General Mathematics Vol. 4 /1996

A property of divided differences with multiple nodes

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Abstract

The aim of this paper is to present a new proof to formula′s of divided differences with multiple nodes, coresponding to a product of two functions.

AMS Subject Classification 65D05

1. Let us consider an arbitrary finite interval $[a, b]$ on the real line, $x_0, x_1, \dots, x_n$ distinct knots in this interval, with the multiplicities $\alpha_0, \alpha_1, \dots, \alpha_n$ respectively, where

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$$

is a sequence of positive integers.

We denote by Π_m the set of all algebraic polynomials of degree at most m,

$$N = \sum_{k=0}^{n} \alpha_k - 1 \ , \tag{1}$$

$$\omega \begin{pmatrix} x_0 \\ \alpha_0 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ \alpha_n \end{pmatrix} (x) = \prod_{k=0}^{n} (x - x_k)^{\alpha_k} \tag{2}$$

$$u_i(x) = \frac{(x - x_i)^{\alpha_i}}{\omega \begin{pmatrix} x_0 \\ \alpha_0 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ \alpha_n \end{pmatrix} (x)} \quad , \quad i = \overline{0, n} \ . \tag{3}$$

Let $L_{k,\nu_k}(x)$ be the fundamental polynomials of Hermite interpolation, $k=\overline{0,n}$, $\nu_k=\overline{0,\alpha_k-1}$, i.e. $L_{k,\nu_k}(x)$ satisfies the condition

$$(4)\quad L_{k,\nu_k}^{(\mu_i)}(x_i)=\delta_{k,i}\delta_{\nu_k,\mu_i},\quad (\forall)i=\overline{0,n}\ ,\ (\forall)\mu_i=\overline{0,\alpha_i-1}$$

where is Kronecher′s delta.

Lemma 1 *(*see [1]*). The fundamental polynomials of Hermite interpolation has the following representation:*

$$(5)\quad L_{k,\nu_k}(x)=\sum_{\ell_k=0}^{\alpha_k-1-\nu_k}\frac{(x-x_k)^{\nu_k+\ell_k}}{(\nu_k)!(\ell_k)!}\left[\frac{1}{u_k(x)}\right]_{|x=x_k}^{(\ell_k)}\cdot u_k(x).$$

Let

$$H\begin{pmatrix}x_0\\ \alpha_0\end{matrix},\ldots,\begin{matrix}x_n\\ \alpha_n\end{matrix};f\end{pmatrix}(x)$$

be Hermite′s interpolating polynomial defined by the following condition

$$(6)\quad \begin{gathered}H^{(\nu_k)}\begin{pmatrix}x_0\\ \alpha_0\end{matrix},\ldots,\begin{matrix}x_n\\ \alpha_n\end{matrix};f\end{pmatrix}(x_k)=f^{(\nu_k)}(x_k),\\ (\forall)k=\overline{0,n}\ ,\ (\forall)\nu_k=\overline{0,\alpha_k-1},\end{gathered}$$

where

$$(7)\quad f\in D^{\alpha}[a,b]=\{g:[a,b]\to R|\exists g^{(\alpha_{k-1})}(x_k),k=\overline{0,n}\}$$

Lemma 2 (see [1]). *Hermite′s interpolating polynomial has the*
representation

$$(8)\quad H\begin{pmatrix}x_0\\ \alpha_0\end{matrix},\ldots,\begin{matrix}x_n\\ \alpha_n\end{matrix};f\end{pmatrix}(x)=\sum_{k=0}^{n}\sum_{\nu_k=0}^{\alpha_k-1}L_{k,\nu_k}(x)f^{(\nu_k)}(x_k)$$

and its degree is at most N.

2. In this section we define the divided differences with multiple nodes and present some properties used in the next section.

Theorem 1 *There is a unique linear functional $A : D^{\alpha}[a,b] \to R$,*

$$A(f) = \sum_{i=0}^{n} \sum_{\nu_i=0}^{\alpha_i-1} A_{i,\nu_i} f^{(\nu_i)}(x_i) \ , \tag{9}$$

where $A_{i,\nu_i} \in R$, which satisfies the conditions:

$$\begin{aligned} &i) \ \ A(e_j) = 0 \quad , \quad (\forall) j = \overline{0, N-1} \ , \\ &ii) \ \ A(e_N) = 1 \ . \end{aligned} \tag{10}$$

Proof. In relation (9) we consider

$$f(x) := L_{j,\mu_j}(x) \ , \tag{11}$$

where $j \in \{0, \ldots, n\}$, $\mu_j \in \{0, \ldots, \alpha_j - 1\}$ are fixed numbers.

Using relation (4) and (9) we deduce

$$A(L_{j,\mu_j}) = \sum_{i=0}^{n} \sum_{\nu_i=0}^{\alpha_i-1} A_{i,\nu_i} L_{j,\mu_j}^{(\nu_i)}(x_i) = A_{j,\mu_j}. \tag{12}$$

From Lemma 1 we find

$$L_{j,\mu_j}(x) = \frac{1}{(\mu_j)!(\alpha_j - 1 - \mu_j)!} \left[\frac{1}{u_j(x)} \right]_{|x=x_j}^{(\alpha_j-1-\mu_j)} x^N + h(x), \tag{13}$$

where $h \in \Pi_{N-1}$.

Taking into account relations (10), (11) and (12) we conclude with

$$A_{j,\mu_j} = \frac{1}{(\mu_j)!(\alpha_j - 1 - \mu_j)!} \left[\frac{1}{u_j(x)} \right]_{|x=x_j}^{(\alpha_j-1-\mu_j)} , \tag{14}$$

$$j = \overline{0, n} \quad , \quad \mu_j = \overline{0, \alpha_j - 1},$$

which proves this theorem. ∎

Definition 1 *The linear functional* $A : D^{\alpha}[a,b] \to R$, *which existence and uniqueness is given by Theorem 1, is called divided difference with multiple nodes, coresponding to the function* f *and to the knots* $x_0, \ldots, x_n$, *with multiplicities* $\alpha_0, \ldots, \alpha_n$.

We denote

$$A(f) = \begin{bmatrix} x_0 \\ \alpha_0 \end{bmatrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; f \Bigg] \ . \tag{15}$$

Theorem 2 *The divided difference coresponding to the function* f *and to the knots* $x_0, \ldots, x_n$, *with multiplicities* $\alpha_0, \ldots, \alpha_n$, *can be expressed as*

$$\left[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; f\right] = \tag{16}$$

$$= \sum_{i=0}^{n} \sum_{\nu_i=0}^{\alpha_i-1} \frac{1}{(\nu_i)!(\alpha_i - 1 - \nu_i)!} \left[\frac{1}{u_i(x)}\right]^{(\alpha_i-1-\nu_i)}_{|x=x_i} \cdot f^{(\nu_i)}(x_i) =$$

$$= \sum_{i=0}^{n} \frac{1}{(\alpha_i - 1)!} \left[\frac{1}{u_i(x)} f(x)\right]^{(\alpha_i-1)}_{|x=x_i} \ .$$

Proof. Directly, from relations (9) and (14). ■

Theorem 3 *The divided difference is equal to the coefficient of* x^N *from the polynomial*

$$H\left(\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; f\right)(x)$$

of Hermite's interpolating.

Proof. Let c_N be the coefficient of x^N of the polynomial

$$H\left(\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; f\right)(x) \ .$$

Using Lemma 1 and Lemma 2 we find

$$c_N = \tag{17}$$

$$= \sum_{i=0}^{n} \sum_{\nu_i=0}^{\alpha_i-1} \frac{1}{(\nu_i)!(\alpha_i-1-\nu_i)!} \cdot \left[\frac{1}{u_i(x)}\right]_{|x=x_i}^{(\alpha_i-1-\nu_i)} f^{(\nu_i)} \cdot (x_i).$$

From Theorem 2 and relation (17) it follows that Theorem 3 is true. ■

Theorem 4 *Hermite's interpolating polynomial has the representation*

$$H\left(\begin{matrix} x_0 \\ \alpha_0\end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n\end{matrix}; f\right)(x) = \sum_{\nu_0=0}^{\alpha_0-1} \left[\begin{matrix} x_0 \\ \nu_0+1\end{matrix}; f\right](x-x_0)^{\nu_0} + \tag{18}$$

$$+ \sum_{i=1}^{n} \sum_{\nu_i=0}^{\alpha_i-1} \left[\begin{matrix} x_0 \\ \alpha_0\end{matrix}, \ldots, \begin{matrix} x_{i-1} \\ \alpha_{i-1}\end{matrix}, \begin{matrix} x_i \\ \nu_i+1\end{matrix}; f\right] \omega_{i,\nu_i}(x) \ ,$$

where

$$\omega_{i,\nu_i}(x) = \prod_{j=0}^{i-1} (x-x_j)^{\alpha_j}(x-x_i)^{\nu_i} \ .$$

Proof. Using Theorem 3 we find the following recurrence relation:

$$H\left(\begin{matrix} x_0 \\ \nu_0+1\end{matrix}; f\right)(x) = \tag{19}$$

$$= H\left(\begin{matrix} x_0 \\ \nu_0\end{matrix}; f\right)(x) + \left[\begin{matrix} x_0 \\ \nu_0+1\end{matrix}; f\right](x-x_0)^{\nu_0} \ ,$$

$$(\forall)\nu_0 = \overline{0, \alpha_0 - 1} \ ,$$

$$H\left(\begin{matrix} x_0 \\ \alpha_0\end{matrix}, \ldots, \begin{matrix} x_{k-1} \\ \alpha_{k-1}\end{matrix}, \begin{matrix} x_k \\ \nu_k+1\end{matrix}; f\right)(x) = \tag{20}$$

$$= H\left(\begin{matrix} x_0 \\ \alpha_0\end{matrix}, \ldots, \begin{matrix} x_{k-1} \\ \alpha_{k-1}\end{matrix}, \begin{matrix} x_k \\ \nu_k\end{matrix}; f\right)(x) +$$

$$+ \left[\begin{matrix} x_0 \\ \alpha_0\end{matrix}, \ldots, \begin{matrix} x_{k-1} \\ \alpha_{k-1}\end{matrix}, \begin{matrix} x_k \\ \nu_k+1\end{matrix}; f\right] \omega_{k,\nu_k}(x),$$

$$(\forall)k = \overline{1,n} \quad , \quad (\forall)\nu_k = \overline{0, \alpha_k - 1} \ .$$

Adding the relation (19) and (20) for $k = 0, 1, \ldots, n$ and $\nu_k = 0, \ldots, \alpha_k - 1$ we deduce (18). ■

Theorem 5 (see [3]). *If $f \in C^N[a,b]$, then there is a point $\xi \in (a,b)$ such that*

$$\begin{bmatrix} x_0 \\ \alpha_0 \end{bmatrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; f \Big] = \frac{f^{(N)}(\xi)}{N!} \ . \tag{21}$$

3. In this section we present a new proof to formula′s of divided differences with multiple nodes, coresponding to a product of two functions.

Lemma 3 *If $A : D^\alpha[a,b] \to R$ is a linear functional,*

$$A(f) = \sum_{i=0}^{n} \sum_{\nu_i=0}^{\alpha_i - 1} c_{i,\nu_i} f^{(\nu_i)}(x_i) \ , \tag{22}$$

where $c_{i,\nu_i} \in R$ is not depending of f, then

$$A(f) = \sum_{\nu_0=0}^{\alpha_0-1} \left[\begin{matrix} x_0 \\ \nu_0 + 1 \end{matrix} ; f \right] B_{0,\nu_0} + \tag{23}$$

$$+ \sum_{i=1}^{n} \sum_{\nu_i=0}^{\alpha_i-1} \left[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_{i-1} \\ \alpha_{i-1} \end{matrix}, \begin{matrix} x_i \\ \nu_i + 1 \end{matrix}; f \right] B_{i,\nu_i} \ ,$$

where $B_{i,\nu_i} \in R$ is not depending of f.

Proof. Using Theorem 4 and the relations

$$f^{(\mu_j)}(x_j) = H^{(\mu_j)} \left(\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; f \right)(x_j) \ ,$$

$$(\forall)j = \overline{0,n} \ , \ (\forall)\mu_j = \overline{0, \alpha_j - 1}$$

we find that Lemma 3 is valid. ■

Theorem 6 *Divided difference with multiple nodes, corresponding to a product of two functions $f, g \in D^{\alpha}[a,b]$, can be expressed as:*

$$\begin{bmatrix} x_0 \\ \alpha_0 \end{bmatrix}, \ldots, \begin{bmatrix} x_n \\ \alpha_n \end{bmatrix}; fg \Bigg] = \tag{24}$$

$$\left[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; fg\right] =$$

$$= \sum_{\nu_0=0}^{\alpha_0-1} \left[\begin{matrix} x_0 \\ \nu_0+1 \end{matrix}; f\right] \left[\begin{matrix} x_0 \\ \alpha_0-\nu_0 \end{matrix}, \begin{matrix} x_1 \\ \alpha_1 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; g\right] +$$

$$+ \sum_{i=1}^{n} \sum_{\nu_i=0}^{\alpha_i-1} \left[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_{i-1} \\ \alpha_{i-1} \end{matrix}, \begin{matrix} x_i \\ \nu_i+1 \end{matrix}; f\right] \cdot$$

$$\cdot \left[\begin{matrix} x_i \\ \alpha_i-\nu_i \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; g\right] .$$

Proof. From relation (16) we find

$$\left[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; fg\right] =$$

$$= \sum_{i=0}^{n} \frac{1}{(\alpha_i-1)!} \left[\frac{1}{u_i(x)} f(x) g(x)\right]^{(\alpha_i-1)}_{|x=x_i} =$$

$$= \sum_{i=0}^{n} \frac{1}{(\alpha_i-1)!} \sum_{\nu_i=0}^{\alpha_i-1} \binom{\alpha_i-1}{\nu_i} \left[\frac{1}{u_i(x)} g(x)\right]^{(\alpha_i-1-\nu_i)}_{|x=x_i} f^{(\nu_i)}(x_i).$$

Therefore

$$\left[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; fg\right] = \sum_{i=0}^{n} \sum_{\nu_i=0}^{\alpha_i-1} c_{i,\nu_i} f^{(\nu_i)}(x_i) , \tag{25}$$

where

$$c_{i,\nu_i} = \frac{1}{(\nu_i)!(\alpha_i-1-\nu_i)!} \left[\frac{1}{u_i(x)} g(x)\right]^{(\alpha_i-1-\nu_i)}_{|x=x_i} , \tag{26}$$

$$i = \overline{0,n} \ , \ \nu_i = \overline{0, \alpha_i - 1},$$

is not depending of f.

Consequently, from Lemma 1 we have the following equality

$$\left[\begin{matrix}x_0\\ \alpha_0\end{matrix},\dots,\begin{matrix}x_n\\ \alpha_n\end{matrix};fg\right]=\sum_{\nu_0=0}^{\alpha_0-1}\left[\begin{matrix}x_0\\ \nu_0+1\end{matrix};f\right]B_{0,\nu_0}+ \tag{27}$$

$$+\sum_{i=1}^{n}\sum_{\nu_i=0}^{\alpha_i-1}\left[\begin{matrix}x_0\\ \alpha_0\end{matrix},\dots,\begin{matrix}x_{i-1}\\ \alpha_{i-1}\end{matrix},\begin{matrix}x_i\\ \nu_i+1\end{matrix};f\right]B_{i,\nu_i}\ ,$$

where $B_{i,\nu_i}\in R$, $i=\overline{0,n}$, $\nu_i=\overline{0,\alpha_i-1}$, is not depending of f.

In the relation (27) we consider

$$f(x):=\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}(x)=(x-x_0)^{\mu_0}\ , \tag{28}$$

where $\mu_0\in\{0,\dots,\alpha_0-1\}$ are fixed, and we find

$$\left[\begin{matrix}x_0\\ \alpha_0\end{matrix},\dots,\begin{matrix}x_n\\ \alpha_n\end{matrix};\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}g\right]= \tag{29}$$

$$=\sum_{\nu_0=0}^{\alpha_0-1}\left[\begin{matrix}x_0\\ \nu_0+1\end{matrix};\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}\right]B_{0,\nu_0}+$$

$$+\sum_{i=1}^{n}\sum_{\nu_i=0}^{\alpha_i-1}\left[\begin{matrix}x_0\\ \alpha_0\end{matrix},\dots,\begin{matrix}x_{i-1}\\ \alpha_{i-1}\end{matrix},\begin{matrix}x_i\\ \nu_i+1\end{matrix};\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}\right]B_{i,\nu_i}\ .$$

Taking into account Theorem 5 and relation (16) we obtain

$$\left[\begin{matrix}x_0\\ \nu_0+1\end{matrix};\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}\right]=\begin{cases}0 & ,\text{if } \ \nu_0\neq\mu_0\\ 1 & ,\text{if } \ \nu_0=\mu_0\end{cases}\ ,(\forall)\nu_0=\overline{0,\alpha_0-1} \tag{30}$$

and

$$\left[\begin{matrix}x_0\\ \alpha_0\end{matrix},\dots,\begin{matrix}x_{i-1}\\ \alpha_{i-1}\end{matrix},\begin{matrix}x_i\\ \nu_i+1\end{matrix};\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}\right]=0, \tag{31}$$

$$(\forall)i=\overline{1,n}\ ,\ (\forall)\nu_i=\overline{0,\alpha_i-1}.$$

Therefore

$$B_{0,\mu_0}=\left[\begin{matrix}x_0\\ \alpha_0\end{matrix},\dots,\begin{matrix}x_n\\ \alpha_n\end{matrix};\omega\begin{pmatrix}x_0\\ \mu_0\end{pmatrix}g\right],\ (\forall)\mu_0=\overline{0,\alpha_0-1}. \tag{32}$$

Using the notation

$$p_{0,\alpha_0-\mu_0}(x)=\omega\begin{pmatrix} x_0 & x_1 & & x_n \\ \alpha_0-\mu_0 & \alpha_1 & \ldots, & \alpha_n\end{pmatrix}(x)\quad,\quad \mu_0=\overline{0,\alpha_0-1},$$

from relations (16) and (32) we deduce

$$B_{0,\mu_0}=\sum_{i=0}^{n}\frac{1}{(\alpha_i-1)!}\left[\frac{(x-x_i)^{\alpha_i}}{\omega\begin{pmatrix} x_0 & & x_n \\ \alpha_0, & \ldots, & \alpha_n\end{pmatrix}(x)}(x-x_0)^{\mu_0}g(x)\right]^{(\alpha_i-1)}_{|x=x_i}=$$

$$=\frac{1}{(\alpha_0-1)!}\left[\frac{(x-x_0)^{\alpha_0-\mu_0}}{p_{0,\alpha_0-\mu_0}(x)}g(x)(x-x_0)^{\mu_0}\right]^{(\alpha_0-1)}_{|x=x_0}+$$

$$+\sum_{i=1}^{n}\frac{1}{(\alpha_i-1)!}\left[\frac{(x-x_i)^{\alpha_i}}{p_{0,\alpha_0-\mu_0}(x)}g(x)\right]^{(\alpha_i-1)}_{|x=x_i}.$$

Applying Newton–Leibnitz formula′s we obtain

$$B_{0,\mu_0}=$$

$$=\frac{1}{(\alpha_0-1)!}\sum_{\nu_0=0}^{\alpha_0-1}\binom{\alpha_0-1}{\nu_0}\left[\frac{(x-x_0)^{\alpha_0-\mu_0}}{p_{0,\alpha_0-\mu_0}(x)}g(x)\right]^{(\alpha_0-1-\nu_0)}_{|x=x_0}\cdot$$

$$\cdot[(x-x_0)^{\mu_0}]^{(\nu_0)}_{|x=x_0}+$$

$$+\sum_{i=1}^{n}\frac{1}{(\alpha_i-1)!}\left[\frac{(x-x_i)^{\alpha_i}}{p_{0,\alpha_0-\mu_0}(x)}g(x)\right]^{(\alpha_i-1)}_{|x=x_i}=$$

$$=\frac{1}{(\alpha_0-\mu_0-1)!}\left[\frac{(x-x_0)^{\alpha_0-\mu_0}}{p_{0,\alpha_0-\mu_0}(x)}g(x)\right]^{(\alpha_0-1-\mu_0)}_{|x=x_0}+$$

$$+\sum_{i=1}^{n}\frac{1}{(\alpha_i-1)!}\left[\frac{(x-x_i)^{\alpha_i}}{p_{0,\alpha_0-\mu_0}(x)}g(x)\right]^{(\alpha_i-1)}_{|x=x_i}.$$

Finally, we find

$$B_{0,\mu_0}=\begin{bmatrix} x_0 & x_1 & & x_n & \\ \alpha_0-\mu_0, & \alpha_1, & \ldots, & \alpha_n & ;g\end{bmatrix},\ (\forall)\mu_0=\overline{0,\alpha_0-1}. \tag{33}$$

Further, in relation (27) we consider

$$f(x) := \omega_{\mu_j}(x) = \omega \begin{pmatrix} x_0 \\ \alpha_0 \end{pmatrix}, \ldots, \begin{matrix} x_{j-1} \\ \alpha_{j-1} \end{matrix}, \begin{matrix} x_j \\ \mu_j \end{pmatrix} (x) \ , \tag{34}$$

where $j \in \{1, \ldots, n\}$, $\mu_j \in \{0, \ldots, \alpha_j - 1\}$ are fixed numbers, and we find

$$\begin{bmatrix} x_0 \\ \alpha_0 \end{bmatrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; \omega_{\mu_j} g \Big] = \sum_{\nu_0=0}^{\alpha_0-1} \begin{bmatrix} x_0 \\ \nu_0+1 \end{bmatrix}; \omega_{\mu_j} \Big] B_{0,\nu_0} + \tag{35}$$

$$+ \sum_{i=1}^{n} \sum_{\nu_i=0}^{\alpha_i-1} \Big[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_{i-1} \\ \alpha_{i-1} \end{matrix}, \begin{matrix} x_i \\ \nu_i+1 \end{matrix}; \omega_{\mu_j} \Big] B_{i,\nu_i} \ .$$

From Theorem 5 and relation (16) it follows that

$$\Big[\begin{matrix} x_0 \\ \nu_0+1 \end{matrix}; \omega_{\mu_j} \Big] = 0 \quad , \quad (\forall)\nu_0 = \overline{0, \alpha_0 - 1} \ , \tag{36}$$

and

$$\Big[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_{i-1} \\ \alpha_{i-1} \end{matrix}, \begin{matrix} x_i \\ \nu_i+1 \end{matrix}; \omega_{\mu_j} \Big] = \delta_{ij}\delta_{\nu_i,\mu_j} = \tag{37}$$

$$= \begin{cases} 1, \ \text{if} \qquad i = j \ \text{ and } \ \nu_i = \mu_j \\ \\ 0, \ \text{unless} \end{cases}$$

Therefore

$$B_{j,\mu_j} = \Big[\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; \omega_{\mu_j} g \Big] \ . \tag{38}$$

Using the notation

$$\Pi_{j,\alpha_j-\mu_j}(x) = \omega \begin{pmatrix} x_j \\ \alpha_j - \mu_j \end{pmatrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix} \Big) (x), \ j = \overline{1, n}, \ \mu_j = \overline{0, \alpha_j - 1},$$

relations (16) and (38) we deduce

$$B_{j,\mu_j} = \sum_{i=0}^{n} \frac{1}{(\alpha_i - 1)!} \left[\frac{(x - x_i)^{\alpha_i}}{\omega \left(\begin{matrix} x_0 \\ \alpha_0 \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix} \right)(x)} \omega_{j,\mu_j}(x) g(x) \right]^{(\alpha_i-1)}_{|x=x_i} =$$

$$= \sum_{i=j}^{n} \frac{1}{(\alpha_i - 1)!} \left[\frac{(x - x_i)^{\alpha_i}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x) \right]^{(\alpha_i - 1)}_{|x=x_i} =$$

$$= \frac{1}{(\alpha_j - 1)!} \left[\frac{(x - x_j)^{\alpha_j-\mu_j}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x)(x - x_j)^{\mu_j} \right]^{(\alpha_j - 1)}_{|x=x_j} +$$

$$+ \sum_{i=j+1}^{n} \frac{1}{(\alpha_j - 1)!} \left[\frac{(x - x_i)^{\alpha_i}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x) \right]^{(\alpha_i - 1)}_{|x=x_i} .$$

Applying Newton–Leibnitz formula′s we find

$$B_{j,\mu_j} = \frac{1}{(\alpha_j - 1)!} \sum_{\nu_j=0}^{\alpha_j-1} \binom{\alpha_j - 1}{\nu_j} \left[\frac{(x - x_j)^{\alpha_j-\mu_j}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x) \right]^{(\alpha_j - 1 - \nu_j)}_{|x=x_j} \cdot$$

$$\cdot [(x - x_j)^{\mu_j}]^{(\nu_j)}_{|x=x_j} +$$

$$+ \sum_{i=j+1}^{n} \frac{1}{(\alpha_i - 1)!} \left[\frac{(x - x_i)^{\alpha_i}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x) \right]^{(\alpha_i - 1)}_{|x=x_i} =$$

$$= \frac{1}{(\alpha_j - 1 - \mu_j)!} \left[\frac{(x - x_j)^{\alpha_j-\mu_j}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x) \right]^{(\alpha_j - 1 - \mu_j)}_{|x=x_j} +$$

$$+ \sum_{i=j+1}^{n} \frac{1}{(\alpha_i - 1)!} \left[\frac{(x - x_i)^{\alpha_i}}{\Pi_{j,\alpha_j-\mu_j}(x)} g(x) \right]^{(\alpha_i - 1)}_{|x=x_i} .$$

Finally, we obtain

$$B_{j,\mu_j} = \left[\begin{matrix} x_j \\ \alpha_j - \mu_j \end{matrix}, \ldots, \begin{matrix} x_n \\ \alpha_n \end{matrix}; g \right] , \tag{39}$$

$$(\forall) j = \overline{1, n} \quad , \quad \mu_j = \overline{0, \alpha_j - 1} \, .$$

From relations (27), (33) and (39) it follows that Theorem 6 is true. ∎

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