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A fundamental property of B–splines

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Abstract

Let $\Delta_n : t_0 \leq t_1 \leq \ldots \leq t_n$ be a division on the real line, where $n \in \mathbb{N}^*$ and $t_0, t_1, \ldots, t_n \in \mathbb{R}$. We denote by $B_{i,k}(x)$, $k = 0, \ldots, n-1$, $i = 0, \ldots, n-1-k$, $x \in \mathbb{R}$, the functions B–spline coresponding to division Δ_n.

Our aim in this paper is to prove a new fundamental property of B–splines.

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1 Preliminaries

In this section we define the B–splines coresponding to division Δ_n and present some known properties of this functions, which will be used in the next section.

Definition 1 *For $\ell = 0, \ldots, n-1$, $j = 1, \ldots, n-\ell$ and $x \in \mathbb{R}$ we define the functions*

$$(1) \qquad \omega_{\ell,j}(x) = \begin{cases} \dfrac{x - t_\ell}{t_{\ell+j} - t_\ell} & , \textit{ if } \ t_\ell < t_{\ell+j} \\ 0 & , \textit{ if } \ t_\ell = t_{\ell+j} \end{cases}$$

Definition 2 *For $k = 0, \ldots, n-1$, $i = 0, \ldots, n-1-k$ and $x \in \mathbb{R}$ we define, using a recurrence relation, the functions B–spline coresponding to division Δ_n*

$$B_{i,0}(x) = \begin{cases} 1 \ , \ \textit{if} \quad x \in [t_i, t_{i+1}) \\ 0 \ , \ \textit{if} \ \ x \in \mathbb{R} \setminus [t_i, t_{i+1}) \end{cases} \tag{2}$$

$$B_{i,k}(x) = \omega_{i,k}(x) \cdot B_{i,k-1}(x) + (1 - \omega_{i+1,k}(x)) \cdot B_{i+1,k-1}(x), \quad k = 1, \ldots, n-1. \tag{3}$$

Remark 1 *If $k \in \{0, \ldots, n-1\}$, $i \in \{0, \ldots, n-1-k\}$ and $t_i = t_{i+1} = \ldots = t_{i+k+1}$, then $B_{i,k}(x) = 0$ for all $x \in \mathbb{R}$.*

Lemma 1 *(see [4]). If $k = 0, \ldots, n-1$, $i = 0, \ldots, n-1-k$ and $x \notin [t_i, t_{i+k+1})$ then*

$$B_{i,k}(x) = 0 \ . \tag{4}$$

Lemma 2 *(see [4]). If $k = 0, \ldots, n-1$, $i = 0, \ldots, n-1-k$ and $x \in [t_i, t_{i+k+1})$ then*

$$B_{i,k}(x) \geq 0 \ . \tag{5}$$

2 Main results

In this section we shall prove a new fundamental property of B–splines.

Further we denote by $a^j_{i,k}(x)$ the functions

$$a^j_{i,k}(x) = \sum_{\ell=0}^{j} B_{i+\ell,k-\ell}(x) = B_{i,k-j}(x) + B_{i+1,k-j}(x) + \ldots + B_{i+j,k-j}(x), \tag{6}$$

where $k = 0, \ldots, n-1$, $i = 0, \ldots, n-1-k$, $j = 0, \ldots, k$ and $x \in [t_i, t_{i+k+1})$.

Theorem 1 *The following inequality holds*

$$B_{i,k}(x) \le a^j_{i,k}(x) \ , \tag{7}$$

for all $k = 1, \ldots, n-1$, $i = 0, \ldots, n-1-k$, $j = 1, \ldots, k$ *and* $x \in [t_i, t_{i+k+1})$.

Proof. We use the induction method.

1. If $j = 1$ we must to show

$$B_{i,k}(x) \le a^1_{i,k}(x) = B_{i,k-1}(x) + B_{i+1,k-1}(x) \ , \tag{8}$$

where $x \in [t_i, t_{i+k+1})$.

From relation (3) we have

$$B_{i,k}(x) = \omega_{i,k}(x) \cdot B_{i,k-1}(x) + (1 - \omega_{i+1,k}(x)) \cdot B_{i+1,k-1}(x). \tag{9}$$

We consider the following situations

1.1. Suppose that $x \in [t_i, t_{i+1})$.

Using relation (1), Lemma 1 and Lemma 2 we deduce

$$\begin{aligned} &0 \le \omega_{i,k}(x) \le 1 \ , \\ &B_{i,k-1}(x) \ge 0 \ , \\ &B_{i+1,k-1}(x) = 0 \ \ (x \notin [t_{i+1}, t_{i+k+1})) \ . \end{aligned} \tag{10}$$

From (9) and (10) it follows that relation (8) is true for $x \in [t_i, t_{i+1})$.

1.2. Suppose that $x \in [t_{i+1}, t_{i+k})$.

Using relation (1), Lemma 1 and Lemma 2 we obtain

$$\begin{aligned} &0 \le \omega_{i,k}(x) \le 1 \ , \\ &B_{i,k-1}(x) \ge 0 \ , \\ &0 \le 1 - \omega_{i+1,k}(x) \le 1 \ , \\ &B_{i+1,k-1}(x) \ge 0 \ . \end{aligned} \tag{11}$$

From (9) and (11) it follows that the relation (8) is valid for $x \in [t_{i+1}, t_{i+k})$.

1.3. Suppose that $x \in [t_{i+k}, t_{i+k+1})$.

Using relation (1), Lemma 1 and Lemma 2 we deduce

$$\begin{aligned}
&B_{i,k-1}(x) = 0\ , \quad (x \notin [t_i, t_{i+k}))\ , \\
(12) \qquad &0 \le 1 - \omega_{i+1,k}(x) \le 1 \\
&B_{i+1,k-1}(x) \ge 0\ .
\end{aligned}$$

From (9) and (12) we obtain that relation (8) is true for $x \in [t_{i+k}, t_{i+k+1})$.

Hence, the relation (8) holds for all $x \in [t_i, t_{i+k+1})$.

2. Now assuming the relation (7) to be true for ℓ, where $\ell \in \{1, \ldots, k-1\}$, i.e

$$\begin{aligned}
&B_{i,k}(x) \le a^{\ell}_{i,k}(x) = \\
(13) \qquad &= B_{i,k-\ell}(x) + B_{i+1,k-\ell}(x) + \ldots + B_{i+\ell,k-\ell}(x), \\
&x \in [t_i, t_{i+k+1})\ ,
\end{aligned}$$

we must to show that it is also valid for $\ell + 1$

$$\begin{aligned}
&B_{i,k}(x) \le a^{\ell+1}_{i,k}(x) = \\
(14) \qquad &= B_{i,k-\ell-1}(x) + B_{i+1,k-\ell-1}(x) + \ldots + B_{i+\ell+1,k-\ell-1}(x), \\
&x \in [t_i, t_{i+k+1}).
\end{aligned}$$

From relations (3) and (13) we deduce

$$\begin{aligned}
&B_{i,k}(x) \le \omega_{i,k-\ell}(x) \cdot B_{i,k-\ell-1}(x) + \\
(15) \qquad &+ B_{i+1,k-\ell-1}(x) + \ldots + B_{i+\ell,k-\ell-1}(x) + \\
&+ (1 - \omega_{i+\ell+1,k-\ell}(x)) \cdot B_{i+\ell+1,k-\ell-1}(x).
\end{aligned}$$

Further we consider the following situations

2.1. If $1 \le \ell < \dfrac{k-1}{2}$ then $i < i+\ell+1 < i+k-\ell < i+k+1$, i.e.

$$t_i \le t_{i+\ell+1} \le t_{i+k-\ell} \le t_{i+k+1}\ .$$

We have tree posibilities

2.1.1. Suppose that $x \in [t_i, t_{i+\ell+1})$.

Using relation (1), Lemma 1 and Lemma 2 we deduce

$$\begin{aligned}
&0 \le \omega_{i,k-\ell}(x) \le 1\ ,\\
(16)\qquad &B_{i,k-\ell-1}(x) \ge 0\ ,\\
&B_{i+\ell+1,k-\ell-1}(x) = 0 \ \ (x \notin [t_{i+\ell+1}, t_{i+k+1}))\ .
\end{aligned}$$

From (15) and (16) we obtain that relation (14) is true for $x \in [t_i, t_{i+\ell+1})$.

2.1.2. Suppose that $x \in [t_{i+\ell+1}, t_{i+k-\ell})$.

Using relation (1), Lemma 1 and Lemma 2 it follows that

$$\begin{aligned}
&0 \le \omega_{i,k-\ell}(x) \le 1\ ,\\
&B_{i,k-\ell-1}(x) \ge 0\ ,\\
(17)\qquad &0 \le 1-\omega_{i+\ell+1,k-\ell}(x) \le 1\ ,\\
&B_{i+\ell+1,k-\ell-1}(x) \ge 0\ .
\end{aligned}$$

Using relations (15) and (17) we deduce that (14) is valid for $x \in [t_{i+\ell+1}, t_{i+k-\ell})$.

2.1.3. Suppose that $x \in [t_{i+k-\ell}, t_{i+k+1})$.

Using relation (1), Lemma 1 and Lemma 2 it follows that

$$
\begin{aligned}
& B_{i,k-\ell-1}(x) = 0 \ , \quad (x \notin [t_i, t_{i+k-\ell})) \\
(18) \qquad & 0 \le 1 - \omega_{i+\ell+1,k-\ell}(x) \le 1 \ , \\
& B_{i+\ell+1,k-\ell-1}(x) \ge 0 \ .
\end{aligned}
$$

Using (15) and (18) we obtain that relation (14) is true for $x \in [t_{i+k-\ell}, t_{i+k+1})$.

Hence, if $1 \le \ell < \dfrac{k-1}{2}$ then relation (14) holds for all $x \in [t_i, t_{i+k+1})$.

2.2. If $\dfrac{k-1}{2} \le \ell \le k-1$ then $i < i+k-\ell < i+\ell+1 < i+k+1$, i.e.

$$t_i \le t_{i+k-\ell} \le t_{i+\ell+1} \le t_{i+k+1} \ .$$

We have two posibilities

2.2.1. Suppose that $x \in [t_i, t_{i+\ell+1})$.

Using relation (1), Lemma 1 and Lemma 2 we obtain

$$
\begin{aligned}
& 0 \le \omega_{i,k-\ell}(x) \le 1 \ , \\
(19) \qquad & B_{i,k-\ell-1}(x) \ge 0 \ , \\
& B_{i+\ell+1,k-\ell-1}(x) = 0 \ \ (x \notin [t_{i+\ell+1}, t_{i+k+1})) \ .
\end{aligned}
$$

From (15) and (19) we deduce that the relation (14) is valid for $x \in [t_i, t_{i+\ell+1})$.

2.2.2. Suppose that $x \in [t_{i+\ell+1}, t_{i+k+1})$.

Using relation (1), Lemma 1 and Lemma 2 it follows that

$$
\begin{aligned}
& B_{i,k-\ell-1}(x) = 0 \ \ (x \notin [t_i, t_{i+k-\ell})) \ , \\
(20) \qquad & 0 \le 1 - \omega_{i+\ell+1,k-\ell}(x) \le 1 \ , \\
& B_{i+\ell+1,k-\ell-1}(x) \ge 0 \ .
\end{aligned}
$$

From (15) and (20) we deduce that the relation (14) is true for $x \in [t_{i+\ell+1}, t_{i+k+1})$.

Hence, if $\dfrac{k-1}{2} \le \ell \le k-1$ then the relation (14) holds for all $x \in [t_i, t_{i+k+1})$.

Therefore, relation (7) is true for all $x \in [t_i, t_{i+k+1})$ and $j \in \{1, \ldots, k\}$. ■

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