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Quasisymmetric and quasimöbius maps on locally convex spaces

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Abstract

We give the definitions of the quasisymmetric and quasimöbius maps, maps defined on a locally convex spaces and we generalise some results of Jussi Väisälä [1].

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Let E, F be two locally convex spaces. We denote by $\mathcal{A}$, $\mathcal{B}$ the families of continuous semi-norms on E and F respectivelly.

Definiton 1 *Let (x_1, x_2, x_3) be a triple of distinct points in E. For $\alpha \in \mathcal{A}$, the α–ration of (x_1, x_2, x_3) is the number $\rho_\alpha(x_1, x_2, x_3)$ defined by*

$$\rho_\alpha(x_1,x_2,x_3) = \begin{cases} \dfrac{|x_2 - x_1|_\alpha}{|x_3 - x_1|_\alpha} \, , & \text{if } |x_3 - x_1|_\alpha \neq 0 \\ 0 \, , & \text{if } |x_2 - x_1|_\alpha = |x_3 - x_1|_\alpha = 0 \\ \infty \, , & \text{if } |x_2 - x_1|_\alpha \neq 0 \, , \ |x_3 - x_1|_\alpha = 0 \end{cases} .$$

Definiton 2 *Let (x_1, x_2, x_3, x_4) be a quadruple of distinct points in E. For $\alpha \in \mathcal{A}$, the α–cross ratio of (x_1, x_2, x_3, x_4) is the number $\tau_\alpha(x_1, x_2, x_3, x_4)$ defined by*

$$\tau_\alpha(x_1, x_2, x_3, x_4) =$$

$$= \begin{cases} \dfrac{|x_1 - x_3|_\alpha}{|x_1 - x_4|_\alpha} \cdot \dfrac{|x_2 - x_4|_\alpha}{|x_2 - x_3|_\alpha} , & \text{if } |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \\ 0 , & \text{if } \begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha = 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha = 0 \end{cases} \\ \infty , & \text{if } \begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha \neq 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha = 0 \end{cases} \end{cases} .$$

Proposition 1 *Let $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$, and $\eta : [0, \infty] \longrightarrow [0, \infty]$ be a homeomorphism ($\eta(\infty) = \infty$). If $f : E \longrightarrow F$ verifies*

$$(1) \qquad |x' - x''|_{\varphi(\beta)} = 0 \Longrightarrow |f(x') - f(x'')|_\beta = 0$$

for any $x', x'' \in E$, then we have

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) = 0 \Longrightarrow \rho_\beta(f(x_1), f(x_2), f(x_3)) = 0$$

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) = \infty \Longrightarrow \rho_\beta(f(x_1), f(x_2), f(x_3)) \in \{0, \infty\}$$

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) \in \{0, \infty\} \Longrightarrow \rho_\beta(f(x_1), f(x_2), f(x_3)) \leq \eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3))$$

Proof. If

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) = 0$$

then

$$\rho_{\varphi(\beta)}(x_1, x_2, x_3) = 0 \ ,$$

and

$$\begin{cases} |x_2 - x_1|_{\varphi(\beta)} = 0 \\ |x_3 - x_1|_{\varphi(\beta)} \neq 0 \end{cases}$$

or

$$\begin{cases} |x_2 - x_1|_{\varphi(\beta)} = 0 \\ |x_3 - x_1|_{\varphi(\beta)} = 0 \ . \end{cases}$$

From (1), we have

$$|f(x_2) - f(x_1)|_\beta = 0 \quad , \quad |f(x_3) - f(x_1)|_\beta \geq 0$$

and

$$\rho_\beta(f(x_1), f(x_2), f(x_3)) = 0$$

If

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) = \infty$$

then

$$\rho_{\varphi(\beta)}(x_1, x_2, x_3) = \infty$$

and

$$\begin{cases} |x_2 - x_1|_{\varphi(\beta)} \neq 0 \\ |x_3 - x_1|_{\varphi(\beta)} = 0 \end{cases}$$

From (1), we have

$$\begin{cases} |f(x_2) - f(x_1)|_\beta \geq 0 \\ |f(x_3) - f(x_1)|_\beta = 0 \end{cases}$$

and

$$\rho_\beta(f(x_1), f(x_2), f(x_3)) \in \{0, \infty\} \ .$$

The last affirmation is a consequence of the previous implications.

Proposition 2 *Let $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ and $\eta : [0,\infty] \longrightarrow [0,\infty]$ be a homeomorphism ($\eta(\infty) = \infty$). If $f : E \longrightarrow F$ verifies*

$$|x' - x''|_{\varphi(\beta)} = 0 \Longrightarrow |f(x') - f(x'')|_\beta = 0$$

for any $x', x'' \in E$, then we have

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) = 0 \Longrightarrow \tau_\beta(f(x_1), f(x_2), f(x_3, f(x_4))) = 0$$

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) = \infty \Longrightarrow \tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) \in \{0, \infty\}$$

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) \in \{0, \infty\} \Longrightarrow \tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) \leq \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4))$$

Proof. We have the implications

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) = 0 \Longrightarrow \tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4) = 0 \Longrightarrow$$

$$\Longrightarrow \begin{cases} |x_1 - x_2|_{\varphi(\beta)} \cdot |x_3 - x_4|_{\varphi(\beta)} = 0 \\ |x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} \geq 0 \end{cases} \Longrightarrow$$

or

$$\Longrightarrow \begin{cases} |f(x_1)-f(x_2)|_\beta \cdot |f(x_3)-f(x_4)|_\beta = 0 \\ |f(x_1)-f(x_3)|_\beta \cdot |f(x_2)-f(x_4)|_\beta \geq 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \tau_\beta(f(x_1),f(x_2),f(x_3),f(x_4)) = 0 \ .$$

Also, if

$$\eta(\tau_{\varphi(\beta)}(x_1,x_2,x_3,x_4)) = \infty \ ,$$

then

$$\tau_{\varphi(\beta)}(x_1,x_2,x_3,x_4) = \infty \Longrightarrow$$

$$\Longrightarrow \begin{cases} |x_1-x_2|_{\varphi(\beta)} \cdot |x_3-x_4|_{\varphi(\beta)} \neq 0 \\ |x_1-x_3|_{\varphi(\beta)} \cdot |x_2-x_4|_{\varphi(\beta)} = 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} |f(x_1)-f(x_2)|_\beta \cdot |f(x_3)-f(x_4)|_\beta \geq 0 \\ |f(x_1)-f(x_3)|_\beta \cdot |f(x_2)-f(x_4)|_\beta = 0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \tau_\beta(f(x_1),f(x_2),f(x_3),f(x_4)) \in \{0,\infty\} \ .$$

The last affirmation is a consequence of the previous implications.

Definiton 3 *Let* $\varphi : \mathcal{B}_0 \longrightarrow \mathcal{A}_0$ *and* $\eta : [0,\infty] \longrightarrow [0,\infty]$ *be a homeomorphism* $(\eta(\infty)=\infty)$*. A map* $f : E_0 \longrightarrow F$*,* $E_0 \subset E$ *is* (η,φ,β)*–quasisymmetric* $((\eta,\varphi,\beta)-QS)$ *if*

(i) $|x_1-x_2|_{\varphi(\beta)} = 0 \Longleftrightarrow |f(x_1)-f(x_2)|_\beta = 0$

(ii) $\rho_\beta(f(x_1), f(x_2), f(x_3)) \leq \eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3))$

for any $x_1, x_2, x_3 \in E_0$.

Definiton 4 *Let* $\varphi : \mathcal{B}_0 \longrightarrow \mathcal{A}_0$ *and* $\eta : [0, \infty] \longrightarrow [0, \infty]$ *be a homeomorphism* ($\eta(\infty) = \infty$). *A map* $f : E_0 \longrightarrow F$, $E_0 \subset E$ *is* (η, φ, β)*–quasimöbius* ($(\eta, \varphi, \beta) - QM$) *if*

(i) $|x_1 - x_2|_{\varphi(\beta)} = 0 \Longleftrightarrow |f(x_1) - f(x_2)|_\beta = 0$

(ii) $\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) \leq \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4))$

for any $x_1, x_2, x_3, x_4 \in E_0$.

Proposition 3 *If* $T \in Isom\,(E, F)$*, then there exists* $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ *so that* T *is* $(\eta, \varphi, \beta) - QS$ *and* $(\eta, \varphi, \beta) - QM$ *for* $\eta : [0, \infty] \longrightarrow [0, \infty]$, $\eta(t) = t$.

Proof. Let $\beta \in \mathcal{B}$. The application $x \longrightarrow |Tx|_\beta$ is a continuous semi–norm on E. There exists $\alpha \in \mathcal{A}$ so that $|x|_\alpha = |Tx|_\beta$. We define $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$, $\varphi(\beta) = \alpha$.

If $x_1, x_2 \in E$

$$|T(x_1) - T_{x_2}|_\beta = |T(x_1 - x_2)|_\beta = |x_1 - x_2|_{\varphi(\beta)}$$

and (i) is satisfied.

Also, we have,

$$\rho_\beta(T(x_1), T(x_2), T(x_3)) = \rho_{\varphi(\beta)}(x_1, x_2, x_3)$$

$$\tau_\beta(T(x_1), T(x_2), T(x_3), T(x_4)) = \tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)$$

for any $x_1, x_2, x_3, x_4 \in E$ and the conditions (ii) from the two definitions are true for $\eta(t) = t$.

Proposition 4 *If $f : D \longrightarrow F$ is $(\eta, \varphi, \beta)-QM$ then $\eta(1) \geq 1$.*

Proof. Let $x_1 = x_2$. We have

$$\tau_\beta(f(x_1), f(x_1), f(x_3), f(x_4)) = 1 \ ,$$

$$\tau_{\varphi(\beta)}(x_1, x_1, x_3, x_4) = 1$$

and from the definition 4 (ii), results $\eta(1) \geq 1$.

Proposition 5 *If*

$$|x_1 - x_3|_\alpha \cdot |x_1 - x_4|_\alpha + |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \ ,$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \frac{1}{\tau_\alpha(x_1, x_2, x_3, x_4)} \ .$$

Proof. If

$$\begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha \neq 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \end{cases}$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \frac{|x_1 - x_3|_\alpha}{|x_1 - x_4|_\alpha} \cdot \frac{|x_2 - x_4|_\alpha}{|x_2 - x_3|_\alpha} =$$

$$= \left(\frac{|x_1 - x_4|_\alpha}{|x_1 - x_3|_\alpha} \cdot \frac{|x_2 - x_3|_\alpha}{|x_2 - x_4|_\alpha} \right)^{-1} = \frac{1}{\tau_\alpha(x_1, x_2, x_4, x_3)} \ .$$

If

$$\begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha = 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \end{cases}$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = 0$$

and

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \infty \ ,$$

and if

$$\begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha \neq 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha = 0 \end{cases}$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \infty$$

and

$$\tau_\alpha(x_1, x_2, x_4, x_3) = 0 \ ,$$

In the last two cases, we have also

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \frac{1}{\tau_\alpha(x_1, x_2, x_3, x_4)} \ .$$

Proposition 6 *If $f : D \longrightarrow F$ is $(\eta, \varphi, \beta)-QM$ then we have*

$$\frac{1}{\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4))} \leq \tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) \leq$$

$$\leq \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) \ ,$$

for any $x_1, x_2, x_3, x_4 \in D$ *with*

$$|x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} + |x_1 - x_4|_{\varphi(\beta)} \cdot |x_2 - x_3|_{\varphi(\beta)} \neq 0 \ .$$

Proof. Firstly, we have

$$|x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} + |x_1 - x_4|_{\varphi(\beta)} \cdot |x_2 - x_3|_{\varphi(\beta)} \neq 0 \Longleftrightarrow$$

$$|f(x_1)-f(x_3)|_\beta \cdot |f(x_2)-f(x_4)|_\beta + |f(x_1)-f(x_4)|_\beta \cdot |f(x_2)-f(x_3)|_\beta \neq 0$$

and using the previous proposition, we can write:

$$\frac{1}{\eta\left(\dfrac{1}{\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)}\right)} = \frac{1}{\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_4, x_3))} \leq$$

$$\leq \frac{1}{\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4))} = \tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)).$$

References

[1] , , .

[2] , , .

[3] , , .

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