-p) g_p(x) dx = \\ &= 2\int_0^p (p-x) g_p(x) dx \ . \end{aligned}$$

Taking into account that $x g_p(x) = p g_{p+1}(x)$, we may write

$$\frac{1}{2p} E(|X - x_0|) = \int_0^p g_p(x) dx - \int_0^p g_{p+1}(x) dx$$

and from (1)

$$E(|X-x_0|) = 2\cdot p^{p+1}\{\Phi(p,p+1;-p) - p\Phi(p+1,p+2;-p)\}\ .$$

By using (3) with $z=-p$, $a=p+1$, $c=p+1$, we observe that

$$E(|X-x_0|) = 2\cdot p^{p+1}\Phi(p+1,p+1;-p)\ .$$

But (4) implies

$$\Phi(p+1,p+1;-p) = e^{-p}\Phi(0,p+1;p) = \frac{e^{-p}}{\Gamma(p+1)}\ .$$

In conclusion

$$E(|X-x_0|) = \frac{2p^p e^{-p}}{\Gamma(p)} = 2x_0 g_p(x_0)\ .$$

We may also consider a gamma–frequency function with two parameters, that is

$$g_{p,\theta}(x) = \begin{cases} 0 & ,\ x\in(-\infty,0] \\ \dfrac{\theta^p}{\Gamma(p)}x^{p-1}e^{-\theta x} & ,\ x\in(0,+\infty)\ . \end{cases}$$

Corollary 1 *If X is a random variable with the distribution $g_{p,\theta}$, $p>0$, $\theta>0$, then*

$$E(|X-x_0|) = \frac{2}{\theta}x_0 g_{p,\theta}(x_0)\ .$$

Proof. In this case $x_0 = \dfrac{p}{\theta}$; we have

$$E(|X-x_0|) = \frac{\theta^p}{\Gamma(p)}\int_0^{+\infty} x^{p-1}e^{-\theta x}\left|x-\frac{p}{\theta}\right|dx =$$

$$= \frac{1}{\theta\Gamma(p)}\int_0^{+\infty} t^{p-1}e^{-t}|t-p|dt =$$

$$= \frac{p^p e^{-p}}{\theta\Gamma(p)} = \frac{2}{\theta}x_0 g_{p,\theta}(x_0)\ .$$

Corollary 2 *Let X be a χ^2 variable with n degrees of freedom, that is the frequency of X is*

$$f_{\chi_n^2}(x) = \begin{cases} 0 & for\ \ x \in (-\infty, 0] \\ \frac{1}{2} g_{\frac{n}{2}}\left(\frac{x}{2}\right) & for\ \ x \in (0, +\infty). \end{cases}$$

Then

$$E(|X - x_0|) = 2n f_{\chi_n^2}(n) = 2x_0 f_{\chi_n^2}(x_0).$$

Theorem 2 *If a continuous random variable X with a beta distribution, with the positive exponents p, q, has the mean x_0 and the variance σ^2, then*

$$E(|X - x_0|) = 2\left(1 + \frac{1}{p+q}\right)\sigma^2 \cdot b_{p,q}(x_0) \ .$$

Proof. Taking into account that

$$x_0 = \frac{p}{p+q} \quad , \quad \sigma^2 = \frac{pq}{(p+q)^2(p+q+1)}$$

we must prove that

$$E(|X - x_0|) = \frac{2p^p q^q}{(p+q)^{p+q+1} B(p,q)} \ .$$

To prove this we use the equality

$$x b_{p,q}(x) = x_0 b_{p+1,q}(x)$$

which give us x_0

$$E(|X - x_0|) = 2\int_0^{x_0} (x_0 - x) b_{p,q}(x) dx =$$

$$= 2x_0 \left\{ \int_0^{x_0} b_{p,q}(x) dx - \int_0^{x_0} b_{p+1,q}(x) dx \right\} \ .$$

From (2) and (5) (with $a = p+1$, $b = 1-q$, $c = p+1$, $x = x_0$) it follows that

$$\begin{aligned} E(|X - x_0|) &= \frac{2x_0^{p+1}}{pB(p,q)} \Big\{ F(p, 1-q; p+1; x_0) - \\ &\quad - \frac{(p+q)}{p+1} x_0 F(p+1, 1-q; p+2; x_0) \Big\} = \\ &= \frac{2x_0^{p+1}}{pB(p,q)} F(p+1, -q; p+1; x_0) \ . \end{aligned}$$

Applying (6) we obtain

$$\begin{aligned} E(|X - x_0|) &= \frac{2x_0^{p+1}}{pB(p,q)} (1-x_0)^q F(0, p+q+1; p+1; x_0) = \\ &= \frac{2p^p q^q}{(p+q)^{p+q+1} B(p,q)} \end{aligned}$$

which completes the proof.

We note that the above equality is equivalent with

$$E(|X - x_0|) = \frac{2}{p+q} x_0 (1-x_0) b_{p,q}(x_0) \ .$$

3 Application

If f is a real function defined at the points $y_0 + kh$, $k = 0, 1, \ldots, n$ we set

$$\Delta_h^n f(y_0) \equiv \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(y_0 + kh),$$

for instance

$$\Delta_h^2 f(y_0) = f(y_0) - 2f(y_0 + h) + f(y_0 + 2h) \ .$$

Theorem 3 *If X is a random variable which has a gamma distribution with the parameter p, $p > 0$, then*

$$P(|X - x_0| < \alpha) = \begin{cases} -\Delta_\alpha^2 G_p(x_0 - \alpha) & \textit{if } x_0 \geq \alpha \\ 2G_p(x_0) - G_p(x_0 + \alpha) & \textit{if } x_0 < \alpha \end{cases}$$

where $G_p(t) \equiv t g_p(t)$.

Proof. Let

$$D_p(t) \equiv \Phi(p, p+1; -t) - t\Phi(p+1, p+2; -t) \ .$$

If view of (3) and (4)

$$D_p(t) = \frac{1}{\Gamma(p+1)} e^{-t} \ .$$

With $v \equiv \max(p - \alpha, 0)$, we have

$$P(|X - x_0| < \alpha) = \int\limits_v^{p+\alpha} |x - p| g_p(x) dx =$$

$$2I(p, p) - I(v, p) - I(p + \alpha, p)$$

where

$$I(s, p) \equiv \int\limits_0^s (p - x) g_p(x) dx \ .$$

From (1) if follows that

$$\begin{aligned} I(s,p) &= p \left\{ \int\limits_0^s g_p(x)dx - \int\limits_0^s g_{p+1}(x)dx \right\} = \\ &= ps^p \left\{ \Phi(p, p+1; -s) - s\Phi(p+1, p+2; -s) \right\} = \\ &= ps^p D_p(s) = G_p(s) \ . \end{aligned}$$

Therefore

$$P(|X - x_0| < \alpha) = 2G_p(p) - G_p(v) - G_p(p + \alpha)$$

and because $p = x_0$ the theorem is proved.

Corollary 3 *If X is a χ^2 variable with n degrees of freedom, then*

$$P(|X - x_0| < \alpha) = \begin{cases} -2\Delta_\alpha^2 F(n-\alpha) & \text{if } n \geq \alpha \\ 4F(n) - 2F(n+\alpha) & \text{if } n < \alpha \end{cases}$$

where $F(t) \equiv t f_{\chi_n^2}(t)$.

Theorem 4 *If a continuous random variable X with a beta distribution $b_{p,q}, p > 0,\ q > 0,$ has the mean x_0, then if $0 < \alpha \leq \min(x_0, 1 - x_0)$*

$$P(|X - x_0| < \alpha) = -\frac{1}{p+q}\Delta_\alpha^2 B_{p,q}(x_0 - \alpha)$$

and for any $\alpha > 0$

$$P(|X - x_0| < \alpha) = \frac{1}{p+q}\left\{2B_{p,q}(x_0) - B_{p,q}(v_1) - B_{p,q}(v_2)\right\}$$

where

$$B_{p,q}(t) \equiv t(1-t)b_{p,q}(t),$$

$$v_1 = \max(x_0 - \alpha, 0),$$

$$v_2 = \min(x_0 + \alpha, 1)\ .$$

Proof. Let

$$D_{p,q}(\beta) \equiv F(p, 1-q; p+1; \beta) - \frac{\beta(p+q)}{p+1} F(p+1, 1-q; p+2; \beta).$$

From (5) and (6)

$$D_{p,q}(\beta) = (1-\beta)^q\ .$$

On the other hand

$$J_{p,q}(\beta) \equiv \int_0^\beta b_{p,q}(x)(x_0 - x)dx = \frac{x_0 \beta^p}{pB(p,q)} D_{p,q}(\beta),$$

that is

$$J_{p,q}(\beta) = \frac{\beta^p(1-\beta)^q}{(p+q)B(p,q)} = \frac{\beta(1-\beta)}{p+q} b_{p,q}(\beta) = \frac{1}{p+q} B_{p,q}(\beta).$$

Furthemore

$$\begin{aligned} P(|X - x_0| \ < \alpha) &= \int_{v_1}^{v_2} b_{p,q}(x)|x - x_0|dx = \\ &= 2J_{p,q}(x_0) - J_{p,q}(v_1) - J_{p,q}(v_2) = \\ &= \tfrac{1}{p+q} \left\{2B_{p,q}(x_0) - B_{p,q}(v_2)\right\} \end{aligned}$$

and the theorem is proved.

References

[1] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G., (1953), *Higher transcedental functions* (Bateman manuscript project), vol. I, New York, McGraw-Hill.

[2] Kamat A.R., (1963), *Incomplete and absolute moments of some discrete distributions*, Proccedings of the International Symposium on Classical and Contagious Discrete Distributions, (Edited by G.P. Patil), McGill University, Montreal, pp. 45–6

Academia Trupelor de Uscat
2400 – Sibiu
Romania