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The fine spectre of some Cesàro generalized operators defined on $\ell^p(p>1)$

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Abstract

The aim of the paper is the study of the fine spectre for a class of Cesàro generalized operators, Rhaly operators, when those operators are defined on the spaces ℓ^p, $p>1$.

AMS Subject Classification

Let s and ℓ^p, $p>1$, be the linear spaces of sequences:

$$s=\{f=(f(n))_{n\in\mathbb{N}}:f(n)\in K\}$$

$$\ell^p=\left\{f=(f(n))_{n\in\mathbb{N}}:\sum_{n=0}^{\infty}|f(n)|^p<\infty\right\},$$

where $K=\mathbb{R}\vee\mathbb{C}$.

For $a=(a_n)\in s$ we define $R_a:s\to s$ by

$$(R_af)(n)=a_n\sum_{i=0}^{n}f(i)\quad,\quad n\in\mathbb{N},$$

for every $f=(f(n))_{n\in\mathbb{N}}\in s$.

The space s may be replaced with other spaces of sequences. In these case, Rhaly operator R_a determines and is determined by an infinite matrix, lower triangular, noted also with R_a:

$$R_a = \begin{bmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The dual of an Rhaly operator $R_a : \ell^p \to \ell^p$ is the operator $R_a^* : \ell^q \to \ell^q$, to which is associated by the infinite matrix, triangular:

$$R_a^* = \begin{bmatrix} \overline{a_0} & \overline{a_1} & \overline{a_2} & \dots \\ 0 & \overline{a_1} & \overline{a_2} & \dots \\ 0 & 0 & \overline{a_2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $a = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}} \in s$ one obtains the discrete Cesàro operator and for $a = \left(\frac{1}{(n+1)^z}\right)_n \in \mathbb{N}$, with $z \in \mathbb{C}$, one obtains the z–Cesàro operator.

G. Leibowitz [1], [2], studies the continuity, compactness of these operators defined on the space ℓ^p, $p > 1$. M.C.Jr. Rhaly [5] studies these Rhaly operators which as the discrete Cesàro operator C_1, are not compact, in ℓ^2. The spectra $\sigma_p(R_a, \ell^2)$ and $\sigma(R_a, \ell^2)$ are investigated.

In this paper $\mathcal{I}$ present some results concerning the spectra of Rhaly operators from $(\ell^p, \ell^p)^*$, where:

$\varphi(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is bijective and $(\lambda I - R_a)^{-1}$ is continuous$\}$;

$\sigma(R_a, \ell^p) = K \setminus \varphi(R_a, \ell^p)$;

$\sigma_p(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a \text{ is not injective}\}$;

$\sigma_c(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is injective, is not surjective and $\overline{(\lambda I - R_a)(\ell^p)} = \ell^p\}$;

$\sigma_r(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is injective, and $\overline{(\lambda I - R_a)(\ell^p)} \neq \ell^p\}$.

A Rhaly operator $R_a : \ell^p \to \ell^p$ $(p > 1)$ is correctly defined if the sequence $((n+1)a_n)$ is bounded, and R_a is continuous.

Let $S = \{a_n : n \in \mathbb{N}\}$.

Lemma 1 *Let R_a be a Rhaly matrix $(a \in s)$, $C = \lambda I - R_a$, such that $c_{jj} \neq 0$ $\forall j \in \mathbb{N}$. Then C^{-1} has the entries:*

$$(1)\qquad \begin{aligned} c_{jj} &= \frac{1}{\lambda - a_j} \quad , \quad j \in \mathbb{N}; \\ c_{ij} &= 0 \quad , \quad i, j \in \mathbb{N}\ ,\ i < j \\ c_{ij} &= c_{j+r,j} = \lambda^{r-1} a_{j+r} \left[\prod_{k=j}^{j+r} (\lambda - a_k)\right]^{-1}, \\ &\qquad r \in \mathbb{N}^*,\ i, j \in \mathbb{N},\ i = j + r. \end{aligned}$$

Proof. The method of the mathematical induction is used.

For the calculation of the elements c_{jj} one uses determinants of some lower triangular matrices with the entries on the diagonal $\lambda - a_k$, $k \neq j$. Obvious $c_{00} = \dfrac{1}{\lambda - a_0}$.

Supposing that, for an $i \in \mathbb{N}^*$, $c_{i0}, c_{i1}, \ldots, c_{i,i-1}$ have the given form, it can be proved the entries $c_{i+1,0}, c_{i+1,1}, \ldots, c_{i+1,i}$ are also given by the above relations.

The condition $c_{i+1,i}(\lambda - a_i) + c_{i+1,i+1}(-a_{i+1}) = 0$ implies:

$$c_{i+1,i} = \frac{a_{i+1}}{(\lambda - a_i)(\lambda - a_{i+1})}.$$

Since $c_{i+1,i-1}(\lambda-a_{i-1})+c_{i+1,i}(-a_i)+c_{i+1,i+1}(-a_{i+1})=0$ is equivalent with

$$c_{i+1,i-1}=\frac{a_i}{\lambda-a_{i-1}}\cdot\frac{a_{i+1}}{\Pi_{k=i}^{i+1}(\lambda-a_k)}+\frac{a_{i+1}}{\lambda-a_{i-1}}\cdot\frac{1}{\lambda-a_{i+1}},$$

we have

$$c_{i+1,i-1}=\frac{\lambda\cdot a_{i+1}}{(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}\,.$$

Also, from the equality

$$c_{i+1,i-2}(\lambda-a_{i-2})+c_{i+1,i-2}(-a_{i-1})+c_{i+1,i}(-a_i)+c_{i+1,i+1}(-a_{i+1})=0,$$

it results that

$$c_{i+1,i-2}=\frac{\lambda a_{i-1}a_{i+1}}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}+$$
$$+\frac{a_ia_{i+1}(\lambda-a_{i-1})}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}+$$
$$+\frac{a_{i+1}(\lambda-a_{i-1})(\lambda-a_i)}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}=$$
$$=\frac{\lambda^2a_{i+1}}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}$$

Ressoning analogously we can find expression of

$$c_{i+1,i-3},c_{i+1,i-4},\ldots,c_{i+1,0}.$$

Let $a=(a_n)_{n\in\mathbb{N}}$ a sequence convergent of real numbers.

Theorem 1 *If R_a is a Rhaly operator with $a_n\neq 0$, $n\in\mathbb{N}$ and $((n+1)a_n)$ bounded, then $0\in\varphi(R_a,\ell^p)$.*

Proof. The operator $0\cdot I-R_a$ is bijective. The sequence $((n+1)a_n)$ being bounded, the Rhaly operator R_a is continuous on ℓ^p for any $p > 1$.

It remains to prove that the operator $(0 \cdot I - R_a)^{-1}$ is not continuous. It is equivalent to the fact that the operator $(0 \cdot I - R_a^*)$ is not surjective.

Be $g \in \ell^2$. It can be shown that the equation $(0 \cdot I - R_a^*)f = g$ is equivalent with $f(n) = \dfrac{g(n-1) - g(n)}{a_n}$, $n \in \mathbb{N}^*$. $g \in \ell^q$ is equivalent with $\sum\limits_{n=0}^{\infty} |g(n)|^2 < \infty$. The sequence (a_n) is bounded (thus, the sequence $((n+1)a_n)$ is unlimited, which contradicts the hypotesis). Hence $\exists M > 0$ one of its superior limits. It follows that

$$\sum_{n=1}^{\infty} |f(n)|^q = \sum_{n=1}^{\infty} \frac{|g(n-1) - g(n)|^q}{|a_n|^q} \geq$$

$$\geq \frac{1}{M^2} \sum_{h=1}^{\infty} |g(n-1) - g(n)|^q \geq \frac{1}{M^2} \sum_{h=1}^{\infty} \frac{1}{n} = \infty,$$

for g chosen so that

$$g(n) - g(n-1) = \frac{1}{n^{1/q}} \quad , \quad n > 1,$$

and

$$g(n-1) + \frac{1}{n^{1/q}} \leq \frac{1}{n-1} \quad , \quad n \geq 1.$$

Then,

$$\sum_{h=1}^{\infty} |g(n)|^q = \sum_{h=1}^{\infty} \left| g(n-1) + \frac{1}{n^{1/q}} \right|^q \leq \sum_{h=1}^{\infty} \frac{1}{(n-1)^q} < \infty.$$

Therefore $f \notin \ell^q$, that is the equation $(0 \cdot I - R_a^*)f = g$ has no solution, and the operator $0 \cdot I - R_a^*$ is not surjective, which remained to be demonstrated.

Theorem 2 *Be R_a a Rhaly operator so that $((n+1)a_n)$ is bounded. If $\lambda \in \sigma(R_a, \ell^p)$ and $\lambda \notin S$, then $\lambda \in \sigma_c(R_a, \ell^p)$.*

Proof. We notice that $\lambda I - R_a$ is a triangular matrix. $\lambda \notin S$ is equivalent with $\lambda - a_n \neq 0$, $n \in \mathbb{N}$; $det\,(\lambda I - R_a) \neq 0$, and $\lambda I - R_a$ is an injective operator.

To study the following equality, $\overline{(\lambda I - R_a)(\ell^p)} = \ell^p$, it can be shown that the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, f non–null. We have

$$f(n+1) = \frac{\lambda}{\lambda - a_n} f(n)\ ; \tag{2}$$

$$f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1})\dots(\lambda - a_0)} f(0)\ ,\ n \geq 0. \tag{3}$$

We assume that $f(0) \neq 0$ (for $f(0) = 0$, $f = 0$). Then

$$\left|\frac{f(n+1)}{f(n)}\right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1.$$

Thus $\Sigma\,|f(n)|^q$ is divergent, that is the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, f non–null.
Therefore, $\lambda \in \sigma_c(R_a, \ell^p)$.

Theorem 3 *Be R_a a Rhaly operator with the quality that the elements of the main diagonal of the matrix R_a are distinct, non–null and the sequence $((n+1)a_n)$ is bounded. If $\lambda = a_n$, $n \in \mathbb{N}$, then $\lambda \in \sigma_p(R_a, \ell^p)$.*

Proof. It must be shown that the operator $\lambda I - R_a$ is not injective. It can be shown that $(\lambda I - R_a)f = 0$, $f \in \ell^p$ and $f \neq 0$.

The equation $(\lambda I - R_a)f = 0$ implies

$$\sum_{k=0}^{n-1} a_n f(k) = (\lambda - a_n) f(k) \quad , \quad n \geq j. \tag{4}$$

If $\lambda = a_j$, $j \geq 1$, we get

$$a_j - a_n = a_n \sum_{k=0}^{n-1} f(k) \quad , \quad n \geq j\ ; \tag{5}$$

$$f(n) = a_j \left(\frac{1}{a_{n+1}} - \frac{1}{a_n} \right). \tag{6}$$

It follows that

$$\sum_{n=0}^{\infty} |f(n)|^p < \infty$$

and

$$|a_j| \sum_{n=j}^{\infty} \left| \frac{1}{a_{n+1}} - \frac{1}{a_n} \right|^p < \infty\ .$$

Be $m < -1$ an inferior limit of the sequence (a_n). Then

$$|a_j|^p \sum_{n=j}^{\infty} \left| \frac{1}{a_{n+1}} - \frac{1}{a_n} \right|^p \leq 2|a_j|^p \sum_{n=j}^{\infty} \left| \frac{1}{a_n} \right|^p <$$

$$< 2|a_j|^p \sum_{n=j}^{\infty} \left| \frac{1}{m} \right|^p = 2 \left| \frac{a_j}{m} \right|^p \frac{|m|}{|m|-1} < \infty.$$

If $\lambda = a_0$, $(\lambda I - R_a)e_0 = 0$.
So the operator $\lambda I - R_a$ is not injective; $\lambda \in \sigma_p(R_a, \ell^p)$.

The following theorems contain the results with reference to Rhaly operators R_a with the quality to

$$0 < \lim_{n\to\infty} (n+1)a_n < \infty.$$

Theorem 4 *Be R_a a Rhaly operator with the quality that a is a sequence of positive real numbers. If $\lambda \in K$ has the qualities:*

$$\left| \lambda - \frac{\alpha p}{2(p-1)} \right| < \frac{\alpha p}{2(p-1)}$$

and $\lambda \in S$, then $\lambda \in \sigma_c(R_a, \ell^p)$.

Proof. $\lambda I - R_a$ is a triangular matrix. $\lambda \notin S$, then all its elements from the main diagonal are non–null, the operator $\lambda I - R_a$ being then injective.

$\overline{(\lambda I - R_a)(\ell^p)} = \ell^p$ is a quality that can be studied using the injectivity of the operator $\lambda I - R_a^*$. For this it can be shown that the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, non–null.

It results the relation of recurence:

$$f(n+1) = \frac{\lambda}{\lambda - a_n} f(n) \quad , \quad n \in \mathbb{N}.$$

From this results the equalities:

$$(7) \quad f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1}) \dots (\lambda - a_0)} f(0), \ n \geq 1.$$

We assume that $f(0) \neq 0$, ($f(0) = 0$ implies $f = 0$). Then,

$$(8) \qquad \left| \frac{f(n+1)}{f(n)} \right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1.$$

Hence $\Sigma\, |f(n)|^q$ is divergente, that is the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, f non–null.

Theorem 5 *Be R_a a Rhaly operator with $a = (a_n)$ a sequence of positive real numbers. If*

$$\left| \lambda - \frac{\alpha p}{2(p-1)} \right| < \frac{\alpha p}{2(p-1)},$$

$\alpha \neq 0$, $\lambda \neq 0$, then $\lambda \in \sigma_c(R_a, \ell^p)$.

Proof. Be $\lambda \in K$ with the qualities from the enunciation, $det\,(\lambda I - R_a) \neq 0$ so the operator $\lambda I - R_a$ is injective ($\lambda \notin S$). Thus, $\lambda \in S$ and $\lambda = a_n$ with $n \in \mathbb{N}$.

The propierty

$$\left| \lambda - \frac{\alpha p}{2(p-1)} \right| < \frac{\alpha p}{2(p-1)}$$

is equivalent with:

i) $\lambda = \dfrac{\alpha p}{p-1} = \alpha q$, and $a_n = q \lim\limits_{n\to\infty}(n+1)a_n$, implies the following equality:

$$\alpha - \alpha = q \lim_{n\to\infty}\left\{(n+1)\lim_{n\to\infty}[(n+1)a_n - (n+1)a_{n+1}]\right\} -$$

$$-q \lim_{n\to\infty}(n+2)a_{n+1}.$$

Then $\lambda = 0$, which contradicts the hypothesis.

ii) $\lambda + \dfrac{\alpha p}{2(p-1)} = \dfrac{\alpha p}{2(p-1)}$ implies $\lambda = 0$, which is false.

We consider the equation $(\lambda I - R_a^*)f = 0$, $f \in 0$, $f \neq 0$. We get the relations

$$f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1})\ldots(\lambda - a_0)} f(0),\ n \in \mathbb{N}. \tag{9}$$

We assume that $f(0) \neq 0$ (if not, $f = 0$). Then

$$\left|\frac{f(n+1)}{f(n)}\right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1.$$

so $\Sigma\,|f(n)|^q$ is divergente, that is the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, f non–null. Then $\lambda \in \sigma_c(R_a, \ell^p)$.

Observation 1 *a) The results above can be particularized for z–Cesàro operators.*

b) The spectrum of a Rhaly operator $R_a : \ell^p \to \ell^p$ with the sequence $((n+1)a_n)$ bounded, where $a = (a_n)$ is a sequence of positive real numbers verifies the equality:

$$\sigma(R_a, \ell^p) = \left\{\left|\lambda - \frac{\alpha p}{2(p-1)}\right| \leq \frac{\alpha p}{2(p-1)} \cup S\right\}.$$

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