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On Newton Interpolation Formula

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Abstract

In the book entitled "Methodus differentialis" published in 1711, Newton indicates in the cases $m \in \{5, 6\}$, without proofs, new formulas for his interpolation polynomial N_m introduced in 1687. Arnold Kowalewski [1] has proved these formulas only in these particular cases. Slight modifications of these proofs are published in 1932 by Gerhard Kowalewski [2]. At the same time, in [2] it is shown that the new formulas are more efficient in the computation than the original formula of Isaac Newton.

The aim of this paper is to establish the interpolation formulas indicated in particular cases by A. and G. Kowalewski brothers, for an arbitrary m.

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We essentially use the fact that

$$N_m(x_0, x_1, \ldots, x_m; f|x) = N_m(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(m)}; f|x)$$

where

$$N_m(x_0, x_1, \ldots, x_m; f|x) =$$
$$= \sum_{k=0}^{m} (x - x_0)(x - x_1)\ldots(x - x_{k-1})[x_0, x_1, \ldots, x_k; f]$$

is the Newton interpolation polynomial and

$$\sigma = \begin{pmatrix} 0 & 1 & \ldots & m \\ \sigma(0) & \sigma(1) & \ldots & \sigma(m) \end{pmatrix}$$

is a permutation of $0, 1, \ldots, m$.

Theorem 1 *(The first formula of Kowalewski brothers). Let $f:[a,b]\to\mathbb{R}$ and $x_0,x_1,\dots,x_{2n}$ be a sistem of distinct points in $[a,b]$. If*

$$\Omega_{n,j}(x)=(x-x_{n-j+1})(x-x_{n-j+2})\dots(x-x_n)\dots(x-x_{n+j-1})$$

then

$$N_{2n}(x_0,x_1,\dots,x_{2n};f|x)=f(x_n)+ \tag{1}$$
$$+\sum_{j=1}^{n}[x_{n-j},\dots,x_{n+j};f]\left(x-\frac{x_{n-j}+x_{n+j}}{2}\right)\Omega_{n,j}(x)+$$
$$+\sum_{j=1}^{n}\frac{[x_{n-j},\dots,x_{n+j-1};f]+[x_{n-j+1},\dots,x_{n+j};f]}{2}\Omega_{n,j}(x).$$

Proof. Let us consider the following permutations σ and τ of the numbers $0,1,\dots,2n$:

$$\sigma(k)=\begin{cases} n+j \ , & k=2j \\ n-j \ , & k=2j-1\end{cases};$$
$$\tau(k)=\begin{cases} n-j \ , & k=2j \\ n+j \ , & k=2j-1\end{cases}.$$

We observe that $\sigma(0)=\tau(0)=n$ and $\sigma(k)+\tau(k)=2n$, $k=0,1,\dots,2n$, and it is clear that

$$\left\{x_{\sigma(0)},x_{\sigma(1)},\dots,x_{\sigma(2j)}\right\}=\left\{x_{\tau(0)},x_{\tau(1)},\dots,x_{\tau(2j)}\right\}=$$
$$=\left\{x_{n-j},x_{n-j+1},\dots,x_{n-1},x_n,x_{n+1},\dots,x_{n+j}\right\},$$
$$\left\{x_{\sigma(0)},\dots,x_{\sigma(2j-1)}\right\}= \tag{2}$$
$$=\left\{x_{n-j},x_{n-j+1},\dots,x_{n-1},x_n,x_{n+1},\dots,x_{n+j-1}\right\},$$
$$\left\{x_{\tau(0)},\dots,x_{\tau(2j-1)}\right\}=$$
$$=\left\{x_{n-j+1},x_{n-j+2},\dots,x_{n-1},x_n,x_{n+1},\dots,x_{n+j-1},x_{n+j}\right\}.$$

Let us denote

$$P_{1,j}(x)=\prod_{\nu=0}^{2j-1}(x-x_{\sigma(\nu)}) \quad , \quad P_{2,j}(x)=\prod_{\nu=0}^{2j-1}(x-x_{\tau(\nu)})$$

$$Q_{1,j}(x)=\prod_{\nu=0}^{2j-2}(x-x_{\sigma(\nu)}) \quad , \quad Q_{2,j}(x)=\prod_{\nu=0}^{2j-2}(x-x_{\tau(\nu)}).$$

From (2) it follows that

$$\begin{aligned} &\frac{P_{1,j}(x)+P_{2,j}(x)}{2}=\left(x-\frac{x_{n-j}+x_{n+j}}{2}\right)\prod_{\nu=n-j+1}^{n+j-1}(x-x_\nu) \\ &Q_{1,j}(x)=Q_{2,j}(x)=\prod_{\nu=n-j+1}^{n+j-1}(x-x_\nu). \end{aligned} \tag{3}$$

Next we have

$$\begin{aligned} &N_{2n}(x_0,x_1,\dots,x_{2n};f|x)= \\ &=N_{2n}(x_{\sigma(0)},x_{\sigma(1)},\dots,x_{\sigma(2n)};f|x)= \\ &=f(x_n)+\sum_{j=1}^{n}[x_{\sigma(0)},x_{\sigma(1)},\dots,x_{\sigma(2j)};f]P_{1,j}(x)+ \\ &+\sum_{j=1}^{n}[x_{\sigma(0)},x_{\sigma(1)},\dots,x_{\sigma(2j-1)};f]Q_{1,j}(x). \end{aligned}$$

It follows from (2) that

$$\begin{aligned} &N_{2n}(x_0,x_1,\dots,x_{2n};f|x)= \\ &=f(x_n)+\sum_{j=1}^{n}[x_{n-j},x_{n-j+1},\dots,x_{n+j};f]P_{1,j}(x)+ \\ &+\sum_{j=1}^{n}[x_{n-j},x_{n-j+1},\dots,x_n,\dots,x_{n+j-1};f]Q_{1,j}(x)\ . \end{aligned} \tag{4}$$

Analoguos to (4) we have

$$N_{2n}(x_0,x_1,\dots,x_{2n};f|x)=N_{2n}(x_{\tau(0)},x_{\tau(1)},\dots,x_{\tau(2n)};f|x)=$$

$$=f(x_n)+\sum_{j=1}^{n}[x_{\tau(0)},x_{\tau(1)},\dots,x_{\tau(2j)};f]P_{2,j}(x)+$$

$$+\sum_{j=1}^{n}[x_{\tau(0)},x_{\tau(1)},\ldots,x_{\tau(2j-1)};f]Q_{2,j}(x),$$

and further

(5)
$$N_{2n}(x_0,x_1,\ldots,x_{2n};f|x)=$$
$$=f(x_n)+\sum_{j=1}^{n}[x_{n-j},\ldots,x_n,\ldots,x_{n+j};f]P_{2,j}(x)+$$
$$+\sum_{j=1}^{n}[x_{n-j+1},x_{n-j+2},\ldots,x_{n+j};f]Q_{2,j}(x).$$

From (4) and (5) and taking into account (3) we find (1).

Theorem 2 *(The second formula of Kowalewski brothers). Let $f:[a,b]\to\mathbb{R}$ and $x_0,x_1,\ldots,x_{2n+1}$ be a sistem of distinct points from $[a,b]$. If*

$$\omega_{n,j}(x)=(x-x_{n-j+1})(x-x_{n-j+2})\ldots(x-x_{n+j})\ \ ,\ \ j=1,2,\ldots,n$$

and

$$\omega_{n,0}(x)=1$$

then

(6)
$$N_{2n+1}(x_0,x_1,\ldots,x_{2n+1};f|x)=$$
$$=\sum_{j=0}^{n}[x_{n-j},\ldots,x_{n+j+1};f]\left(x-\frac{x_{n-j}+x_{n+j+1}}{2}\right)\omega_{n,j}(x)+$$
$$+\sum_{j=0}^{n}\frac{[x_{n-j},\ldots,x_{n+j};f]+[x_{n-j+1},\ldots,x_{n+j+1};f]}{2}\omega_{n,j}(x).$$

Proof. If λ is a permutation of the numbers $0,1,\ldots,2n+1$, then

$$N_{2n+1}(x_0,x_1,\ldots,x_{2n+1};f|x)=$$
$$=\sum_{k=0}^{n}(x-x_{\lambda(0)})\ldots(x-x_{\lambda(2k-1)})[x_{\lambda(0)},x_{\lambda(1)},\ldots,x_{\lambda(2k)};f]+$$
$$+\sum_{k=0}^{n}(x-x_{\lambda(0)})\ldots(x-x_{\lambda(2k)})[x_{\lambda(0)},x_{\lambda(1)},\ldots,x_{\lambda(2k+1)};f].$$

Let σ_1 and τ_1 be the permutations defined by

$$\sigma_1(k) = \begin{cases} n+j+1 & , \ k = 2j \\ n-j & , \ k = 2j+1 \end{cases};$$

$$\tau_1(k) = \begin{cases} n-j & , \ k = 2j \\ n+j+1 & , \ k = 2j+1 \end{cases}.$$

For σ_1 and τ_1 we have

$$\{\sigma_1(0), \sigma_1(1), \ldots, \sigma_1(2k-1)\} = \{\tau_1(0), \tau_1(1), \ldots, \tau_1(2k-1)\} =$$

$$= \{n-k+1, n-k+2, \ldots, n, \ldots, n+k\},$$

$$\{\sigma_1(0), \ldots, \sigma_1(2k-1), \sigma_1(2k)\} =$$

$$= \{n-k+1, n-k+2, \ldots, n, \ldots, n+k, n+k+1\}$$

and

$$\{\tau_1(0), \ldots, \tau_1(2k-1), \tau_1(2k)\} = \{n-k, n-k+1, \ldots, n+k\}.$$

The following equalities are true

$$N_{2n+1}(x_0, x_1, \ldots, x_{2n+1}; f|x) = f(x_{n+1}) +$$

$$+ \sum_{k=1}^{n} (x - x_{n-k+1}) \ldots (x - x_{n-k})[x_{n-k+1}, \ldots, x_{n+k+1}; f] +$$

$$+ \sum_{k=0}^{n} (x - x_{n-k}) \ldots (x - x_{n+k+1})[x_{n-1}, \ldots, x_{n+k+1}; f]$$

and consequently

$$N_{2n+1}(x_0, x_1, \ldots, x_{2n+1}; f|x) = f(x_n) +$$

$$+ \sum_{k=1}^{n} (x - x_{n-k+1})(x - x_{n-k+2}) \ldots (x - x_{n+k})[x_{n-k}, \ldots, x_{n+k}; f] +$$

$$+ \sum_{k=0}^{n} (x - x_{n-k}) \ldots (x - x_{n+k})[x_{n-k}, \ldots, x_{n+k+1}; f].$$

The interpolation formulas (1) and (6) may be written in the following form that includes both of them. If $n=\left[\frac{m}{2}\right]$ then

$$(7)\qquad N_m(x_0,x_1,\ldots,x_m;f|x)=\frac{f(x_n)+f(x_{m-n})}{2}+$$

$$+\sum_{j=1}^{m-n}\Delta_1(f)\left(x-\frac{x_{m-n-j}+x_{n+j}}{2}\right)(x-x_{m-n-j+1})\ldots(x-x_{n+j-1})+$$

$$+\sum_{j=1}^{n}\frac{\Delta_2(f)+\Delta_3(f)}{2}(x-x_{n-j+1})\ldots(x-x_{m-n+j-1}).$$

where

$$\Delta_1(f)=[x_{m-n-j},\ldots,x_{n+j};f]$$

$$\Delta_2(f)=[x_{n-j},\ldots,x_{m-n+j-1;f}]$$

$$\Delta_3(f)=[x_{n-j+1},\ldots,x_{m-n+j};f]\ .$$

References

[1] Kowalewski, A., *Newton, Cotes, Gauss, Jacobi: Vier grundgelegende Abhandlungen über Interpolation und genäherte Quadratur*, Leipzig, 1917.

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