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On the inequality of Mathieu

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Abstract

In this paper we obtain a new generalization for Mathieu s inequality.
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Mathieu [6] conjectured in 1890 that the inequality

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} < \frac{1}{x^2} \tag{1}$$

where x is a real number, $x \neq 0$, is valid.

The inequality (1) is called Mathieu′s. I was proved only in 1952 by L. Berg [2].

E. Makai [5] gave a very elegant and elementary proof for (1) and obtained the following lower estimation:

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} > \frac{1}{x^2+\frac{1}{2}}\ . \tag{2}$$

Another improvements where given by D.S. Mitrinović în [7], Gh. Costovici in [3], P.H. Diananda in [4] and D. Acu in [1].

The main purpose of this paper is to give a new generalization for the inequalities (1) and (2).

Theorem 1 *Let $a_1, a_2, \ldots, a_n, \ldots$ be an arithmetic progression, $a_i > 0$, $i = 1, 2, \ldots$ and let x be real number $x \neq 0$. Then*

$$\frac{1}{a_1^2 + x^2 + \frac{r^2}{2} - a_1 r} < \tag{3}$$

$$< \sum_{n=1}^{\infty} \frac{2a_n r}{(a_n^2 + x^2)^2} < \frac{1}{a_1^2 + x^2 - a_1 r}$$

where r is the ration of progression.

Proof. We have

$$\frac{1}{\left(a_n - \frac{r}{2}\right)^2 + x^2 - \frac{r^2}{4}} - \frac{1}{\left(a_n + \frac{r}{2}\right)^2 + x^2 - \frac{r^2}{4}} =$$

$$= \frac{2a_n r}{(a_n^2 + x^2)^2 - (a_n r)^2} > \frac{2a_n r}{(a_n^2 + x^2)^2}\ .$$

Summing for $n = 1, 2, \ldots$, we obtain:

$$\sum_{n=1}^{\infty} \frac{2a_n r}{(a_n^2 + x^2)^2} < \frac{1}{\left(a_1 - \frac{r}{2}\right)^2 + x^2 - \frac{r^2}{4}} = \frac{1}{a_1^2 + x^2 - a_1 r}$$

that is the inequality (3) of the righr side.

On the other hand, we can write:

$$\frac{1}{\left(a_n - \frac{r}{2}\right)^2 + x^2 + \frac{r^2}{4}} - \frac{1}{\left(a_n + \frac{r}{2}\right)^2 + x^2 + \frac{r^2}{4}} =$$

$$= \frac{2a_n r}{(a_n^2 + x^2)^2 + x^2 r^2 + \frac{r^4}{4}} < \frac{2a_n r}{(a_n^2 + x^2)^2}\ .$$

Summing for $n = 1, 2, \ldots$, we obtain:

$$\sum_{n=1}^{\infty} \frac{2a_n r}{(a_n^2 + x^2)^2} > \frac{1}{\left(a_1 + \frac{r}{2}\right)^2 + x^2 - \frac{r^2}{4}} = \frac{1}{a_1^2 + x^2 + \frac{r^2}{2} - a_1 r}$$

that is the inequality (3) of the left side.

Remark. For $a_1 = 1$ and $r = 1$ the inequalities (3) give the inequalities (1) and (2).

References

[1] D. Acu, *On the inequality of Mathieu*, General Mathematics, Vol.4 1996, Nr.1–2, 11–14.

[2] L. Berg, *Uber eine Abschätrung von Mathieu*, Math. Nachr. 7(1952), 257–259.

[3] Gh. Costovici, *Some inequalities of Mathieu type*, Symposium septimum tiraspolensegeneralis topologiae et suae applicationum, Chişinău, MCMXCVI (1996), 82–84.

[4] P.H. Diananda, *On some inequalities related to Mathieu's*, Univ. Beograd Publ. Electrotehn. Fak., ser. mat. fiz., nr. 544–576 (1976), 77–80.

[5] E. Makai, *On the inequality of Mathieu*, Publ. Math. (Debrecen) 5(1957), 204–205.

[6] E. Mathieu, *Traité de physique mathematique*, Vol. VI–VII, part. 2 (Paris), 1890.

[7] D.S. Mitrinović (in cooperation with P.M. Vasić), *Analytic inequalities*, Springer–Verlag Berlin–Heidelberg–New York, 1970.

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