General Mathematics Vol. 10, No. 1–2 (2002), 71–74

About one fixed point theorem

Amelia Bucur

Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

The aim of this paper is the study of a fixed point property for pairs of classes of topological spaces defined as follows:

for a class of sets φ and a set X we shall denote by

$$\mathcal{C}(X) = \{C \in \mathcal{C} : C \subset X\} \text{ and } \mathcal{C}^*(X) = \{C \in \mathcal{C}(X) : C \neq \varnothing\}.$$

We say that a map $T : X \to Y$ has $\mathcal{C}$ (resp. $\mathcal{C}^*$) values if for each $x \in X$, $T(x) \in \mathcal{C}(X)$ (resp. $T(x) \in \mathcal{C}^*(X)$).

2000 Mathematical Subject Classification: 46E99

1 Introduction

In what follows we use the following definition:

Definition 1. *We say that a pair $(\mathcal{T}, \mathcal{C})$ consisting of two classes of compact Hausdorff topological spaces has the fixed point property provided:*

i) $X, Y \in \mathcal{C}$ *implies* $X \times Y \in \mathcal{T}$;

ii) $C \in \mathcal{C}(X)$, $D \in \mathcal{C}(Y)$ *implies* $C \times D \in \mathcal{C}(X \times Y)$, *for each* $X, Y \in \mathcal{T}$*;*
iii) for each $X \in \mathcal{T}$, *any upper semicontinuous map* $T : X \to X$ *with* $\mathcal{C}^*$ *values has a fixed point.*

Example 1. Both $\mathcal{T}$ and $\mathcal{C}$ are the class of all compact convex subsets of all Hausdorff locally convex topological vector spaces. In this case condition (iii) in Definition is satisfied according to the Kakutami - Tan - Glicksberg fixed point theorem (see [2], [3]).

2 Fixed points

Let $\{X_i\}_{1 \leq i \leq n}$ be a finite family of sets ($n \geq 2$). Let

$$X = \prod_{i=1}^{n} X_i \text{ and } X^i = \prod_{\substack{j=1 \\ j \neq 1}}^{n} X_j.$$

Any $x = (x_1, x_2, ..., x_n) \in X$ can be expressed as $x = (x^i, x_i)$ for any $i \in \{1, 2, ..., n\}$, where x^i denotes the canonical projection of x on X^i.

Theorem 1. *Let* $X_i \in \mathcal{T}(1 \leq i \leq n)$ *and for each* $i \in \{1, 2, ..., n\}$ *let* $T_i : X \to X_i$ *be an upper semicontinuous map with* $\mathcal{C}^*$ *values. Then there exists an* $\widehat{x} \in X$ *such that* $\widehat{x}_i \in T_i(\widehat{x})$ *for each* $i \in \{1, 2, ..., n\}$.

The proof can be found in [1].

Theorem 2. *Let* $X_i \in \mathcal{T}(1 \leq i \leq n)$ *and* $S_i : X_i \to X_{i+1}(1 \leq i \leq n-1)$, $S_n : X_n \to X_1$ *be upper semicontinuous map with* $\mathcal{C}^*$ *values. Then the composite* $S_n \circ ... \circ S_1$ *has a fixed point.*

Proof. Let $T_i : X \to X_i(1 \leq i \leq n)$ be the maps defined by

$$T_1(x_1, x_2, ..., x_n) = S_n(x_n)$$

$$T_i(x_1, x_2, ..., x_n) = S_{i-1}(x_{i-1}) \text{ for } 2 \leq i \leq n.$$

By Theorem 1 there exists $\widehat{x} = (\widehat{x}_1, ..., \widehat{x}_n) \in X$ such that $\widehat{x}_1 \in T_1(\widehat{x}) = S_n(\widehat{x}_n)$, $\widehat{x}_2 \in T_2(\widehat{x}) = S_1(\widehat{x}_1)$, ..., $\widehat{x}_1 \in T_n(\widehat{x}) = S_{n-1}(\widehat{x}_{n-1})$, whence $\widehat{x}_1 \in (S_n \circ ... \circ S_1)(\widehat{x}_1)$.

Theorem 1 can be reformulated to the form of quasi-equilibrium theorem as follows:

Theorem 3. *For each $i \in \{1, 2, ..., n\}$ let $X_i \in \mathcal{T}$, $S_i : X \to X_i$ closed map and $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ upper semicontinuous functions. Suppose that for each $i \in \{1, 2, ..., n\}$ the following conditions are satisfied:*
a) $g_i(x) \leq f_i(x)$ for each $x \in X$;
b) the function M_i defined on X by $M_i(x) = \max\limits_{y \in S_i(x)} g_i(x_i, y)$ is lower semicontinuous;
c) for each $x \in X$, $\{y \in S_i(x) : f_i(x^i, y) \geq M_i(x)\} \in \mathcal{C}(X)$. Then there exists an $\widehat{x} \in X$ such that for each $i \in \{1, 2, ..., n\}$, $\widehat{x}_i \in S_i(\widehat{x})$ and $f_i(\widehat{x}_i, \widehat{x}_i) \geq M_i(\widehat{x})$.

Proof. For each $i \in \{1, 2, ..., n\}$ define a map $T_i : X \to X_i$ by $T_i(x) = \{y \in S_i(x) : f_i(x^i, y) \geq M_i(x)\}$ for $x \in X$.

Note that each $T_i(x)$ is non-empty by (i), since $S_i(x)$ is compact and $g_i(x^i, \cdot)$ is upper semicontinuous on $S_i(x)$. We prove that the graph of T_i is closed in $X \times X_i$. In view of this let $(x_\alpha, y_\alpha) \in$ graph T_i such that $(x_\alpha, y_\alpha) \to (x, y)$.

Then

$$f_i(x^i, y) \geq \limsup_{\alpha} f_i(x^i_\alpha, y_\alpha) \geq \limsup_{\alpha} M_i(x_\alpha) \geq \liminf_{\alpha} M_i(x_\alpha) \geq M_i(x)$$

and, since graph S_i is closed in $X \times X_i$, $y_\alpha \in S_i(x_\alpha)$ implies $y \in S_i(x)$. Hence $(x, y) \in$ graph T_i. Since X_i is compact, T_i is upper semicontinuous. By Theorem 1, there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for each

$i \in \{1, 2, ..., n\}$; that is $\widehat{x}_i \in S_i(\widehat{x})$ and $f_i(\widehat{x}^i, \widehat{x}_i) \geq M_i(\widehat{x})$. This completes the proof.

The origin of Theorem 3 is the Nash equilibrium theorem [4].

References

[1] Balaj M., *On fixed point property*, The 27th Annual Congress of the American Romanian Academy of Arts and Sciences, Oradea, 2002, 595 - 597.

[2] Fan K., *Fixed - point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci., U.S.A., 38, 1952, 121 - 126.

[3] Glicksberg I. L., *A further generalization of the Kakutami fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc., 3, 1952, 170 - 174.

[4] Nash J., *Non - cooperative games*, Ann. Math., 54, 1951, 286 - 295.

Department of Mathematics
"Lucian Blaga" University of Sibiu
Str. Dr. I. Raţiu, nr. 5 - 7
550012 - Sibiu, România
E-mail: *amelia.bucur@ulbsibiu.ro*