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On the inequality of Mathieu

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Abstract

In this paper we obtain a new generalization for Mathieu′s inequality.

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Mathieu [5] conjectured in 1890 that the inequality

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2}\ , \tag{1}$$

where c is a real number, $c \neq 0$, is valid.

The inequality (1) is called Mathieu′s. It was proved only in 1952 by L. Berg [2].

E. Makai [4] gave a very elegant and elementary proof for (1) and obtained the following lower estimation:

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} > \frac{1}{c^2+\frac{1}{2}}\ . \tag{2}$$

P.A. Diananda [3] refined Mathieu′s inequality to

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2} - \frac{1}{(2c^2+2c+1)(8c^2+8c+3)}\ .$$

Gh. Costovici [2] has recently proved the following inequalities of Mathieu type:

$$\sum_{n=1}^{\infty} \frac{4n(n+1)(n+2)}{(n^2+2n+c^2)^4} < \frac{1}{c^4}$$

and

$$\sum_{n=1}^{\infty} \frac{6n(n+1)(n+2)(n+3)(n+4)}{(n^2+5n+c^2)^6} < \frac{1}{c^6}\ .$$

The main purpose of this short paper is to generalize Mathieu′s inequality (1) and the inequality (2).

Theorem 1 *If c is a real number, $c \neq 0$, and p is a natural number, $p \geq 1$, then we have*

$$\frac{1}{\frac{(p-1)!(p+1)}{4}+c^2} < \sum_{n=1}^{\infty} \frac{(p+1)n_{[p]}}{\left[n_{[p]}\left(n+\frac{p-1}{2}\right)+c^2\right]^2} < \frac{1}{c^2}, \tag{3}$$

where $n_{[p]} = n(n+1)(n+2)\dots(n+p-1)$.

Proof. Using the notation

$$A_{n,p} = nn_{[p]} + \frac{p-1}{2}n_{[p]} + c^2 ,$$

we have:

$$\frac{1}{(n-1)n_{[p]}+c^2} - \frac{1}{n_{[p]}(n+p)+c^2} =$$

$$= \frac{(p+1)n_{[p]}}{[nn_{[p]}-n_{[p]}+c^2][nn_{[p]}+pn_{[p]}+c^2]} =$$

$$= \frac{(p+1)n_{[p]}}{\left[A_{n,p}-\frac{p+1}{2}n_{[p]}\right]\left[A_{n,p}+\frac{p+1}{2}n_{[p]}\right]} =$$

$$= \frac{(p+1)n_{[p]}}{A_{n,p}^2-\left(\frac{p+1}{2}\right)^2 n_{[p]}^2} > \frac{p+1}{A_{n,p}^2} =$$

$$= \frac{(p+1)n_{[p]}}{\left[n_{[p]}\left(n+\frac{p-1}{2}\right)+c^2\right]^2} .$$

Summing for $n = 1, 2, \dots,$ we obtain, for $c \neq 0$, and $p = 1, 2, \dots$:

$$\sum_{n=1}^{\infty} \frac{(p+1)n_{[p]}}{\left[n_{[p]}\left(n+\frac{p-1}{2}\right)+c^2\right]^2} < \frac{1}{c^2} ,$$

that is the inequality (3) of the right side.

Now, putting $B_{n,p} = \frac{n_{[p]}}{2n+p-1}\left(\frac{p+1}{2}\right)^2$, we get:

$$\frac{1}{(n-1)n_{[p]}+B_{n,p}+c^2} - \frac{1}{(n+p)n_{[p]}+B_{n,p}+c^2} =$$

$$= \frac{(p+1)n_{[p]}}{\left[A_{n,p}-\frac{p+1}{2}n_{[p]}+B_{n,p}\right]\left[A_{n,p}+\frac{p+1}{2}n_{[p]}+B_{n,p}\right]} =$$

$$= \frac{(p+1)n_{[p]}}{A_{n,p}^2 + 2c^2 B_{n,p} + B_{n,p}^2} < \frac{(p+1)n_{[p]}}{A_{n,p}^2} =$$

$$= \frac{(p+1)n_{[p]}}{\left[n_{[p]}\left(n + \frac{p-1}{2}\right) + c^2\right]^2} .$$

Summing for $n = 1, 2, \ldots$, we obtain

$$\sum_{n=1}^{\infty} \frac{(p+1)n_{[p]}}{\left[n_{[p]}\left(n + \frac{p-1}{2}\right) + c^2\right]^2} > \frac{1}{\frac{(p-1)!(p+1)}{2} + c^2},$$

that is the inequality (3) of the left side.

Remark 1 . *For $p = 1$ the inequalities (3) give the inequalities (1) and (2).*

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