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On a diophantine equation involving factorials

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Abstract

The aim of this note is to give an answer to a question posed by D. Acu in [1].

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In the second part of his paper [1] Acu considered the diophantine equation

$$x_1! + \ldots + x_k! = z! \quad , \quad k \geq 2. \tag{1}$$

He posed the question, whether there is $k \in \mathcal{N}$, such that (1) has a solution such that $x_1 < \ldots < x_k < z$ or not. In this note we want to show that such a solution never exists.

If $(x_1, \ldots, x_k, z) \in \mathcal{N}^{k+1}$ is a solution of (1) we will in the sequel always assume $x_1 \leq \ldots \leq x_k \leq z$.

Our first proposition is theorem 2 of [1] with minor additions.

Proposition 1 *a) If $(x_1, \ldots, x_k, z)$ is a solution of (1), then $x_k \leq k-1$ and $(x_k+1)! \leq z! \leq kx_k! \leq k!$ holds. Hence the number of solutions of (1) is finit.*

b) (1) always has, what we call the maximal solution:

$$x_1 = \ldots = x_k = k-1 \quad , \quad z = k.$$

c) *If $S = (x_1, \ldots, x_k, z)$ is a solution of (1) with $x_k = k - 1$, then S is the maximal solution.*

For the sake of completeness we repeat the proof of theorem 2 of [1]:

a) If $(x_1, \ldots, x_k, z)$ is a solution of (1) we have $x_k! < z!$ and hence $(x_k + 1)! \le z!$. On the other hand $z! \le kx_k!$. (Note, that we always assume $x_1 \le \ldots \le x_k$). Hence $(x_k + 1)! \le kx_k!$ and $x_k \le k - 1$. The rest of the inequalities is now obvious.

b) Is also obvious.

c) If $x_k = k - 1$ we have by a) $z = k$ and S must be the maximal solution.

Remark 1 *From c) follows, that $x_k \le k - 2$ for each solution, which is not the maximal solution. Hence there is no solution of (1) such that $x_1 < \ldots < x_k$.*

In the next proposition we want to give a sharper estimate for x_k in a non maximal solution of (1).

Proposition 2 *If $n = \left[\frac{k-1}{2}\right]$ (= greatest natural number $\le \frac{k-1}{2}$), and if $S = (x_1, \ldots, x_k, z)$ is a non maximal solution of (1), then $x_k \le k - n - 1$.*

Proof. If $k = 2$ or $k = 3$ there is only the maximal solution by [1]. If $k = 4$ we have $n = 1$ and the proposition is true by an example in [1]. Hence we assume $k \ge 5$. In this case we want to prove by induction on m the following statement: If $m \in \mathcal{N}$ satisfies $2m + 1 \le k$, then $x_k \le k - m - 1$.

For $m = 1$ this is true by preceeding remark. Assume now, that $x_k \leq k-m-1$ for some $m \in \mathcal{N}$ with $2m+1 \leq k$. If $2(m+1)+1 > k$ we are done. Hence we assume $2(m+1)+1 \leq k$. In this case we have $2m+1 \leq k$ and we know already $x_k \leq k-m-1$. We want to show $x_k \leq k-m-2$ by contradiction.

Assume $x_k = k-m-1$. Then we have by a) of proposition 1

$$(x_k+1)! = (k-m)! \leq z! \leq k(k-m-1)!.$$

Hence $z \geq k-m$. If $z \geq k-m+1$ would hold, we had

$$(k-m+1)! \leq k(k-m-1)!$$

and

$$(k-m+1)(k-m) \leq k.$$

But this is impossible since both factors on the left side are greater than $\dfrac{k}{2}$ and $k \geq 5$. Hence $z = k-m$.

The equation (1) now has the form

$$x_1! + \ldots + x_{k-1}! + (k-m-1)! = (k-m)! \; . \tag{2}$$

Since $k(k-m-1)! > (k-m)!$ we get $x_1 \leq k-m-2$. Now take $\ell \in \{1, \ldots, k-1\}$ such that $x_i \leq k-m-2$ for $1 \leq i \leq \ell$ and $x_{\ell+1} = \ldots = x_k = k-m-1$. Putting this into (2) we obtain

$$x_1! + \ldots + x_\ell! = (k-m-1)!(\ell-m) \; . \tag{3}$$

Since the left side of (3) is strictly positive we get $\ell \geq m+1$ since $x_i \leq k-m-2$ for $1 \leq i \leq \ell$ we get from (3)

$$\ell(k-m-2)! \geq (k-m-1)!(\ell-m)$$

and

$$m(k-m-1) \geq \ell(k-m-2) \; .$$

Putting $\ell = m+j$ for some $j \in \mathcal{N}$ we obtain

$$m \geq j(k-m-2) \; .$$

Since $k \geq 2m+3$ we have

$$m \geq j(m+1)$$

and hence $\ell = m + j < m + 1$. This is a contradiction and we have shown $x_k \leq k - (m+1) - 1$.

Remark 2 *If k is even, $k = 2\ell$, we have $x_k \leq \ell$ by Proposition 2. On the other hand*

$$x_1 = \ldots = x_\ell = \ell - 1 \quad , \quad x_{\ell+1} = \ldots = x_k = \ell \quad , \quad z = \ell + 1$$

is non maximal solution of (1).

Proposition 3 *a) If $k = 2\ell$, $\ell \geq 2$, then the solution mentioned in the above remark is the solution of (1) with $x_k = \ell$.*

b) If $k = 2\ell - 1$, $\ell \geq 5$, there is no solution of (1) such that $x_k = \ell - 1$, i.e. for any non maximal solution we have $x_k \leq \ell - 2$.

Proof. a) Assume $(x_1, \ldots, x_k, z)$ is a solution of (1) such that $x_k = \ell$. As in the preceeding proff we obtain $z = \ell + 1$. Take $m \in \{1, \ldots, k-1\}$ such that $x_i \leq \ell - 1$ for $1 \leq i \leq m$ and $x_{m+1} = \ldots = x_k = \ell$. As in the proof of proposition 2 we get $\ell \leq m$. Comparing with the solution in the preceeding remark one sees $\ell = m$ and $x_1 = \ldots = x_\ell = \ell - 1$.
b) If $k = 2\ell - 1$ we have

$$n = \left[\frac{k-1}{2}\right] = \ell - 1$$

and by proposition 2 $x_k \leq \ell - 1$. Assume $x_k = \ell - 1$. As in the preeceding proofs we obtain $z = \ell$ in this case. Since

$$x_1 = \ldots = x_k = \ell - 1$$

is no solution of (1) there is $m \in \{1, \ldots, k-1\}$ such that $x_i \leq \ell - 2$ for $1 \leq i \leq m$ and $x_{m+1} = \ldots = x_k = \ell - 1$. Putting this into equation (1) we get

$$x_1! + \ldots + x_m! = (\ell - 1)!(m + 1 - \ell) .$$

Hence $m \geq \ell$. Assume $m \geq \ell + 1$. Put $m = \ell + j$ for some $j \in \{1, \ldots, \ell - 1\}$. Then we obtain

$$x_1! + \ldots + x_{\ell+j}! = (\ell - 1)!(j + \ell) .$$

From $x_i \leq \ell - 2$ for $1 \leq i \leq \ell + j$ it follows

$$x_1! + \ldots + x_{\ell+j}! \leq (\ell - 2)!(j + \ell) .$$

Since $\ell \geq 5$ the right side of this inequality is strictly smaller that $(\ell - 1)!(j + 1)$ and we have a contradiction. Hence $\ell = m$.

Now the above equation has the form

$$x_1! + \ldots + x_\ell! = (\ell - 1)! .$$

We have to show that this equation has no solution. If $x_1 = 1$, $x_2 = \ldots = x_{\ell-2}$ it follows $x_1! + \ldots + x_\ell! > (\ell - 1)!$. Hence $x_1, x_2 \leq \ell - 3$. But then for $x_3 = \ldots = x_\ell = \ell - 2$ we have

$$x_1! + \ldots + x_\ell! \leq 2(\ell - 3)! + (\ell - 2)(\ell - 2)! < (\ell - 1)!$$

(Note $\ell > 5$), a contradiction. Hence there is no solution with $x_k = \ell - 1$.

Remark 3 *In the preceeding propositions we have shown that for $k \geq 8$ and any solution $(x_1, \ldots, x_k, z)$ of (1) the following holds:*
If k is even the x_j take at most $\frac{k}{2} - 1$ different values, and if k is odd the x_j take at most $\frac{k-3}{2}$ different values. So for a solution of (1) such that $x_1 \leq \ldots \leq x_k$ we must in fact at many places have "$=$" and not "$<$".

Finally we want to show by two easy examples that the x_j really can take $\frac{k-2}{2}$ respectivelly $\frac{k-3}{2}$ different values.

1. For $k = 8$ we have the solutions

   (a) $x_1 = \ldots = x_8 = 7$, $z = 8$ the maximal solution

   (b) $x_1 = \ldots = x_4 = 3$, $x_5 = \ldots = x_8 = 4$, $z = 5$ (cf. proposition 3)

   (c) $x_1 = \ldots = x_4 = 1$, $x_5 = 2$, $x_6 = x_7 = x_8 = 3$, $z = 4$

   (d) $x_1 = \ldots = x_6 = 2$, $x_7 = x_8 = 3$, $z = 4$.

   Hence there is a solution with 3 different values for the x_j.

2. For $k = 9$ we have the solutions

   (a) The maximal solution

   (b) $x_1 = x_2 = 1$, $x_3 = \ldots = x_7 = 2$, $x_8 = x_9 = 3$, $z = 4$

   (c) $x_1 = \ldots = x_6 = 1$, $x_2 = x_8 = x_9 = 3$, $z = 4$

   Hence we have a solution with 3 different values for the x_j.

These two examples are rather exceptional. In fact one can prove with the same methods we used here, that for k "big enough" the following holds: If $(x_1, \ldots, x_k, z)$ is a solution of (1) which is not the maximal one, and, in case k is even, is not the solution of Proposition 3a), then x_k is strictly smaller than $\frac{k-3}{2}$ respectively $\frac{k-2}{2}$.

References

[1] Acu D., *Some diophantine equations involving factorials*, General Mathematics 3(1994), no.1, 35–38.

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