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A sufficient condition for univalency

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Abstract

In this note we obtain a sufficient condition for the analyticity and the univalence of the function defined by an integral operator which generalizes the result of paper [1].

Also we will give an example in which the univalece of function can be established according to this new criterion.

AMS Subject Classification

1 Introduction

We denote by U_r the disc of z–plane, $U_r = \{z \in \mathbb{C} : |z| < r\}$, where $r \in (0, 1]$, $U_1 = U$, $U^* = U \setminus \{0\}$ and $I = [0, \infty)$.

Let A be the class of functions f which are analytic in U with $f(0) = 0$ and $f'(0) = 1$.

Theorem A. ([1]) *Let $g \in A$ and let α be a complex number, $|\alpha - 1| < 1$. If*

$$(1) \qquad (1 - |z|^2) \left|(\alpha - 1)\frac{zg'(z)}{g(z)}\right| \leq 1 \quad , \quad (\forall) z \in U,$$

then the function G,

$$(2) \qquad G(z) = \left(\alpha \int\limits_0^z g^{\alpha-1}(u)du\right)^{\frac{1}{\alpha}}$$

is analytic and univalent in U.

Theorem B. ([2]) *Let $f \in A$. If there exists an analytic function p with positive real part in U, $p(0) = 1$, such that the inequality*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^2 - (1-|z|^2)\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1 \tag{3}$$

is true for all $z \in U$, then the function f is univalent in U.

2 Preliminaries

Definition 1 . *Let f and g be two analytic functions in U. We say that f is subordinate to g, written $f \prec g$, if there exists a function φ analytic in U which satisfies $\varphi(0) = 0$, $|\varphi(z)| < 1$ and*

$$f(z) = g(\varphi(z)) \quad , \quad (\forall) z \in U \ .$$

Definition 2 *A function $L : U \times I \to \mathbb{C}$ is called a Loewner chain if*

$$L(z,t) = e^t z + a_2(t)z^2 + \ldots \quad , \quad (\forall) z \in U \tag{4}$$

is analytic and univalent in U for each $t \in I$ and if

$$L(z,s) \prec L(z,t) \quad , \quad 0 \le s \le t < \infty \ ,$$

where by $\prec$ we denote the relation of subordination.

Theorem D. ([3]) *Let r be real number, $r \in (0,1]$. Let*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots + a_1(t) \neq 0,$$

be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r.

For almost all $t \in I$ suppose

$$z\frac{\partial L(z,t)}{\partial z} = P(z,t)\frac{\partial L(z,t)}{\partial t} \quad , \quad (\forall) z \in U_r \tag{5}$$

where $P(z,t)$ is analytic in U and satisfies $Re\, P(z,t) > 0$, $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, $L(z,t)$ has an analytic and univalent extension to the whole disc U.

3 Main results

Theorem 1 *Let $g \in A$ and let α be a complex number, $|\alpha - 1| < 1$. If there exists an analytic function p with positive real part in U, $p(0) = 1$, such that*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^2 - (1-|z|^2)\left[(\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)+1}\right]\right| \leq 1, \tag{6}$$

for all $z \in U$, ten the function G,

$$g(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)du\right)^{\frac{1}{\alpha}} \tag{7}$$

is analytic and univalent in U.

Proof. Let p be an analytic function in U, $Re\, p(z) > 0$, for all $z \in U$, $p(0) = 1$.

Let us consider the function h,

$$h(z) = \frac{2}{p(z)+1}. \tag{8}$$

It results that the function h is also analytic in U, $h(z) \neq 0$ for all $z \in U$, $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$.

Let's prove that there exists a real number r_0, $r_0 \in (0,1]$ such that the function $L : U_r \times I \to \mathbb{C}$, defined formally by

$$(9) \quad L(z,t) = \left[\alpha \int_0^{e^{-t}z} g^{\alpha-1}(u)du + \alpha(e^t - e^{-t})z\frac{g^{\alpha-1}(e^{-t}z)}{h(e^{-t}z)}\right]^{\frac{1}{\alpha}}$$

is analytic in U_{r_0} for all $t \in I$.

Since $g \in A$, the function f, $f(z) = \dfrac{g(z)}{z}$ is analytic in U and $f(0) = 1$. Then there is a disc U_r, $0 < r \leq 1$, in which $f(z) \neq 0$ for any $z \in U_r$ and we can choose the uniform branch of $(f(z))^{\alpha-1}$, equal to 1 the origin, analytic in U_r,

$$(10) \qquad f_1(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1} = 1 + a_1 z + \ldots + a_n z^n + \ldots,$$

$(\forall) z \in U_r$,

Let us denote by

$$(11) \qquad f_2(z,t) = \alpha \int_0^{e^{-t}z} u^{\alpha-1} f_1(u)du.$$

We obtain

$$(12) \qquad f_2(z,t) = z^\alpha f_3(z,t),$$

where

$$(13) \qquad f_3(z,t) = e^{-\alpha t} + \sum_{n=1}^{\infty} \frac{a_n \alpha}{\alpha + n} e^{-t(n+\alpha)} z^n.$$

The function f_3 is analytic in U_r for all $t \in I$, since

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{a_n \alpha}{\alpha + n} e^{-t(n+\alpha)}\right|} = e^{-t} \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}.$$

It is clear that, if $z \in U_r$, then $e^{-t}z \in U_r$ for all $t \in I$ and from the analyticity of h in U_r, it follows that the function f_4 is also analytic in U_r, where

$$f_4(z,t) = f_3(z,t) + \alpha(e^t - e^{-t})e^{-t(\alpha-1)}\frac{f_1(e^{-t}z)}{h(e^{-t}z)}. \tag{14}$$

We have

$$f_4(0,t) = e^{-\alpha t}(1 - \alpha + \alpha e^{2t}) \neq 0$$

for any $t \in I$ and it results that there exists $r_0 \in (0, r]$ such that $f_4(z,t) \neq 0$ in U_{r_0} for all $t \in I$. Then we can choose an uniform branch of $[f_4(z,t)]^{\frac{1}{\alpha}}$ analytic in U_{r_0}, denoted by $f_5(s,t)$, which is equal to $\gamma = e^{-t}(1 - \alpha + \alpha e^{2t})^{\frac{1}{\alpha}}$ at the origin, and for γ we fix a determination

$$f_5(z,t) = [f_4(s,t)]^{\frac{1}{\alpha}} \tag{15}$$

From (10)—(15) it results that the relation (9) may be wtitten as

$$L(z,t) = z \cdot f_5(z,t) \tag{16}$$

and hence we obtain that the function $L(z,t)$ is analytic in U_{r_0},

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots \quad , \quad (\forall)z \in U_{r_0} \ , \ (\forall)t \in I$$

where

$$a_1(t) = e^{t\frac{2-\alpha}{\alpha}}\left[(1-\alpha)e^{-2t} + \alpha\right]^{\frac{1}{\alpha}} \quad , \quad a_1(t) \neq 0$$

for all $t \in I$.

Since $|\alpha - 1| < 1$, which is equivalent with $Re\,\dfrac{2-\alpha}{\alpha} > 0$, it results that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

$L(z,t)$ is an analytic function in U_{r_0} for all $t \geq 0$ and then it follows that there exists a number r_1, $0 < r_1 < r_0$ and a positive constant $K = K(r_1)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K \quad , \quad (\forall) z \in U_{r_1} \ , \ t \geq 0.$$

Then, by Montel′s theorem, it results that $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_1}.

From (16) we have

$$\frac{\partial L(z,t)}{\partial t} = a\frac{\partial f_5(z,t)}{\partial t} \ . \tag{17}$$

It is clear that $\dfrac{\partial f_5(z,t)}{\partial t}$ is an analytic function in U_{r_1} and then $\dfrac{\partial L(z,t)}{\partial t}$ is also. Then, for all fixed numbers T and r_2, $T > 0$, $0 < r_2 < r_1$, there exists a constant $K_1 > 0$ (which depends on T and r_2) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1 \ , \ (\forall) z \in U_{r_2} \ , \ t \in [0,T].$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniformly with respect to U_{r_2}.

Since $\dfrac{\partial L(z,t)}{\partial t}$ is analytic in U_{r_2}, from (17) it results that there is a number r_3, $0 < r_3 < r_2$, such that

$$\frac{1}{z}\cdot\frac{\partial L(z,t)}{\partial t} \neq 0 \ , \ (\forall) z \in U_{r_3}$$

and then the function

$$P(z,t) = z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}$$

is analytic in U_{r3} for all $t \geq 0$.

In order to prove that the function $P(z,t)$ has an analytic extension with positive real part in U, for all $t \geq 0$ it is sufficient to prove that the function $w(z,t)$ defined in U_{r3} by

$$w(z,t) = \frac{P(z,t)-1}{P(z,t)+1}$$

can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$.

After computation we obtain

$$\begin{aligned} w(z,t) &= (h(e^{-t}s)-1)e^{-2t}+ \\ &+(1-e^{-2t})e^{-t}z\left[(\alpha-1)\frac{g'(e^{-t}z)}{g(e^{-t}z)} - \frac{h'(e^{-t}z)}{h(e^{-t}z)}\right]. \end{aligned} \tag{18}$$

In view of (8) we have

$$h(z)-1 = \frac{1-p(z)}{1+p(z)} \quad \text{and} \quad \frac{h'(z)}{h(z)} = -\frac{p'(z)}{p(z)+1}.$$

Therefore, the relation (18) becomes:

$$\begin{aligned} w(z,t) &= \frac{1-p(e^{-t}z)}{1+p(e^{-t}z)} \cdot e^{-2t}+ \\ &+(1-e^{-2t})e^{-t}z\left[(\alpha-1)\frac{g'(e^{-t}z)}{g(e^{-t}z)} + \frac{p'(e^{-t}z)}{p(e^{-t}z)+1}\right] \end{aligned} \tag{19}$$

From (6) and since $Re\, p(z) > 0$ for $z \in U$, we deduce that $g(z) \neq 0$ for all $z \in U^*$ and then the function $w(z,t)$ is analytic in the unit disc U.

For $z \neq 0$ and $t = 0$ we obtain

$$|w(z,0)| = \left|\frac{1-p(z)}{1+p(z)}\right| < 1 \tag{20}$$

for all $z \in U$ since $Re\, p(z) > 0$ in U. Also, we have

$$|w(0,t)| = |\alpha - 1|(1 - e^{-2t}) < 1. \tag{21}$$

Let now a fixed number $t > 0$, $s \in U$, $z \neq 0$. In this case the function $w(z,t)$ is analytic in $\bar{U}$ because $|e^{-t}z| \leq e^{-t} < 1$, for all $z \in \bar{U}$. Using the maximum principle, for all $z \in U$ and $t > 0$ we have

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|, \tag{22}$$

where $\theta = \theta(t)$ is a real number.

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and from (19) we obtain

$$|w(e^{i\theta},t)| =$$

$$= \left| \frac{p(u)-1}{p(u)+1}|u|^2 - (1-|u|^2)\left[(\alpha-1)\frac{ug'(u)}{g(u)} + \frac{up'(u)}{p(u)+1}\right] \right|.$$

Because $u \in U$, the relation (6) implies $|w(e^{i\theta},t)| \leq 1$ and from (20), (21) and (22) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$.

From Theorem D, it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disc U, for each $t \in I$. For $t = 0$ we conclude that the function

$$L(z,0) = \left(\alpha \int_0^z g^{\alpha-1}(u)du\right)^{\frac{1}{\alpha}} \equiv G(z)$$

is analytic and univalent in U.

Remark 1 *For $p(z) \equiv 1$ Theorem 1 becomes Theorem A.*

Theorem 2 *Let $f \in A$ and let α be a complex number, $|\alpha - 1| < 1$. If there exists an analytic function p with positive* *real part in U, $p(0) = 1$, such that*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^2 - -(1-|z|^2)\left[\frac{zf'(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1} + \alpha - 1\right]\right| \leq 1. \tag{23}$$

for all $z \in U$, then the function F_α,

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha} \tag{24}$$

is analytic and univalent in U.

Proof. From (23) we have $f'(z) \neq 0$, $(\forall) z \in U$. For $\alpha \in \mathbb{C}$, $\alpha \neq 1$ we can select an analytic branch in U of the function $[f'(z)]^{1/(\alpha-1)}$ equal to 1 at the origin.

Let us consider the function g, $g \in A$

$$g(z) = z[f'(z)]^{1/(\alpha-1)} \tag{25}$$

We have

$$(\alpha-1)\frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f'(z)} + (\alpha-1).$$

The condition (6) of Theorem 1 becomes (23). From Theorem 1 it results that the function

$$\left(\alpha \int_0^z g^{\alpha-1}(u)du\right)^{1/\alpha} = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha} \equiv F_\alpha(z)$$

is analytic and univalent in U.

For $\alpha = 1$, from (24) we have $F_1 = f$ and the relation (23) is identical with (3). From Theorem B it results that f is univalent in U.

We will give an example in which the function f does not satisfy the condition (1) of the Theorem A but it satisfies the conditions of the Theorem 1.

Example. Let α be a complex number, $|\alpha - 1| < 1$. Then the function

$$F_\alpha(z) = z\left[1 + \frac{\alpha}{4(\alpha+1)}z - \frac{2\alpha}{3(\alpha+2)}z^2\right]^{\frac{1}{\alpha}} \tag{26}$$

is analytic and univalent in U.

Proof. Let us consider the function f, $f \in A$,

$$f(z) = z + \frac{z^2}{8} - \frac{2z^3}{9}. \tag{27}$$

For the function g, $g(z) = z[f'(z)]^{1/(\alpha)}$, $\alpha \neq 1$ the condition (1) of the Theorem A becomes:

$$(1-|z|^2)\left|(\alpha-1)\frac{zg'(z)}{g(z)}\right| =$$

$$= (1-|z|^2)\left|\frac{z(3-16z)}{12+3z-8z^2} + \alpha - 1\right|$$

In the point $z_0 = -9/10$ and for $\alpha = 0,5$, the inequality (1) is not true

$$(1-|z_0|^2)\left|(\alpha-1)\frac{z_0 g'(z_0)}{g(z_0)}\right| = 1,15\ldots > 1$$

Let us consider the function

$$p(z) = \frac{12-3z+8z^2}{12+3z-8z^2}$$

It is easy to see that $Re\, p(z) > 0$ for all $z \in U$. For this choyce, from (6) we get

$$\left|\frac{8z^2 - 3z}{12}|z|^2 - (1 - |z|^2)(\alpha - 1)\right| < 1 \tag{28}$$

and in view of assertion $|\alpha - 1| < 1$, the relation (28) is true. It follows that the function F_α defined by (26) is analytic and univalent in U.

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