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A general class of starlike integral operators defined on the class of functions with bounded argument rotation

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Abstract

Let $\mathcal{U}$ be the open complex unit disc and let $\mathcal{A}$ be the class of all functions f, analytic in $\mathcal{U}$ and normed with the conditions $f(0) = 0$, $f'(0) = 1$. Let $\mathcal{B}$ be the class of all analytic functions h, defined on $\mathcal{U}$ and with $h(0) = 1$. If γ is a real number, greated then -1 and $g \in \mathcal{A}$, $h \in \mathcal{B}$, we define the integral operator $I_{g,h} : \mathcal{A} \to \mathcal{A}$ by $F(z) = \mathcal{I}(f)(z)$, where:

$$F(z) = \frac{\gamma+1}{g^{\gamma}(z)} \int_0^z f(u)g^{\gamma-1}(u)h(u)du \quad , \quad z \in \mathcal{U} \ .$$

If $\mathcal{R} = \{f \in \mathcal{A} : Re\, f'(z) > 0, z \in \mathcal{U}\}$ is the class of functions with bounded rotation in $\mathcal{U}$ and $S^* = \left\{f \in \mathcal{A} : \dfrac{Re\, zf'(z)}{f(z)} > 0, z \in \mathcal{U}\right\}$ is the class of starlike functions in $\mathcal{U}$, we will obtain sufficient conditions on γ, g and h so that $\mathcal{I}_{g,h}(\mathcal{R}) \subset S^*$.

Keywords and phrases: bounded rotation, convexity of order α, starlike function, subordination.

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1 Introduction

Let $\mathcal{C}$ be the complex plane and $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$ the open unit disc in $\mathcal{C}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the subclasses of analytic functions f in $\mathcal{U}$ normed with the conditions $f(0) = 0$, $f'(0) = 1$ and, respectively $f(0) = 1$. Let now:

$$\mathcal{R} = \{f \in \mathcal{A} : Re\, f'(z) > 0, z \in \mathcal{U}\}$$

be the class of functions with bounded rotation. Let:

$$S^* = \left\{ f \in \mathcal{A} : Re\frac{zf'(z)}{f(z)} > 0, z \in \mathcal{U} \right\}$$

be the class of stalike functions in $\mathcal{U}$.

A clasical result of J. Krzys ([4]) states that $\mathcal{R}$ is not a subclass of S^*.

In [13] and [14] R. Singh and S. Singh showed that if $f \in \mathcal{R}$, then $F \in S^*$, where

$$F(z) = \int_0^z \frac{f(t)}{t} dt \ .$$

This result and another one, of P.T. Mocanu ([9]), who proved that the result of R. Singh and S. Singh remains true when F is replaced by G, where

$$G(z) = \frac{2}{z} \int_0^z f(t)dt \tag{1}$$

where generalized in [1] by considering the integral operator $I : \mathcal{A} \to \mathcal{A}$ defined by

$$I(f)(z) = \frac{\gamma + 1}{g^\gamma(z)} \int_0^z f(t) g^{\gamma-1}(t) g'(t) dt \quad , \quad f \in \mathcal{A} \, , \; z \in \mathcal{U}$$

where $\gamma > -1$, $g \in \mathcal{A}$ and $g(z)g'(z) \neq 0$ in $\mathcal{U}$. The result, due to H. Al-Amiri, Kit C. Chan, P.T. Mocanu and S. Suebert, states that, under suitable conditions on γ and g the following inclusion:

$$I(\mathcal{R}) \subset S^*$$

holds. This result generalize also a result of M. Iovanov and P.T. Mocanu in [10].

On the other hand, the author considers in [2] the integral operator $L : \mathcal{A} \to \mathcal{A}$ given by:

$$(2) \qquad L(f)(z) = \frac{\gamma+1}{g^{\gamma}(z)} \int_0^z \frac{f(t)g^{\gamma}(t)}{t} dt \quad , \quad f \in \mathcal{A} \, , \; z \in \mathcal{U}$$

and showed that, under suitable conditions on $\gamma > 1$ and on g, we have

$$L(\mathcal{R}) \subset S^*.$$

In this paper we will consider the functions $g \in \mathcal{A}$ and $h \in \mathcal{B}$ and the integral operator $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ defined by:

$$F(z) = \mathcal{I}(f)(z)$$

where:

$$(3) \quad F(z) = \frac{\gamma+1}{g^{\gamma}(z)} \int_0^z f(u) g^{\gamma-1}(u) h(u) du \, , \; f \in \mathcal{A} \, , \; z \in \mathcal{U}$$

With the suitable conditions on $\gamma > 1$ and on the functions $g \in \mathcal{A}$ and $h \in \mathcal{B}$ we will obtain the inclusion:

$$\mathcal{I}(\mathcal{R}) \subset S^*.$$

The main theorem is an extension of all the above-mentioned results.

2 Preliminaries

Definition 1 *Let f and g be analytic functions in $\mathcal{U}$ and g univalent. We say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if $f(0) = g(0)$ and $F(\mathcal{U}) \subset G(\mathcal{U})$.*

Lemma 1 [6] *Let the function $\psi : \mathcal{C}^2 \times \mathcal{U} \to \mathcal{C}$ satisfy to the condition:*

$$Re\, \psi(ix, y; z) \leq 0 \quad , \quad z \in \mathcal{U}$$

for all real x and $y \leq -\dfrac{(1+x^2)}{2}$

If the function p is analytic in $\mathcal{U}$, with $p(0) = 1$ and if

$$Re\, \psi[p(z), zp'(z); z] > 0 \quad , \quad z \in \mathcal{U}$$

then $Re\, p(z) > 0$ in $\mathcal{U}$.

More general forms of this lemma can be found in [10].

Lemma 2 [3] *If $f \in \mathcal{R}$, then:*

$$f(z) \prec q(z) = \frac{2}{z}\log(1+z) - 1$$

and

$$\left|\arg \frac{f(z)}{z}\right| < \phi_0 = \max\{|\arg q(z)| : |z| = 1\} = 0,911\ldots$$

This result is due to Mac Gregor and is also mentioned in [1].

Definition 2 *An analytic function f in $\mathcal{U}$ is said to be convex of order α, with $0 \leq \alpha < 1$, if, for all $z \in \mathcal{U}$, we have:*

$$Re\, \frac{zf'(z)}{f'(z)} + 1 > \alpha \ .$$

Definition 3 *An analytic function f in $\mathcal{U}$, with $f(0) = 0$ and $f'(0) \neq 0$ is called starlike of order β, with $0 \leq \beta < 1$, if, for all $z \in \mathcal{U}$, we have:*

$$Re\, \frac{zf'(z)}{f(z)} + 1 > \beta \ .$$

The next lemma, also mentioned in [1], establishes a result that follows from the fact that starlike functions are mapped by the

Libera transform defined in [3] into starlike functions of order α and:

$$-zg'(z) = \frac{2}{z}\int_0^z \frac{t}{(1+t^2)}dt \ ,$$

where

$$q(z) = \frac{2}{z}\log(1+z) - 1.$$

The result was first obtained by P.T. Mocanu, M.O. Reade and D. Ripeanu in [11].

Lemma 3 [1] *The function*

$$q(z) = \frac{2}{z}\log(1+z) - 1$$

is convex of order α, where:

$$\alpha = \frac{3 - 4\ln 2}{2(2\ln 2 - 1)} = 0,294351\dots \tag{4}$$

Lemma 4 [1] *Let h be a convex function of order α, with $h(0) = 0$. Let $B(z)$, $C(z)$ and $D(z)$ be analytic functions in $\mathcal{U}$ and:*

$$\begin{aligned} &Re\, B(z) \geq \\ &\geq \tfrac{1}{2\beta}\{|C(z) - 1| - Re\,[C(z) - 1]\} + \tfrac{2^{2(1-\alpha)}}{|h'(0)|}|D(z)| \end{aligned} \tag{5}$$

where

$$\beta = \beta(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2 - 2^{2(1-\alpha)}} & : \ \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2\ln 2} & : \ \alpha = \dfrac{1}{2} \end{cases} \tag{6}$$

If p is analytic in $\mathcal{U}$, with $p(0) = 0$, and satisfies

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z)$$

then $p(z) \prec h(z)$.

In the case $\alpha = 0$, the proof of this lemma may be found in [7] or in [8]. In the general case the proof is similar. In [1] the general form of this lemma is also used for proving the main results. The real number $\beta(\alpha)$ is the order of starlikeness of a function $f \in A$, convex of order α. This result was first conjectured by Mac Gregor in [5] and was then proved by Wilken and Feng in [15]. Another proof then that of Wilken and Feng can be found in [12].

3 Main result

Theorem 1 *Let* $\alpha = 0,294\ldots$ *be given by* (**??**) *and let* $\delta = 2\beta = 1,2468\ldots$ *where* $\beta = \beta(\alpha)$ *is given by* (6). *Let* $\omega = 1 + \delta 2^{2(1-\alpha)} = 4,3162\ldots$ *and* $a = \tan\phi_0 = 1,2906\ldots$ *where* ϕ_0 *is given in Lemma 2. Let* $g \in \mathcal{A}$ *with* $\dfrac{g(z)g'(z)}{z} \neq 0$ *in* $\mathcal{U}$, *let* $h \in \mathcal{B}$ *and* $\gamma > -1$. *If we denote:*

$$G(z) = (\gamma+1)\frac{zg'(z)}{g(z)} - \left(1 + \frac{zh'(z)}{h(z)}\right) \tag{7}$$

$$H(z) = \frac{zh'(z)}{h(z)} + (\gamma-1)\left(\frac{zg'(z)}{g(z)} - 1\right) \tag{8}$$

and if the following relations are satisfied for all $z \in \mathcal{U}$:

$$\begin{aligned} &Re\,\gamma\frac{zg'(z)}{g(z)}\left[\frac{zg''(z)}{g'(z)} - \frac{zh'(z)}{h(z)}\right] + \\ &+a\left|Im\,\gamma\frac{zg'(z)}{g(z)}\left[\frac{zg''(z)}{g'(z)} - \frac{zh'(z)}{h(z)}\right]\right| \leq 0 \end{aligned} \tag{9}$$

(where $a = \tan\phi_0 = 1,290685$ *with* ϕ_0 *given in Lemma 2)*

$$a|Re\,G(z)| + |Im\,G(z)| \leq \sqrt{3} \tag{10}$$

$$Re\,H(z) - \omega|H(z)| + \delta \geq 0 \tag{11}$$

then $\mathcal{I}_{g,h}(\mathcal{R}) \subset S^*$, where the integral operator $\mathcal{I}_{g,h}$ is defined by (3).

Proof. Let $f \in \mathcal{R}$ and $F = \mathcal{I}_{g,h}(f)$. Denote by

$$Q(z) = \frac{g(z)}{zh(z)} \qquad R(z) = \frac{g'(z)}{h(z)} \ .$$

From (3) we deduce after some calculations:

$$\frac{g(z)}{zh(z)} zF'(z) + \gamma \frac{g'(z)}{h(z)} F(z) = (\gamma + 1) f(z) \ . \tag{12}$$

Let $P(z) = \dfrac{F(z)}{z}$. Then

$$F(z) = zP(z) \quad \text{and} \quad F'(z) = P(z) + zP'(z).$$

Using this notations, the relation (12) becomes:

$$zP'(z)Q(z) + P(z)[Q(z) + \gamma R(z)] = (\gamma + 1)\frac{f(z)}{z} \ .$$

Let $K(z) = P(z)Q(z)$. Then we have

$$K'(z) = P'(z)Q(z) + P(z)Q'(z),$$

and, by

$$|Im\, K(z)| < aRe\, K(z) \ \text{ where } \ a = \tan \phi_0 = 1,290685 \ldots$$

Because

$$|Im\, G(z)K(z)| \le |Re\, G(z)|\ |Im\, K(z)| + |Re\, K(z)|\ |Im\, G(z)| <$$
$$Re\, K(z)[a|Re\, G(z)| + |Im\, G(z)|] \ ,$$

we deduce from (10):

$$|Im\, G(z)K(z)| < \sqrt{3} Re\, K(z) \qquad z \in \mathcal{U} \ . \tag{13}$$

Let now $p(z)=\dfrac{zF'(z)}{F(z)}$. From (13) we have that P does not vanish in $\mathcal{U}$ and it is obvious that $p(z)=1+\dfrac{zP'(z)}{P(z)}$. From (12) we have:

$$F(z)[p(z)Q(z)+\gamma R(z)]=(\gamma+1)f(z)\ . \tag{14}$$

Differentiating the relation (14) and taking into account that

$$\frac{zQ'(z)}{Q(z)}+\gamma\frac{R(z)}{Q(z)}=G(z)$$

and $K(z)=P(z)Q(z)$ we easily deduce:

$$K(z)\left[zp'(z)+p^2(z)+G(z)p(z)+\gamma\frac{zR'(z)}{Q(z)}\right]= \tag{15}$$

$$=(\gamma+1)f'(z)\ .$$

But $\gamma>-1$ and $f\in\mathcal{R}$ imply that the left–hand side of (15) has positive real part. In order to use Lemma 1 for proving that $Re\,p(z)>0$ in $\mathcal{U}$ we denote the left–hand side of (15) with $\psi[p(z),zp'(z);z]$. For $x\in\mathbb{R}$ and $y\le-\dfrac{(1+x^2)}{2}$ we have:

$$Re\,\psi(ix,y;z)=Re\,K(z)\left[y-x^2+ixG(z)+\gamma\frac{zR'(z)}{Q(z)}\right]=$$

$$=-x^2Re\,K(z)+yRe\,K(z)+xRe\,iK(z)G(z)+$$

$$+\gamma Re\,K(z)\frac{zR'(z)}{Q(z)}=-x^2Re\,K(z)+yRe\,K(z)$$

$$-x\,Im\,K(z)G(z)+\gamma Re\,K(z)\frac{zR'(z)}{Q(z)}$$

Because the relation (13) implies $Re\,K(z)>0$ in $\mathcal{U}$, we deduce:

$$Re\,\psi(ix,y;z)\le-x^2Re\,K(z)-\frac{1+x^2}{2}Re\,K(z)-$$

$$-x\, Im\, K(z)G(z) + \gamma Re\, K(z)\frac{zR'(z)}{Q(z)} =$$

$$= \frac{1}{2}\Big[3x^2 Re\, K(z) + 2x\, Im\, K(z)G(z) +$$

$$Re\, K(z)\left(1 - 2\gamma\frac{zR'(z)}{Q(z)}\right)\Big]$$

We have

$$\Delta = 4\left[(Im\, K(z)G(z))^2 - 3Re\, K(z) Re\, K(z)\left(1 - 2\gamma\frac{zR'(z)}{Q(z)}\right)\right]$$

From (13) we have:

$$\text{(16)} \quad \Delta \leq 4\left\{3[Re\, K(z)]^2 - 3[Re\, K(z)]^2\left(1 - 2\gamma\frac{zR'(z)}{Q(z)}\right)\right\} \leq$$

$$\leq 12 Re\, K(z) Re\, 2\gamma K(z)\frac{zR'(z)}{Q(z)}\ .$$

We know that $Re\, K(z) > 0$ in $\mathcal{U}$ and we will prove the following inequality:

$$\text{(17)} \qquad Re\, 2\gamma K(z)\frac{zR'(z)}{Q(z)} \leq 0\ , \quad \text{in } \mathcal{U}.$$

It is clear that inequality (17) is equivalent to:

$$\text{(18)} \qquad Re\, \gamma\frac{zR'(z)}{Q(z)} \leq \frac{Im\, K(z)}{Re\, K(z)} \cdot Im\, \gamma\frac{zR'(z)}{Q(z)}\ .$$

Because

$$\tan(arg\, K(z)) = \frac{Im\, K(z)}{Re\, K(z)}$$

and

$$|arg\, K(z)| < \phi_0 = 0.911622\ldots$$

we deduce immediately:

$$\left|\frac{Im\, K(z)}{Re\, K(z)}\right| < a = tan\, \phi_0 \ .$$

Let $m = Re\, \gamma \dfrac{zR'(z)}{Q(z)}$, $n = Im\, \gamma \dfrac{zR'(z)}{Q(z)}$ and $t = tan\,(arg\, K(z))$. It is obvious that $t \in (-a, a)$. We will find conditions on $m, n \in \mathbb{R}$ so that, with our notations, the relation (18) holds, this means:

$$\text{(19)} \qquad nt - m \geq 0 \quad \text{for all} \quad t \in (-a, a).$$

For $n > 0$ we have $nt - m \geq 0$ for all $t \geq \dfrac{m}{n}$.
For $n < 0$ we have $nt - m \geq 0$ for all $t \leq \dfrac{m}{n}$.
If $nt - m \geq 0$ for all $t \in (-a, a)$ we must have:

$$t \geq \tfrac{m}{n} \ \ , \text{ for all } \ t \in (-a, a) \ \text{ if } \ n > 0$$

$$t \leq \tfrac{m}{n} \ \ , \text{ for all } \ t \in (-a, a) \ \text{ if } \ n < 0$$

The first inequality holds if $\dfrac{m}{n} \leq -a$, this means $m \leq -na$ and $n > 0$.
The second inequality holds if $\dfrac{m}{n} \geq a$, this means $m \leq na$ and $n < 0$.
It is clear that the two above inequalities have the same following form for $n \neq 0 : m + |n|a \leq 0$, this means:

$$Re\, \gamma \frac{zR'(z)}{Q(z)} + a \left| Im\, \gamma \frac{zR'(z)}{Q(z)} \right| \leq 0 \ .$$

But the last inequality is equivalent to (9). For $n = 0$, from (9) we obtain again that (18) holds. It follows that $\Delta < 0$ and thus, $Re\, \psi(ix, y; z) < 0$ for all $x, y \in \mathbb{R}$, $y \leq -\dfrac{(1 + x^2)}{2}$. From

(15) and Lemma 2.1 we obtain that p has positive real part and, consequently, $F \in S^*$, which completes the proof.

Corollary 1 *If ω and δ are defined in Theorem 1 and if $g \in \mathcal{A}$, $\dfrac{g(z)g'(z)}{z} \neq 0$ in $\mathcal{U}$, $h \in \mathcal{B}$, then $\mathcal{I}(\mathcal{R}) \subset S^*$, where*

$$\mathcal{I}(f)(z) = \int_0^z \frac{f(t)h(t)}{g(t)} dt \quad , \quad f \in \mathcal{A} \ , \ z \in \mathcal{U} \ .$$

The proof of the corollary is immediate. It suffices to apply Theorem 3.1 for $\gamma = 0$ and to check all the required conditions.

4 Particular cases

Remark 1 *If we take $h(z) = g'(z)$ in Theorem 1, we obtain Theorem 1 in* [1].

Remark 2 *If we take $h(z) = \dfrac{g(z)}{z}$ in Theorem 1, we obtain Theorem 1 in* [2].

Remark 3 *If we take $h(z) = \dfrac{g(z)}{z}$ in Corollary 1, we obtain the result of R. Singh and S. Singh* ([13] and [14]) *mentioned in the introduction.*

Remark 4 *If we take $h(z) = 1$ in Corollary 1, we obtain the following result:*

Corollary 2 *If δ and ω are the numbers defined in Theorem 3.1 and if $g \in \mathcal{A}$, $\omega\delta \in \mathbb{R}$, $\delta > -1$, $z \in \mathcal{U}$ satisfies the condition:*

$$Re \frac{zg'(z)}{g(z)} + \omega \left| \frac{zg'(z)}{g(z)} - 1 \right| \leq \delta + 1$$

then $F \in S^*$ *whenever* $f \in \mathcal{R}$, *where:*

$$F(z) = \int_0^z \frac{f(t)}{g(t)} dt \ .$$

For $g(z) = z$ *in* $\mathcal{U}$, *we obtain from Corollary 1 the mentioned result of R. Singh and S. Singh.*

Remark 5 *If we take* $h(z) = 1$ *and* $\gamma = 1$ *in Theorem 3.1, we obtain the following result:*

Corollary 3 *If* a *is the real number defined in Theorem 3.1 and if* $g \in \mathcal{A}$, *with* $\dfrac{g(z)g'(z)}{z} \neq 0$ *in* $\mathcal{U}$ *satisfies the conditions:*

$$\begin{array}{l} a\left|2Re\,\frac{zg'(z)}{g(z)} - 1\right| + 2\left|Im\,\frac{zg'(z)}{g(z)} - 1\right| \leq \sqrt{3} \quad , \quad z \in \mathcal{U} \\ Re\,\frac{z^2 g''(z)}{g'(z)} + a\left|Im\,\frac{z^2 g'(z)}{g'(z)} - 1\right| \leq 0 \quad , \quad z \in \mathcal{U} \end{array}$$

then $I_g(\mathcal{R}) \subset S^*$, *where:*

$$I_g(f)(z) = \frac{2}{g(z)} \int_0^z f(t) dt \ .$$

For $g(z) = z$ in $\mathcal{U}$, we obtain from Corollary 2 the result of P.T. Mocanu ([9]) mentioned in the introduction.

Remark 6 *If we take* $h(z) = g'(z)$ *and* $\gamma = 1$ *in Theorem 1, we obtain the result of P.T. Mocanu and M. Iovanov in* [10].

References

[1] H. Al-Amiri, K. Chan, J. Gresser, P.T. Mocanu, S. Suebert, *Integral operators on certain subclasses of univalent functions*, Rev. Roum. Math. Purres Appl. (to appear).

[2] Eugen Drăghici, *Starlikeness of some integral operators defined on the class of bounded turning functions*, Rendiconti del Seminario Matematico di Messina, to appear.

[3] D.J. Hallenbech, T.H. Mac Gregor, *Linear problems and convexity techniques in geometric function theory*, Pitman Publ. 1984 N.Y.

[4] J. Krzys, *A conterexample concerning univalent functions*, Folia. Soc. Scient. Lubliniensis, 2(1962), pp. 57–58.

[5] T.H. Mac Gregor, *A subordination for convex functions of order α*, J. London Math. Soc. 9(1975), pp. 530–536.

[6] S.S. Miller, P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. Journal. 28 (1981), pp. 157–171.

[7] S.S. Miller and P.T. Mocanu, *The theory and applications of second–order differential subordinations*, Studia Univ. Babeş–Bolyai, Mathematica, XXXIV, 4, (Cluj, Romania), 1989, pp. 3–33.

[8] S.S. Miller, P.T. Mocanu, *Univalence of Gaussian and confluent hypergeometric functions*, Proc. Amer. Math. Soc., 110, 2(1990), pp. 333–342.

[9] P.T. Mocanu, *On starlikeness of Libera transform*, Mathematica (Cluj– Romania), 28(51), 1986, pp. 153–155.

[10] P.T. Mocanu, M. Iovanov, *The effect of certain integral operator on functions of bounded turning and starlike functions*, Seminar of Geometric Function Theory, Babeş–Bolyai Univ. Fac. of Math. Preprint No.5, Cluj–Napoca (Romania), (1986), pp.83–90.

[11] P.T. Mocanu, M.O. Reade, D. Ripeanu, *The order of starlikeness of a Libera integral operator*, Mathematica (Cluj–Romania), 19(42), 1(1977), pp. 67–73.

[12] P.T. Mocanu, D. Ripeanu, I. Şerb, *On the order of starlikeness of convex functions of order alpha*, L'Analise numérique et la théorie de l'approximation, 102(1981), pp. 195–199.

[13] R. Singh, S. Singh, *Starlikeness and convexity of certain integrals*, Ann. Univ. Curie–Sklodowska Sectio A, 16(1981), pp. 145–148.

[14] R. Singh, S. Singh, *Starlikeness of close–to–convex functions*, Indian J. Pure Appl. Math., 13(1982), pp. 190–194.

[15] D. Wilken, J. Feng, *A remark on convex and starlike functions with integral representation*, J. Math. Anal. Appl. 64(1978), pp. 458–466.

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