On Rational Maps with Two Critical Points

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This is a preliminary investigation of the geometry and dynamics of rational maps with only two critical points.

INTRODUCTION

We study rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $n \geq 2$ that are *bicritical*, that is, have only two critical points. Every rational map of degree two is bicritical; this case is discussed in [Milnor 1993; Rees 1990; $\geq$ 2000; Silverman 1998; Stimson 1993]. For $n > 2$, bicriticality is a very strong restriction. In fact bicritical maps seem to behave much more like quadratic rational maps than like general rational maps of degree n.

It is shown that the moduli space $\mathcal{M}$, consisting of all holomorphic conjugacy classes of bicritical maps of degree n, is biholomorphic to $\mathbb{C}^2$. Furthermore, the Julia set of a bicritical map is either connected, or totally disconnected and isomorphic to the one-sided shift on n symbols. In the latter case this Julia set can be either hyperbolic or parabolic. Correspondingly the moduli space splits as the disjoint union of the *connectedness locus*, the *hyperbolic shift locus*, and the *parabolic shift locus*:

$$\mathcal{M} = \mathcal{C} \cup \mathcal{S}_{\mathrm{hyp}} \cup \mathcal{S}_{\mathrm{par}}$$

$\mathcal{S}_{\mathrm{hyp}}$ is a connected open subset of $\mathcal{M}$ with free cyclic fundamental group, while $\mathcal{S}_{\mathrm{par}}$ is a codimension one subset, conformally isomorphic to $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$.

Remark 0.1. There is another interesting trichotomy obtained by considering the multipliers $\lambda_1, \ldots, \lambda_{n+1}$ at the various fixed points (see Section 2). If we assume that $|\lambda_j| \neq 1$ for all of these multipliers, there are three possibilities, as follows. If two of the fixed points are attracting, then we are in the *principal hyperbolic component*, and the Julia set is a quasicircle. If there is just one attracting fixed point, we are in the *polynomial-like* case, and can

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reduce to the polynomial case by a quasiconformal surgery. (If N is a compact neighborhood of the attracting point which contains exactly one critical value, with $f(N)$ compactly contained in N, then f carries $\widehat{\mathbb{C}} \smallsetminus f^{-1}(N)$ onto $\widehat{\mathbb{C}} \smallsetminus N$ by a map which is polynomial-like in the sense of [Douady and Hubbard 1985b].) Note that the hyperbolic shift locus is included here. Finally, it may happen that all $n+1$ fixed points are strictly repelling. This *essentially non-polynomial-like* case is the most interesting, since this is where we must look for any new or exotic behavior. (Compare Remark 3.2.)

In order to understand limiting behavior as the rational map becomes degenerate, it is convenient to introduce a partial compactification of moduli space by adding a line $L_\infty \cong \mathbb{C}$ of "points at infinity". The resulting *extended moduli space* $\widehat{\mathcal{M}} = \mathcal{M} \cup L_\infty$ fibers as a complex line bundle

$$\mathbb{C} \hookrightarrow \widehat{\mathcal{M}} \xrightarrow{X} \widehat{\mathbb{C}}$$

over the Riemann sphere, with Chern number equal to $n-1$. Here $X : \mathcal{M} \to \mathbb{C}$ is a certain conjugacy class invariant which can be described up to sign as a cross-ratio (see Lemma 1.7), and $X(L_\infty) = \infty$. The connectedness locus $\mathcal{C} \subset \mathcal{M}$ has compact closure within $\widehat{\mathcal{M}}$.

For each $\lambda \in \mathbb{C} \smallsetminus \{0\}$ the curve $\mathrm{Per}_1(\lambda)$ consisting of conjugacy classes of maps with a fixed point of multiplier λ forms a holomorphic section of the line bundle $\widehat{\mathcal{M}} \to \widehat{\mathbb{C}}$. Any two such sections have exactly $n-1$ intersections, counted with multiplicity. On the other hand, for $\lambda = 0$ the locus $\mathrm{Per}_1(0)$ is not a section, but rather coincides with a fiber

$$L_0 = \{\, (f) \in \widehat{\mathcal{M}} : X(f) = 0 \,\}.$$

This fiber can be identified with the set of conjugacy classes of unicritical polynomial maps $z \mapsto z^n + \text{constant}$.

The moduli space $\mathcal{M} \cong \mathbb{C}^2$ contains a *real subspace* $\mathcal{M}_{\mathbb{R}} \cong \mathbb{R}^2$. This consists not only of conjugacy classes of maps with real coefficients but also, when the degree n is odd, of a more exotic region consisting of conjugacy classes of maps f which commute with the antipodal map $z \mapsto -1/\bar{z}$ of the Riemann sphere. Such f give rise to dynamical systems on the nonorientable surface which is obtained by identifying z with $-1/\bar{z}$. Similarly the extended moduli space $\widehat{\mathcal{M}} \supset \mathcal{M}$ contains a real subset $\widehat{\mathcal{M}}_{\mathbb{R}}$ which fibers as a real line bundle

$$\mathbb{R} \hookrightarrow \widehat{\mathcal{M}}_{\mathbb{R}} \xrightarrow{X} \mathbb{R} \cup \{\infty\}$$

over the circle $\mathbb{R} \cup \{\infty\}$. Topologically, $\widehat{\mathcal{M}}_{\mathbb{R}}$ is either a cylinder or Möbius band according as n is odd or even.

1. CONJUGACY INVARIANTS AND THE MODULI SPACE $\mathcal{M}$

Let $\mathbf{Bicrit}_n$ be the space of all bicritical maps of degree $n \geq 2$. It is not hard to check that $\mathbf{Bicrit}_n$ is a smooth 5-dimensional complex manifold. By definition, two rational maps f and g are (holomorphically) *conjugate* if there exists a Möbius automorphism φ of the Riemann sphere so that $g = \varphi \circ f \circ \varphi^{-1}$. We are interested in the *moduli space* $\mathcal{M}$ consisting of all conjugacy classes (f) of degree n bicritical maps. The first two sections will provide a rather formal algebraic description of this space.

First consider the *marked* moduli space $\mathcal{M}'$, consisting of conjugacy classes of f with numbered critical points c_1, c_2. Here, by definition, the *conjugacy class* of (f, c_1, c_2) consists of all triples

$$\bigl(\varphi \circ f \circ \varphi^{-1},\ \varphi(c_1),\ \varphi(c_2)\bigr),$$

where φ ranges over Möbius automorphisms of the Riemann sphere $\widehat{\mathbb{C}}$. In order to construct a complete set of invariants for such an (f, c_1, c_2), we proceed as follows.

Lemma 1.1. *If we put the critical points c_1 at infinity and c_2 at zero, then f must have the form*

$$f(z) = \frac{az^n + b}{cz^n + d}, \tag{1–1}$$

with derivative

$$f'(z) = \frac{nz^{n-1}(ad - bc)}{(cz^n + d)^2}.$$

(There should be no confusion between the coefficient c and the critical points c_j.) Here the determinant $ad - bc$ must be nonzero. Note that this transformation depends only on the ratios $(a : b : c : d)$.

Proof. Write $f(z)$ as the quotient $p(z)/q(z)$ and look at the equation $f(z) = v$ or $p(z) - vq(z) = 0$. Suppose, to fix our ideas, that the two critical values $v_1 = f(0)$ and $v_2 = f(\infty)$ are finite. Then the polynomial $p(z) - v_2 q(z)$ has no finite roots, and hence must be constant. Similarly, $p(z) - v_1 q(z)$ has no nonzero roots, hence must have the form kz^m. Solving the resulting linear equations for $p(z)$ and $q(z)$, the conclusion follows easily. The case where v_1 or v_2 is infinite can be handled by a similar argument. □

Remark. The Julia set of a bicritical map of degree n always has an n-fold rotational symmetry about its critical points. If we use the normal form (1–1), this symmetry is expressed by the equation $f(\omega z) = f(z)$ and hence $J = \omega J$, whenever $\omega^n = 1$.

Theorem 1.2. *A complete set of conjugacy invariants for a map f in the normal form* (1–1), *with marked critical points at zero and infinity, is given by the expressions*

$$X = \frac{bc}{ad-bc}, \quad Y_1 = \frac{a^{n+1}b^{n-1}}{(ad-bc)^n}, \quad Y_2 = \frac{c^{n-1}d^{n+1}}{(ad-bc)^n}. \tag{1–2}$$

These invariants are subject to the relation

$$Y_1 Y_2 = X^{n-1}(X+1)^{n+1}, \tag{1–3}$$

but to no other relations. Hence the moduli space $\mathcal{M}'$ consisting of all such marked conjugacy classes is homeomorphic to the affine algebraic variety consisting of all $(X, Y_1, Y_2) \in \mathbb{C}^3$ satisfying equation (1–3).

As an example, the polynomial function $f(z) = z^n + b$ corresponds to the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

with invariants

$$X = Y_2 = 0, \quad Y_1 = b^{n-1}.$$

Figure 1(a) and Figure 2 on page 492 show the Y_1-coordinate plane and the b-coordinate plane for degree $n = 4$, in this locus $X = Y_2 = 0$ of polynomial maps.

Proof of Theorem 1.2. Multiplying the four coefficients in (1–1) by a common factor, we can normalize so that $ad - bc = 1$. The coefficients are then uniquely determined up to a common change of sign. Note that the expressions (1–2) are all invariant under this transformation, since numerator and denominator are homogeneous of the same degree. With this normalization, we can write (1–2) in the simpler form

$$\begin{aligned} X + 1 &= ad, & X &= bc, \\ Y_1 &= a^{n+1}b^{n-1}, & Y_2 &= c^{n-1}d^{n+1}. \end{aligned} \tag{1–4}$$

Such a normal form with critical points at zero and infinity is not unique, since we are still free to conjugate by a Möbius automorphism that fixes both zero and infinity. If we write such an automorphism as $\varphi(z) = z/t^2$, we must replace f by $\varphi \circ f \circ \varphi^{-1}(z) = f(t^2 z)/t^2$. A brief computation shows that the four coefficients, normalized so that the determinant remains $+1$, are then transformed by the rule

$$(a, b, c, d) \mapsto (t^{n-1}a,\ b/t^{n+1},\ t^{n+1}c,\ d/t^{n-1}). \tag{1–5}$$

It is clear that the three expressions (1–4) are invariant under this transformation (1–5), and also under a simultaneous change of sign for a, b, c, d, and that they satisfy the required relation (1–3).

Conversely, given (X, Y_1, Y_2) satisfying (1–3), we must show that there is one and only one corresponding choice of $\pm(a, b, c, d)$, up to the transformation (1–5).

Case 1. Suppose that $Y_2 \neq 0$, hence $c \neq 0$ and $d \neq 0$. Then using (1–5) we can make a linear change of variables so that $d = 1$. It follows from (1–4) that

$$a = X + 1, \qquad b = X/c, \qquad c^{n-1} = Y_2.$$

Thus we obtain a normal form which is uniquely determined by X and Y_2, up to a choice of $(n-1)$-st root for Y_2. With this choice of a, b, c, d, note that the relation $Y_1 = a^{n+1}b^{n-1}$ follows from (1–3). But applying (1–5) again with t equal to any $(2n-2)$-nd root of unity, since $t^{n-1} = \pm 1$, we see that

$$(a, b, c, d) \mapsto \pm(a,\ b/t^2,\ t^2 c,\ d),$$

where t^2 can be an arbitrary $(n-1)$-st root of unity. Therefore, the conjugacy class does not depend on a particular choice of $(n-1)$-st root c.

Case 2. If $Y_1 \neq 0$ the argument is similar.

Case 3. If $Y_1 = Y_2 = 0$, then X must be either 0 or -1 by equation (1–3). If $X = Y_1 = Y_2 = 0$, then making use of the hypothesis that $ad - bc = 1$ we see that $b = c = 0$ and that f is conjugate to

$$z \mapsto z^n.$$

On the other hand, if $X + 1 = Y_1 = Y_2 = 0$, then it follows similarly that $a = d = 0$ and that f is conjugate to the map

$$z \mapsto 1/z^n.$$

(Remark: These two exceptional points in moduli space will often require special attention.) This completes the proof. □

Now consider the quotient space of $\mathcal{M}'$ under the involution

$$(f, c_1, c_2) \leftrightarrow (f, c_2, c_1),$$

which interchanges the two critical points.

Corollary 1.3. *This quotient space* $\mathcal{M}$, *consisting of all holomorphic conjugacy classes of degree n bicritical maps, is biholomorphic to* $\mathbb{C}^2$, *with coordinates* X *and* $Y = Y_1 + Y_2$.

We will use the notation L_{X_0} for the complex line consisting of all $(f) \in \mathcal{M}$ with $X(f) = X_0$.

Proof. If f is given by (1–1), then a holomorphically conjugate map with critical points interchanged is given by

$$\frac{1}{f(1/z)} = \frac{dz^n + c}{bz^n + a}.$$

Thus $(a, b, c, d) \leftrightarrow (d, c, b, a)$ and $Y_1 \leftrightarrow Y_2$, with X fixed. We must form the quotient of $\mathcal{M}'$ under this involution. Since the product $Y_1 Y_2$ can be expressed as a smooth function of X, it is easy to check that the two quantities X and $Y = Y_1 + Y_2$ form a complete and independent set of invariants for the quotient variety $\mathcal{M}$. □

Remark. The algebraic variety (1–3) has a singular point at $X = -1$, $Y_1 = Y_2 = 0$, and (if $n \geq 3$) another singular point at $X = Y_1 = Y_2 = 0$. However, by passing to the quotient variety in which we unmark the critical points, these two singular points miraculously disappear.

Corollary 1.4. *The symmetry locus* $\Sigma \subset \mathcal{M}$, *consisting of all conjugacy classes of* f *which commute with some Möbius automorphism, is the variety defined by the equation*

$$Y^2 = 4\,X^{n-1}(X + 1)^{n+1}.$$

Proof. First suppose that there exists a non-trivial automorphism which fixes the two critical points. Then it must also fix the critical values, hence the set of critical points must coincide with the set of critical values. There are only two possibilities: Either f fixes both critical points hence (f) is the conjugacy class of $z \mapsto z^n$ with $X = Y = 0$, or else f interchanges the two critical points, hence (f) is the class of $z \mapsto 1/z^n$ with $X + 1 = Y_1 = Y_2 = 0$. (In these two exceptional cases, the group of automorphims fixing the critical points is cyclic of order $n-1$ or $n+1$ respectively, and the full group of automorphisms is dihedral of order $2(n-1)$ or $2(n+1)$ respectively.)

If we exclude these two cases, then a non-trivial automorphism ι commuting with f must be an involution which interchanges the two critical points, and must be unique. It is easy to see that such an involution exists if and only if $Y_1 = Y_2 = Y/2$. Since the two exceptional cases also satisfy this equation, the conclusion then follows from (1–3). □

Remark 1.5. When n is odd, this symmetry locus is a reducible variety, splitting as $\Sigma_+ \cup \Sigma_-$ where $\Sigma_\pm$ is defined by

$$Y = \pm 2\,X^{(n-1)/2}(X + 1)^{(n+1)/2}.$$

In fact a class $(f) \in \mathcal{M}$ can be symmetric in two essentially different ways when n is odd. Let ι be the (usually unique) involution which commutes with f. Then the two fixed points of ι are also fixed by f when $(f) \in \Sigma_+$, but are interchanged by f when $f \in \Sigma_-$. This can be proved by using the normal form (1–1), taking $\iota(z)$ to be $1/z$, so that

$$f(z) = \pm\frac{az^n + b}{bz^n + a}. \tag{1–6}$$

(Note that the two exceptional conjugacy classes, where the involution ι is not uniquely determined, constitute the intersection $\Sigma_+ \cap \Sigma_-$.) For $(f) \in \Sigma_+$, computation shows that the two invariant fixed points have multipliers λ_1 and λ_2 with sum $\lambda_1 + \lambda_2 = 2n(2X + 1)$ and with product $\lambda_1\lambda_2 = n^2$. On the other hand, for $(f) \in \Sigma_-$ the two fixed points of ι constitute a period two orbit for f with multiplier n^2. It follows that Σ_- is contained as one irreducible component in the curve $\mathrm{Per}_2(n^2)$ of Section 8.

Evidently, the two halves of the symmetry locus represent quite different dynamic behavior. For example if $(f) \in \Sigma_-$ then there are either two nonrepelling fixed points or none. In either case, it follows that the Julia set is connected. On the other hand, if we use the normal form (1–6) with the positive choice of sign, a brief computation shows that the multiplier at the fixed point $1 = f(1) = \iota(1)$ equals $n(a-b)/(a+b)$. Whenever this fixed point is attracting, it follows by symmetry that both critical points must lie in its immediate basin. Using Theorem B.5 (Appendix B), it follows that the Julia set is totally disconnected.

For n even, the symmetry locus is irreducible, conformally isomorphic to a punctured plane. In fact each $(f) \in \Sigma$ has a unique fixed point which is invariant under the involution. The multiplier λ at this fixed point can take any nonzero value, and computation shows that $X = (\lambda/n + n/\lambda - 2)/4$ is then uniquely determined. (The correspondence $\lambda \mapsto X$ is two-to-one, since a generic fiber L_X intersects Σ in two different points.)

Remark 1.6. Adam Epstein made the following observation. Again let ι be the (usually unique) involution commuting with f. Then there is a natural involution of the symmetry locus given by $(f) \mapsto (f^\star)$, where $f^\star = f \circ \iota = \iota \circ f$ has the same Julia set as f. The map $f^\star$ has invariants $X^\star = -1 - X$ and $Y^\star = (-1)^n YX/(1 + X)$. When n is odd, note that this involution maps each irreducible component $\Sigma_\pm$ to itself. Using the normal form (1–6), this involution interchanges the coefficients a and b.

An interesting involution of the entire moduli space $\mathcal{M}$ is given by the correspondence $(f) \mapsto (J \circ f)$, where $J = J_f$ is the unique involution of the Riemann sphere which fixes the two critical values. If f is given by (1–1), then

$$J \circ f(z) = \frac{az^n - b}{cz^n - d},$$

with $X(J \circ f) = X(f)$ and $Y_i(J \circ f) = -Y_i(f)$. Thus this involution maps the symmetry locus to itself, interchanging Σ_+ and Σ_- in the odd degree case.

The correspondence $(f) \mapsto X(f)$ is rather natural, and can be defined in several different ways. For example, we will see in Section 2 that $X(f)$ is linearly related to the sum of the multipliers at the various fixed points of f. The next lemma offers another example, involving a cross-ratio formula. (Compare Appendix C.)

Lemma 1.7. *If f is a rational map with critical points c_1, c_2 and with critical values $v_j = f(c_j)$, then the invariant $X = X(f)$ is equal to the negative of the cross-ratio*

$$\frac{(c_1 - v_1)\,(c_2 - v_2)}{(c_1 - c_2)\,(v_1 - v_2)}.$$

Proof. This cross-ratio is clearly well defined and invariant under conjugation. (Note that the denominator never vanishes.) Putting the critical points at $c_1 = \infty$ and $c_2 = 0$, the left hand factors cancel and the cross-ratio reduces to

$$\frac{0 - v_2}{v_1 - v_2} = \frac{-b/d}{a/c - b/d} = \frac{-bc}{ad - bc},$$

as required. □

For further cross-ratio formulas, see Appendix C.

Corollary 1.8. *Denote the modulus of an annulus $A \subset \widehat{\mathbb{C}}$ by $\mathrm{mod}A$, and let $\mathrm{mod}f \ge 0$ be the largest possible modulus of an annulus in $\widehat{\mathbb{C}}$ which separates the critical values of f from the critical points of f (taking $\mathrm{mod}f = 0$ when there is no such annulus). Given a sequence of conjugacy classes $(f_i) \in \mathcal{M}$, the invariants $|X(f_i)|$ tend to infinity if and only if the invariants $\mathrm{mod}f_i$ tend to infinity.*

Proof. In fact we will show that

$$\frac{\log r}{2\pi} \le \mathrm{mod}f \le \mathrm{mod}\big(\widehat{\mathbb{C}} \smallsetminus ([-1, 0] \cup [r, +\infty])\big),$$

where $r = |X|$, and where both the upper and the lower bound tend to infinity as $r \to \infty$. After a Möbius automorphism, we may assume that the critical points are located at $0, -1$ and the corresponding critical values at X, ∞. Now the lower bound is obtained by using the round annulus $\{z : 1 < |z| < r\}$, while the upper bound follows from [Ahlfors 1966, Ch. III]. □

The space $\mathbb{C}^2$ is an extremely flabby object, with a very large group of holomorphic automorphisms. In Section 6 we will impose a much more rigid structure on the moduli space $\mathcal{M} \cong \mathbb{C}^2$ by partially compactifying it. The invariant X will play a key role in this partial compactification, since it will serve as the projection map of a canonical fibration, with typical fiber

$$L_{X_0} = \{(f) : X(f) = X_0\}.$$

2. FIXED POINTS AND THE CURVES $\mathrm{Per}_1(\lambda)$

Recall that the *multiplier* of a rational map f at a finite fixed point $z = f(z)$ is defined to be the first derivative $\lambda = f'(z)$. (In the case of a fixed point at infinity the multiplier is equal to the limit of $1/f'(z)$ as $z \to \infty$.) We first prove the following.

Lemma 2.1. *Let f be a bicritical map of degree n with invariants X and Y. If f has a fixed point of multiplier λ, then the product $\lambda^n Y$ can be expressed as a polynomial function of degree $2n$ in the variables X and λ.*

Definition. Let $\mathrm{Per}_1(\lambda) \subset \mathcal{M}$ be the set of all conjugacy classes of bicritical maps which admit a fixed point of multiplier λ.

Corollary 2.2. *For $\lambda \neq 0$, the curve $\mathrm{Per}_1(\lambda)$ can be described as the graph of a polynomial function $Y =$ $\mathrm{polynomial}_\lambda(X)$. In particular, for $\lambda \neq 0$, each fiber $L_{X_0} = \{(f) : X(F) = X_0\}$ contains one and only one conjugacy class (f) of maps which have a fixed point of multiplier λ.*

Proof of Lemma 2.1 and Corollary 2.2. We will use the normal form (1–1). First suppose that $\lambda \neq 0$. Then the fixed point of multiplier λ must be distinct from the two critical points 0 and ∞. After a linear change of coordinates, we may assume that this fixed point is $z = 1$. Thus $1 = f(1) = (a+b)/(c+d)$. Multiplying the coefficients by a common factor, we may assume that $a + b = c + d = 2$. If we define parameters μ and ξ by the equations

$$\begin{aligned} a &= 1+\mu+\xi, & b &= 1-\mu-\xi, \\ c &= 1-\mu+\xi, & d &= 1+\mu-\xi, \end{aligned} \tag{2–1}$$

a straightforward computation shows that $ad - bc = 4\mu$,

$$\lambda = f'(1) = \frac{n(ad-bc)}{(c+d)^2} = n\mu, \quad X = \frac{(1-\mu)^2 - \xi^2}{4\mu},$$

and that Y equals

$$\frac{(1+\mu+\xi)^{n+1}(1-\mu-\xi)^{n-1} + (1-\mu+\xi)^{n-1}(1+\mu-\xi)^{n+1}}{(4\mu)^n}.$$

Thus $\mu^n Y$ is equal to a polynomial of degree $2n$ in the variables μ and ξ. Note that this expression for Y is unchanged if we replace ξ by $-\xi$. (The involution $\xi \leftrightarrow -\xi$ corresponds to the conjugation $f(z) \leftrightarrow 1/f(1/z)$.) Hence $\mu^n Y$ can be expressed as a polynomial function of μ and ξ^2. Substituting $\xi^2 = (1-\mu)^2 - 4\mu X$ and $\mu = \lambda/n$, we obtain the required polynomial expression for $\lambda^n Y$, of degree n in λ and X. This proves Corollary 2.2.

To prove Lemma 2.1, we must also check that this same polynomial relation remains valid when f has a fixed point (necessarily 0 or ∞) of multiplier $\lambda = 0$. In that case, the product $\lambda^n Y$ is certainly zero, and $X = 0$ so that $\xi = \pm 1$, and the numerator of the expression for Y is identically zero, as required. □

We can describe the form of these polynomial relations more precisely as follows. We will continue to work with the quotient $\mu = \lambda/n$.

Theorem 2.3. *For each n there are polynomials $P_0(X)$, $P_1(X)$, $P_2(X)$, $\dots$, $P_{n+1}(X)$ in $\mathbb{Z}[X]$, each $P_k(X)$ having degree $\leq k$, so that*

$$\mu Y = P_{n+1}(X) - \mu P_n(X) + \mu^2 P_{n-1}(X) - + \cdots + (-\mu)^n P_1(X) + (-\mu)^{n+1} P_0(X).$$

As explicit examples, we have

$$\mu\, Y = X - \mu\, X(2X-1) + \mu^2(4X+1) - \mu^3$$

for $n = 2$, and

$$\mu\, Y = X^2 - 2\mu\, X^2(X-1) + \mu^2(9X^2+4X+1) - 2\mu^3(3X+1) + \mu^4$$

for $n = 3$.

Proof of Theorem 2.3. From the proof of Lemma 2.1, it follows easily that we can define polynomials $P_j(X)$ by the formula

$$\mu Y = \sum_{i+j=n+1} (-\mu)^i P_j(X), \tag{2–2}$$

where $0 \leq j \leq 2n$ or equivalently $n+1 \geq i \geq 1-n$, and where each $P_j(X)$ has degree $\leq j$. What is new in Theorem 2.3 is the statement that the P_j have integer coefficients, and that $P_j = 0$ for $j > n+1$. To prove this, we will derive the same formula (2–2) in a different way. Again we use the normal form (1–1) with a fixed point of multiplier $\lambda \neq 0$ at $z = 1$, but now we normalize so that $ad - bc = 1$, and set $u = a + b = c + d$. Then computation shows that $\mu = \lambda/n = 1/u^2$. Furthermore

$$1 = ad - bc = (u-b)(u-c) - bc = u^2 - (b+c)u,$$

or in other words

$$b + c = u - 1/u.$$

Since $bc = X$, it follows that b and c are the two roots of the equation

$$b^2 - (u - 1/u)b + X = 0.$$

Thus

$$2b = u - 1/u \pm \sqrt{(u - 1/u)^2 - 4X}.$$

For $|u|$ large, each of the two solutions b and c can be expressed as a Laurent series in u with coefficients depending on X. One solution has the form

$$b = \frac{X}{u} + \frac{X(1{+}X)}{u^3} + \frac{X(1{+}X)(1{+}2X)}{u^5} + \cdots,$$

tending to zero as $u \to \infty$, where the successive coefficients are polynomials in X with integer coefficients which can be computed by a straightforward induction. The other solution is then given by $c = u - 1/u - b = u - (1 + X)/u - \cdots$, and is asymptotic to u. From these we can compute

$$Y = (u - b)^{n+1} b^{n-1} \; + \; (u - c)^{n+1} c^{n-1}$$

as a Laurent series in u. This series begins as

$$\begin{aligned} Y &= u^2 X^{n-1} - X^{n-1}(2X{+}1{-}n) \\ &\qquad + (n^2 X^{n-1}{+}\cdots{+}1)/u^2 + O(1/u^4) \\ &= X^{n-1}/\mu - X^{n-1}(2X{+}1{-}n) \\ &\qquad + (n^2 X^{n-1}{+}\cdots{+}1)\,\mu + O(\mu^2), \end{aligned}$$

or in other words as

$$\begin{aligned} \mu\, Y = X^{n-1} - X^{n-1}(2X{+}1{-}n)\,\mu \\ + (n^2 X^{n-1}{+}\cdots{+}1)\,\mu^2 + O(\mu^3) \end{aligned} \quad (2\text{–}3)$$

as $\mu \to 0$. Thus there are no terms in μ^i with $i < 0$. This proves that formula (2–2) reduces to the required form. ☐

Here is some more precise information about the polynomials $P_k(X)$. It will be convenient to introduce the abbreviation

$$\beta(m,k) = \binom{m-k}{k} \; + \; \binom{m-k-1}{k-1}$$

for the sum of two binomial coefficients.

Lemma 2.4. *Each $P_k(X)$ with $0 \le k \le n$ is a polynomial of degree k, however $P_{n+1}(X)$ is a polynomial of degree $n - 1$. We have*

$$\begin{aligned} P_0(X) &= 1, \\ P_1(X) &= 2nX + (n-1), \\ &\cdots \\ P_{n-1}(X) &= n^2 X^{n-1} + \sum_{j=0}^{n-2} \binom{n+1}{j} X^j, \\ P_n(X) &= 2X^n \, + \, (1-n)\, X^{n-1}, \\ P_{n+1}(X) &= X^{n-1}. \end{aligned}$$

The constant term $P_k(0)$ in each of these polynomials is equal to the binomial coefficient $\binom{n-1}{k}$, while the coefficient of the degree k term is equal to the sum $\beta(2n,k)$.

Proof outline. The explicit formulas for $P_k(X)$ with $k \ge n-1$ can be derived from the computation (2–3). For the remaining information, we again use the normal form (1–1) with $ad - bc = 1$ and with $u = a + b = c + d$. Let $s_k = b^k + c^k$, where $s_1 = b + c = u - 1/u$. Starting from the Newton formula

$$s_{k+1} = (b+c)s_k \, - \, (bc)\, s_{k-1} = (u - u^{-1})s_k - X\, s_{k-1},$$

it follows inductively that we can express each s_k as a polynomial function of $u - u^{-1}$ and X. The precise formula is

$$s_k = \sum_{0 \le j \le k/2} \beta(k,j)\,(-X)^j\,(u - u^{-1})^{k-2j}.$$

(Note that this computation is independent of the degree n.) Equivalently, recalling that $\mu = 1/u^2$, we can write

$$\frac{s_k}{u^k} = \sum_{0 \le j \le k/2} \beta(k,j)\,(-\mu\, X)^j\,(1-\mu)^{k-2j}. \quad (2\text{–}4)$$

Now we can compute

$$\begin{aligned} Y &= a^{n+1} b^{n-1} + c^{n-1} d^{n+1} \\ &= (u-b)^{n+1} b^{n-1} + c^{n-1}(u-c)^{n+1} \\ &= u^{n+1} s_{n-1} - \tbinom{n+1}{1} u^n s_n + \tbinom{n+1}{2} u^{n-1} s_{n+1} \\ &\qquad\qquad - + \cdots \pm s_{2n}, \end{aligned}$$

or equivalently

$$\begin{aligned} \mu^n Y = \frac{s_{n-1}}{u^{n-1}} - \binom{n+1}{1}\frac{s_n}{u^n} + \binom{n+1}{2}\frac{s_{n+1}}{u^{n+1}} \\ - + \cdots \pm \frac{s_{2n}}{u^{2n}}. \end{aligned}$$

Substituting (2–4) into this last equation, we obtain a fairly explicit formula for $\mu^n Y$. Further details will be left to the reader. ☐

As an application, we can give a more precise form of Corollary 2.2.

Corollary 2.5. *Each* $\mathrm{Per}_1(\lambda) \subset \mathcal{M}$ *with* $\lambda \neq 0$ *can be described a smooth curve of the form*

$$Y = -2X^n + \big(n(\lambda+\lambda^{-1}+1)-1\big)X^{n-1} + \cdots + \lambda(n-\lambda)^{n-1}/n^n.$$

If $\lambda \neq \lambda'$ *with* $\lambda\lambda' \neq 0, 1$, *then it follows easily that the curves* $\mathrm{Per}_1(\lambda)$ *and* $\mathrm{Per}_1(\lambda')$ *have exactly* $n-1$ *points of intersection, counted with multiplicity.*

Proof. The first statement is proved by plugging the explicit values from Lemma 2.4 into the equation of Theorem 2.3. It follows that each intersection $\mathrm{Per}_1(\lambda) \cap \mathrm{Per}_1(\lambda')$ with $\lambda \neq \lambda'$ is described by a polynomial equation of the form

$$n\Big(\lambda + \frac{1}{\lambda} - \lambda' - \frac{1}{\lambda'}\Big) X^{n-1} + \text{(lower terms)} = 0.$$

In the generic case where $\lambda\lambda' \neq 0$, 1, this equation has degree $n-1$, and the assertion follows. □

(On the other hand, if $\lambda\lambda' = 1$ the leading coefficient of this polynomial equation is zero. In this case, we must count one or more "intersections at infinity" in order to get the right number. (See Section 6.) A more significant exception occurs when $\lambda' = 0$. In fact, to make the count come out right, we should identify $\mathrm{Per}_1(0)$ with the curve $X^{n-1} = 0$, or in other words with the locus L_0 counted $n-1$ times. (Compare Section 8.) In fact, as $\lambda' \to 0$ with Y bounded, the locus $\mathrm{Per}_1(\lambda')$ degenerates towards an $(n-1)$-sheeted covering of the locus $X = 0$. towards the curve $X^{n-1} = 0$ of multiplicity $n-1$.

Definition. Every rational map f of degree n has $n+1$ fixed points counted with multiplicity. Let $\lambda_1, \ldots, \lambda_{n+1}$ be the multipliers at these fixed points, and let

$$\sigma_k = \sum_{1 \le i_1 < \cdots < i_k \le n+1} \lambda_{i_1} \cdots \lambda_{i_k}$$

be the k-th *elementary symmetric function* of these multipliers. It is convenient to set $\sigma_0 = 1$. Note that the quotient σ_k/n^k can be described as the k-th elementary symmetric function of the quotients $\mu_i = \lambda_i/n$.

Theorem 2.6. *These elementary symmetric functions can be computed by the formula*

$$\begin{aligned} \sigma_k/n^k &= P_k(X) \quad \text{for } \ 0 \le k \le n{+}1, \ k \neq n, \\ \sigma_n/n^n &= P_n(X) + Y. \end{aligned} \tag{2–5}$$

Proof. Each of the $n+1$ multipliers $\lambda = \lambda_k$ trivially satisfies the polynomial equation

$$\lambda^{n+1} - \sigma_1\lambda^n + \sigma_2\lambda^{n-1} - + \cdots \pm \sigma_{n+1} = \prod_{k=1}^{n+1} (\lambda - \lambda_k) = 0.$$

Hence the quotient $\mu = \lambda/n$ satisfies

$$\mu^{n+1} - \sigma_1\mu^n/n + \sigma_2\mu^{n-1}/n^2 - + \cdots \pm \sigma_{n+1}/n^{n+1} = 0.$$

On the other hand, from Theorem 2.3 we see that

$$\mu^{n+1} - P_1(X)\mu^n + P_2(X)\mu^{n-1} - + \cdots \mp (P_n(X) + Y)\mu \pm P_{n+1}(X) = 0.$$

The difference of these two polynomial equations is a polynomial of degree n in μ which vanishes at all $n+1$ of the μ_i. If the μ_i are pairwise distinct (or in other words if the multipliers $\lambda_i = n\mu_i$ are pairwise distinct), then it follows immediately that corresponding coefficients are equal, which proves (2–5). These identities follow in the general case by continuity or by analytic continuation, since for generic $(f) \in \mathcal{M}$ the $n+1$ multipliers are indeed distinct. To prove this, we need only construct a single example where the multipliers are distinct. For example if $f(z) = z^n + b$ then the multipliers are distinct provided that we exclude n very special values of the parameter b. First we must guarantee that $(f) \neq \mathrm{Per}_1(1)$, in order to be sure that the $n+1$ fixed points are distinct. But if $(f) \in \mathrm{Per}_1(1)$, then the fixed point equation $b = z - z^n$ together with the multiplier equation $nz^{n-1} = 1$ imply that the invariant $Y = b^{n-1}$ is equal to $(n-1)^{n-1}/n^n$. Finally, we must choose $b \neq 0$ to guarantee that two distinct fixed points, say $z \neq \omega z$, cannot have the same multiplier $\lambda = nz^{n-1} = n(\omega z)^{n-1}$. But this would imply that $\omega^{n-1} = 1$, and the fixed point equation $z - z^n = b$ would then yield $(\omega z) - (\omega z)^n = \omega b \neq b$, provided that $b \neq 0$. Also, no finite fixed point has multiplier zero provided that $b \neq 0$. Thus generically the multipliers are distinct, which completes the proof. □

Remark 2.7. As an immediate corollary of Lemma 2.4 and Theorem 2.6: *We could equally well use the two invariants σ_1 and σ_n as coordinates for the moduli space $\mathcal{M} \cong \mathbb{C}^2$, in place of the invariants X and Y of Section* 1. (In practice, in Section 6, it will be convenient to use X and σ_n as coordinates.)

Remark 2.8. It seems surprising that every one of the elementary symmetric functions σ_k with $k \neq n$ can be expressed as a function of X alone. Only σ_n depends also on the coordinate Y. As an example to illustrate this statement, consider the family of unicritical polynomials

$$f(z) = z^n + b,$$

with invariants $X = 0$ and $Y = b^{n-1}$. For the special case $b = 0$, there are two fixed points of multiplier zero and $n-1$ fixed points of multiplier n, hence

$$\sigma_k/n^k = \binom{n-1}{k}$$

for every k. It follows from Theorem 2.6 that this same formula holds for any value of the parameter b, provided that $k \neq n$. On the other hand for $k = n$, since this binomial coefficient is zero, it follows that

$$\sigma_n/n^n = Y = b^{n-1}.$$

For example in the quadratic case $f(z) = z^2 + b$, it follows that the multipliers at the finite fixed points satisfy $\lambda_1 + \lambda_2 = 2$ and $\lambda_1\lambda_2 = 4b$.

Remark 2.9. The *holomorphic fixed point formula* asserts that

$$\sum_1^{n+1} \frac{1}{1-\lambda_j} = 1$$

if $\lambda_j \neq 1$ for all j. (See [Milnor 1999], for example.) This gives rise to a linear relation between the σ_k, or equivalently between the P_k, which takes the form

$$\sum_0^{n+1} (-1)^k (n-k)\,\sigma_k = \sum_0^{n+1} (-n)^k (n-k)\,P_k(X) = 0.$$

It follows by continuity that this relation still holds also when some of the λ_j equal 1. Note that the invariant Y is not involved, since the coefficient of σ_n in this formula is zero.

Remark 2.10. It is sometimes convenient to consider the moduli space for bicritical maps with one marked fixed point. In this case, a complete set of invariants is provided by X and Y together with the multiplier $\lambda = n\,\mu$ at this marked point. These are subject only to the relation $\mu Y = X^{n-1} - \mu P_n(X) + \mu^2 P_{n-1}(X) - + \cdots + (-\mu)^{n+1}$. We can understand the topology of the resulting variety better by introducing a new coordinate $Y_\mu = Y + P_n(X) - \mu P_{n-1}(X) + - \cdots + (-\mu)^n$ in place of Y. Then X, Y_μ and μ are subject only to the relation $\mu\, Y_\mu = X^{n-1}$. For $n \geq 3$ this variety has a singular point at $X = Y = Y_\mu = \mu = 0$.

3. SHIFT LOCUS OR CONNECTEDNESS LOCUS

By definition, a conjugacy class (f) of degree n rational maps belongs to the *connectedness locus* $\mathcal{C}$ if the Julia set J_f is connected; and belongs to the *shift locus* $\mathcal{S}$ if J_f is totally disconnected with $f|_{J_f}$ topologically conjugate to the one-sided shift on n symbols. This section will prove that every conjugacy class of maps with only two critical points must belong to one or the other:

Theorem 3.1. *Every $(f) \in \mathcal{M}$ belongs either to the connectedness locus or to the shift locus.*

Note. If (f) belongs to the shift locus, then evidently both critical points belong to the Fatou set $\widehat{\mathbb{C}} \smallsetminus J_f$, which is connected but far from simply connected. There are two possibilities. If f has an attracting fixed point, and hence is hyperbolic on its Julia set, then we will say that (f) belongs to the *hyperbolic shift locus* $\mathcal{S}_{\mathrm{hyp}}$. Otherwise, f must have a parabolic fixed point, and we will say that (f) belongs to the *parabolic shift locus* $\mathcal{S}_{\mathrm{par}}$. Thus the moduli space partitions as a disjoint union

$$\mathcal{M} = \mathcal{C} \cup \mathcal{S}_{\mathrm{hyp}} \cup \mathcal{S}_{\mathrm{par}}.$$

We will explore this partition of $\mathcal{M}$ further in Sections 4 and 7.

Remark 3.2. In contrast with the polynomial case, we will see that the connectedness locus is neither closed nor bounded in $\mathcal{M}$ (although it has compact closure in the extended moduli space $\widehat{\mathcal{M}}$). In analogy with the polynomial case, one might be tempted to conjecture that the interior of the connectedness locus consists only of hyperbolic maps. In fact this conjecture is true if we restrict attention to the open subset consisting of (f) with at least one attracting fixed point. (Compare Remark 0.1.) However, the connectedness locus also contains an "essentially non-polynomial-like" region $\mathcal{C}_{\mathrm{NP}}$ consisting of maps for which all $n+1$ fixed points are strictly repelling. This region is certainly contained in the interior of the connectedness locus, and yet contains many non-hyperbolic maps. (Compare [Rees 1986].)

Here is one example. If $f(z) = \kappa + (1-\kappa)/z^n$, with κ an n-th root of unity distinct from 1, the critical points are 0 and ∞ with $0 \mapsto \infty \mapsto \kappa \mapsto 1$, so that both critical orbits eventually land at the repelling fixed point 1. It follows that the Julia set is the entire Riemann sphere, and that all periodic orbits are strictly repelling. (This map lies in the locus $X = -1$, where one critical point maps directly to the other. Compare [Bamón and Bobenrieth 1999].) *It is conjectured that the region* $\mathcal{C}_{\mathrm{NP}}$ *is a topological* 4*-cell.* When $n = 2$, this can be proved as follows. Let $I_j = 1/(1-\lambda_j)$ be the holomorphic fixed point index at the j-th fixed point. (Compare Remark 2.9.) Then this region in moduli space can be identified with the star shaped region consisting of unordered triples of complex numbers I_j with $0 < \operatorname{Re} I_j < \frac{1}{2}$ and $I_1 + I_2 + I_3 = +1$. On the other hand, for $n > 2$, I don't know even whether $\mathcal{C}_{\mathrm{NP}}$ is simply connected. Evidently an understanding of the topology and dynamics associated with this region $\mathcal{C}_{\mathrm{NP}}$ would be fundamental in reaching an understanding of bicritical maps.

The two key ingredients in the proof of Theorem 3.1 are the following:

Theorem A.1 (Shishikura). *A rational map with two critical points cannot have any Herman rings.*

A proof of this statement can be extracted from [Shishikura 1987], although it is not explicitly stated there. See Appendix A for a proof that does not use quasiconformal surgery.

Theorem B.5 (Przytycki and Makienko). *If a map f with two critical points has the property that both critical values lie in a common Fatou component, then* $\langle f \rangle$ *belongs to the shift locus.*

More generally, it is shown in [Przytycki 1996] and in [Makienko 1995] that any rational map with all critical values in a single Fatou component is isomorphic, when restricted to its Julia set, to the one-sided shift. However, since their argument is rather complicated, and since we need only the bicritical case, a proof of Theorem B.5 is given in Appendix B.

Remark 3.3. Here is an alternative statement: *Suppose that both critical orbits are eventually absorbed by an invariant Fatou component, $\Omega = f(\Omega)$. Then* $\langle f \rangle$ *belongs to the shift locus.* In fact such a Fatou component must contain at least one of the two critical points, and hence must be fully invariant, $\Omega = f^{-1}(\Omega)$. Hence it contains both critical values, and Theorem 3.1 applies. (By way of contrast, a map with three critical points may well have connected Julia set, even though all critical orbits are eventually absorbed by an invariant Fatou component. For example the map $f(z) = 2 + 2z^3/(27(2-z))$ has critical points $0, 3, \infty$ with orbit $3 \mapsto 0 \mapsto 2 \mapsto \infty$ ending on a superattractive fixed point. The immediate basin of infinity contains no other critical point, since no critical orbit converges non-trivially to infinity, hence this basin is simply connected. It follows, as in the proof of Theorem 3.1, that every Fatou component is simply connected.)

The proof of Theorem 3.1 will also use the following elementary observation.

Lemma 3.4. *Let $P \subset \widehat{\mathbb{C}}$ be a region bounded by a simple closed curve which passes through neither critical value. Then the pre-image of P under f can be described as follows.*

1. *If P contains no critical value, $f^{-1}(P)$ consists of n disjoint simply-connected regions bounded by n disjoint simple closed curves.*
2. *If P contains just one critical value, $f^{-1}(P)$ is a single simply connected region bounded by a simple closed curve, and maps onto P by a ramified n-fold covering.*
3. *If P contains both critical values, $f^{-1}(P)$ is a multiply connected region with n boundary curves, and maps onto P by a ramified n-fold covering.*

Proof. The proof is straightforward. Cases 0 and 2 correspond to the "inside" and "outside" of the same simple closed curve. In the case of just one critical point in P, the set $f^{-1}(P)$ must be connected since f is locally n-to-one near a critical point, and the branched covering $f^{-1}(P) \to P$ is unique up to isomorphism, since the fundamental group of $P \smallsetminus$ (critical value) is free cyclic, so that there is only one n-fold covering of this set up to isomorphism. $\square$

Proof of Theorem 3.1. Suppose that (f) is not in the shift locus, and hence that no Fatou component contains more than one critical value. If L is a loop in an arbitrary Fatou component, then using Sullivan's Non-Wandering Theorem we see that some forward image $f^{\circ k}(L)$ lies in a simply connected region U which is either

(1) a linearizing neighborhood of some geometrically attracting periodic point,
(2) a Böttcher neighborhood of some superattracting periodic point,
(3) an attracting petal for a parabolic point, or
(4) a Siegel disk.

Here we are using the fact that there are no Herman rings (Theorem A.1). Using Lemma 3.4, it follows by induction on k that each component of $f^{-k}(U)$ is simply connected. This proves that every Fatou component is simply connected, and hence that the Julia set is connected. □

4. THE PARABOLIC SHIFT LOCUS $\mathcal{S}_{\rm par} \cong \mathbb{C} \smallsetminus \overline{\mathbb{D}}$

Recall from Section 3 that the moduli space $\mathcal{M}$ splits as a disjoint union

$$\mathcal{M} = \mathcal{C} \cup \mathcal{S}_{\rm hyp} \cup \mathcal{S}_{\rm par}.$$

Evidently $\mathcal{S}_{\rm hyp}$ is an open subset of moduli space, disjoint from the curve $\mathrm{Per}_1(1) \cong \mathbb{C}$, while $\mathcal{S}_{\rm par}$ is a relatively open subset of this curve $\mathrm{Per}_1(1)$.

Lemma 4.1. *The parabolic shift locus is contained in the common topological boundary $\partial\mathcal{S}_{\rm hyp} = \partial\mathcal{C}$. Hence the closure $\overline{\mathcal{C}} \subset \mathcal{M}$ is equal to $\mathcal{C} \cup \mathrm{Per}_1(1)$, with complement $\mathcal{S}_{\rm hyp}$.*

Proof. For every (f) in the parabolic shift locus, we must show that f can be approximated arbitrarily closely by a map with connected Julia set, and also by a hyperbolic map with totally disconnected Julia set. Let $\{f_t\}$ be a holomorphic one-parameter family of maps with $f_0 = f$. We will assume that each f_t has two critical points, and that this family is not contained in $\mathrm{Per}_1(1)$. Then for $|t|$ small but nonzero, the parabolic fixed point for f_0 splits into two nearby fixed points, with multipliers say λ_1 and λ_2. As t traverses a loop around $t = 0$, these two fixed points may be interchanged. However, if we set $t = u^2$, both λ_1 and λ_2 can certainly be expressed as single valued holomorphic functions of u, with $\lambda_1(0) = \lambda_2(0) = 1$. Since these functions are nonconstant, we can choose u close to zero so that $\lambda_1(u)$ takes any required value close to 1.

First we choose u so that $\lambda_1(u) = e^{2\pi i/q}$ with $q > 1$. Then the corresponding fixed point is parabolic, with at least two attracting petals. Hence the associated Fatou set is not connected, and (f_{u^2}) must belong to the connectedness locus $\mathcal{C}$.

Now we choose u so that $\lambda_1(u)$ is real, with $\lambda_1 < 1$, so that the corresponding fixed point is strictly attracting. For u sufficiently close to zero, we will show that (f_{u^2}) belongs to the hyperbolic shift locus. Choose a simple arc A joining the two critical points with the Fatou set for f_0. Then for large k the image $f_0^{\circ k}(A)$ lies close to the parabolic point, and within a sector of small angular size about the attracting direction for this parabolic point. An easy perturbation argument then shows that the same description holds for $f_{(u^2)}$, provided that $\lambda_1(u) < 1$ with u close to zero. (See [Milnor 2000, § 4], for example.) Thus both critical values lie in a common Fatou component, and it follows that $(f_{u^2}) \in \mathcal{S}_{\rm hyp}$. □

Theorem 4.2. *The intersection $\mathcal{C} \cap \mathrm{Per}_1(1)$ is a compact, connected, full subset of the curve $\mathrm{Per}_1(1) \cong \mathbb{C}$. Equivalently, the parabolic shift locus $\mathcal{S}_{\rm par} = \mathrm{Per}_1(1) \smallsetminus (\mathcal{C} \cap \mathrm{Per}_1(1))$ is always non-vacuous, conformally isomorphic to a punctured disk.*

Compare Figure 1(d) and [Milnor 1993, Figure 4].

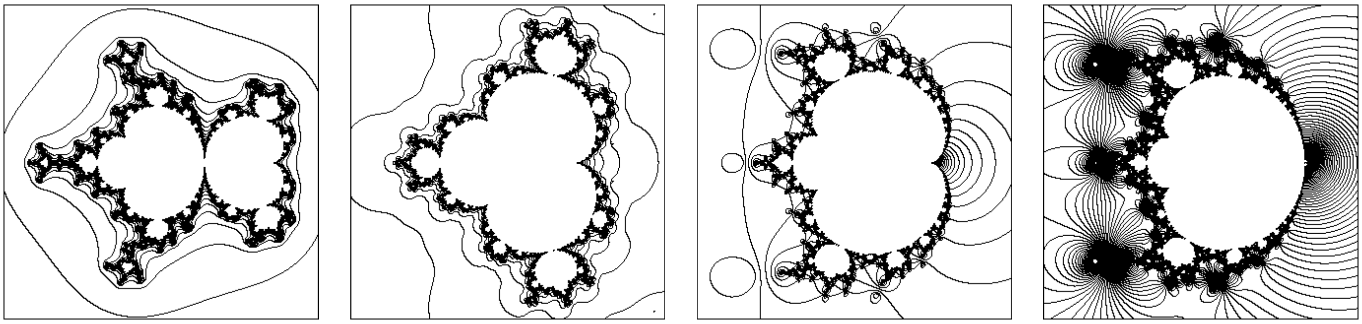
I will outline two proofs of Theorem 4.2, one suggested by conversations with Schleicher, and the other suggested by Shishikura. The first begins as follows.

Proof of compactness. It will be convenient to use the normal form (1–1), choosing the matrix of coefficients to have the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} n{+}1{+}\alpha & n{-}1{-}\alpha \\ n{-}1{+}\alpha & n{+}1{-}\alpha \end{bmatrix}, \tag{4–1}$$

where $\alpha \in \mathbb{C}$ is a parameter. Then it is easy to check that the associated rational function f satisfies $f(1) = f'(1) = 1$, and that the invariant $X = X(f)$ of Section 1 is given by $X = ((n-1)^2 - \alpha^2)/4n$, so that $|X| \to \infty$ as $|\alpha| \to \infty$.

Remark. The special case $\alpha = 0$ corresponds to the unique conjugacy class (f) such that f has a fixed



(a) (b) (c) (d)

FIGURE 1. Pictures of $\mathcal{C} \cap \mathrm{Per}_1(\lambda)$ for $\lambda = 0$, 0.01, 0.5, and 1 respectively, for the degree $n = 4$. The configuration deforms continuously for $0 < \lambda < 1$, and conjecturally as $\lambda \to 1$ also. However, there is a qualitative difference between the first and second pictures due to the fact that as $\lambda \to 0$ the curve $\mathrm{Per}_1(\lambda)$ converges not towards $\mathrm{Per}_1(0)$ but rather towards an $(n-1)$-fold branched covering of $\mathrm{Per}_1(0)$, as shown in Figure 2. Thus Figure 1b is a somewhat squashed version of Figure 2. The surrounding curves in Figures 5a, 5b, 5c represent equal rates of convergence towards the attracting fixed point for the more slowly converging critical point. (For the corresponding curves in Figure 1d, see Remark 4.3.)

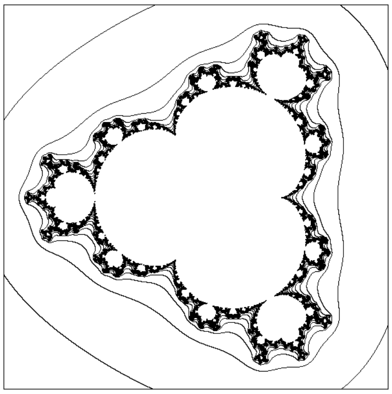


FIGURE 2. The "Multibrot set" for degree $n = 4$, that is, the connectedness locus in the b-parameter plane for the family of unicritical polynomial maps $z \mapsto z^4 + b$. (See [Lau and Schleicher 1996], for example.) Note the $(n-1)$-fold rotational symmetry. The corresponding figure in the b^3-plane is shown in Figure 1(a).

point of multiplier $+1$ with two attracting petals. Using this normal form (4–1), the corresponding Julia set is the unit circle.

Assuming that $\alpha \neq 0$, set

$$z = 1 + \frac{2}{\alpha w}, \qquad w = \frac{2}{\alpha(z-1)}.$$

Then a straightforward computation shows that the map $z \mapsto f(z)$ corresponds to

$$\begin{aligned} w \mapsto F(w) &= \frac{2}{\alpha f(1 + 2/\alpha w) - \alpha} \\ &= w + 1 + O\Big(\frac{1}{\alpha w}\Big), \end{aligned} \tag{4–2}$$

where the error estimate holds uniformly provided that both $|\alpha|$ and $|\alpha w|$ are sufficiently large. In particular, it follows that the region $\{z : \mathrm{Re}\, w > \frac{1}{2}\}$ maps holomorphically into itself, and hence is contained in the Fatou set of f, provided that $|\alpha|$ is sufficiently large. On the other hand, it is not hard to check that

$$f(0) = \frac{b}{d} = 1 + \frac{2}{\alpha w_1}, \qquad f(\infty) = \frac{a}{c} = 1 + \frac{2}{\alpha w_2},$$

where

$$w_1 = 1 - \frac{n+1}{\alpha}, \qquad w_2 = 1 + \frac{n-1}{\alpha}. \tag{4–3}$$

Thus, for $|\alpha|$ sufficiently large, both critical values belong to the half-plane $\mathrm{Re}\, w > \frac{1}{2}$, and hence belong to the same Fatou component, so that (f) belongs to the parabolic shift locus. Therefore the closed set $\mathcal{C} \cap \mathrm{Per}_1(1)$ is bounded and hence compact, as asserted. □

Remark 4.3. We can construct a holomorphic function $\Phi : \mathcal{S}_{\mathrm{par}} \to \mathbb{C}$ as follows. For any $(f) \in \mathcal{S}_{\mathrm{par}}$, let $P \subset \widehat{\mathbb{C}} \smallsetminus J_f$ be an attracting petal for the parabolic fixed point, and let φ be a Fatou coordinate, mapping P biholomorphically into $\mathbb{C}$, and satisfying

$$\varphi(f(z)) = \varphi(z) + 1.$$

Then φ extends canonically to a holomorphic map which carries the entire parabolic basin onto $\mathbb{C}$, satisfying this same functional equation. In particular, if c_1 and c_2 are the critical points, then the difference $\varphi(c_1) - \varphi(c_2)$ is a well defined complex number, independent of the choice of petal and Fatou coordinate. In order to make this construction independent of the numbering of the critical points, we set

$$\Phi(f) = \big(\varphi(c_1) - \varphi(c_2)\big)^2.$$

then it is not difficult to check that

$$\Phi : \mathcal{S}_{\mathrm{par}} \to \mathbb{C}$$

is well defined and holomorphic. (Compare the construction of Fatou coordinates as given in [Steinmetz 1993].) The curves $|\Phi| =$ constant are shown in Figure 1(d), and in a much larger region of $\mathrm{Per}_1(1)$ in Figure 3. The asymptotic formula

$$\Phi(f) \simeq \left(\frac{2n}{\alpha}\right)^2 \simeq \frac{-n}{X(f)} \tag{4–4}$$

as $|X(f)| \to \infty$ can be verified as follows. The inequality

$$\frac{dF(w)}{dw} = 1 \;+\; O\left(\frac{1}{\alpha w^2}\right)$$

follows from (4–2) together with Schwarz's Lemma. Hence

$$\frac{F(w_2) - F(w_1)}{w_2 - w_1} = 1 \;+\; O\left(\frac{1}{\alpha w_j^2}\right)$$

provided that w_2/w_1 is reasonably close to 1. Setting

$$\varphi(c_2) - \varphi(c_1) = \lim \big(F^{\circ m}(w_2) - F^{\circ m}(w_1)\big)$$

as in [Steinmetz 1993], it follows that

$$\varphi(c_2) - \varphi(c_1) = (w_2 - w_1)\,(1 + O(1/\alpha)).$$

But $w_2 - w_1 = 2n/\alpha$ by (4–3); and (4–4) follows.

The first proof of Theorem 4.2 continues as follows. To show that $\mathcal{C} \cap \mathrm{Per}_1(1)$ is connected, we study the limit, in the Hausdorff topology, of the intersection $\mathcal{C} \cap \mathrm{Per}_1(\lambda)$ as λ tends to 1 through real values $\lambda < 1$. First note that

$$\limsup_{\lambda \nearrow 1} \mathcal{C} \cap \mathrm{Per}_1(\lambda) \subset \mathcal{C} \cap \mathrm{Per}_1(1). \tag{4–5}$$

Suppose that $\langle f \rangle$ can be approximated arbitrarily closely by elements of $\mathcal{C} \cap \mathrm{Per}_1(\lambda)$ with $\lambda \nearrow 1$. If $\langle f \rangle$ did not belong to $\mathcal{C} \cap \mathrm{Per}_1(1)$, then it would have to belong to the parabolic shift locus. From the proof of Lemma 4.1, it would follow that any approximating map in $\mathrm{Per}_1(\lambda)$ with $\lambda < 1$ must belong to the hyperbolic shift locus, contradicting our assumption.

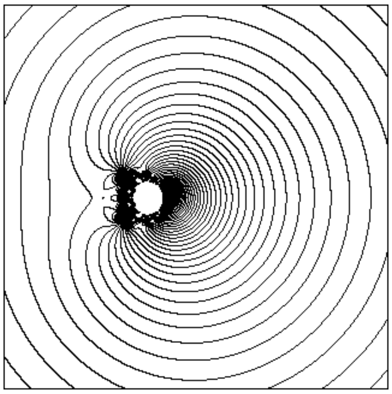


FIGURE 3. Another picture of $\mathcal{C} \cap \mathrm{Per}_1(1)$ in the case $n = 4$, showing a much larger region in order to illustrate behavior near infinity.

On the other hand, we will show that

$$\liminf_{\lambda \nearrow 1}\; \partial(\mathcal{C} \cap \mathrm{Per}_1(\lambda)) \supset \partial(\mathcal{C} \cap \mathrm{Per}_1(1)). \tag{4–6}$$

We again use the normal form (4–1) with marked critical points 0 and ∞, and with parabolic fixed point $z = 1$, writing $f = f_\alpha$, where α is the parameter. Consider a boundary point f_{α_0} of $\mathcal{C} \cap \mathrm{Per}_1(1)$. We will show that f_{α_0} can be approximated arbitrarily closely by maps f_α such that one critical orbit of f_α lands on a repelling periodic orbit. To fix our ideas, suppose that the critical point ∞ lies in the parabolic basin for f_{α_0}, and consider the sequence of maps

$$\alpha \mapsto f_\alpha^{\circ k}(0)$$

for $k = 1, 2, 3, \ldots$, where α ranges over some neighborhood of α_0.

Case 1. Suppose that the $f_\alpha^{\circ k}(0)$ do not form a normal family throughout any neighborhood of α_0, and choose a repelling periodic orbit for f_{α_0} with period ≥ 3. This orbit varies holomorphically with the parameter α, throughout some neighborhood of α_0. By non-normality, we can choose α arbitrarily close to α_0 so that the orbit of 0 under f_α eventually lands on this periodic orbit. Now perturbing slightly, we can preserve this critical orbit relation but replace the multiplier at the fixed point $z = 1$ by some number $\lambda < 1$. The resulting map must belong to $\mathcal{C} \cap \mathrm{Per}_1(\lambda)$, as required.

Case 2. Now suppose that the $f_\alpha^{\circ k}(0)$ do form a normal family throughout some neighborhood of α_0. The hypothesis that every neighborhood of f_{α_0} intersects the parabolic shift locus then guarantees that this family of maps must converge uniformly near α_0 to the constant map $\alpha \mapsto 1$ as $k \to \infty$. In particular, $f_{\alpha_0}^{\circ k}(0)$ must converge to 1. If 0 lies in the same parabolic basin as ∞, then it follows that f_{α_0} lies in the shift locus, contradicting our hypothesis. The only other possibilities are that either

(a) $\alpha_0 = 0$, so that f_{α_0} has two distinct parabolic basins and $z = 1$ is a fixed point of higher multiplicity, or

(b) some forward image $f_{\alpha_0}^{\circ k}(0)$ is precisely equal to the parabolic fixed point $z = 1$.

In case (a), under a slight perturbation within $\mathrm{Per}_1(1)$ this fixed point splits into one fixed point of multiplier $+1$. together with a second fixed point which can have any multiplier close to $+1$. In particular, if we perturb so that this second fixed point is attracting, then we must be within the connectedness locus. Therefore f_{α_0} is not an isolated point of $\mathcal{C} \cap \mathrm{Per}_1(1)$. Hence it is not an isolated boundary point, and, after a slight perturbation, we can obtain a contradiction by the argument above.

In case (b), there is only one parabolic basin. Suppose that (f_{α_0}) were an isolated point of $\mathcal{C} \cap \mathrm{Per}_1(1)$. Let α range over a small circle centered at α_0, and assume that the corresponding maps f_α all belong to the (parabolic) shift locus. Then the corresponding images $f_\alpha^{\circ k}(0)$ must loop around the parabolic fixed point $z = 1$ one or more times, without ever hitting the Julia set $J(f_\alpha)$. By Mañé–Sad–Sullivan or Lyubich, this Julia set must vary continuously as we go around around the loop. Choose a repelling periodic point of period ≥ 2 which is close enough to $z = 1$ so that it remains inside this loop in the z-plane, for all parameter values in the circle. A priori, we might worry that this periodic point comes back to a different periodic point as we go around the circle. However this cannot happen since we can deform the circle in $\mathrm{Per}_1(1)$ into a circle in $\mathrm{Per}_1(1-\varepsilon)$ which deforming the Julia set homeomorphically. The corresponding disk in $\mathrm{Per}_1(1-\varepsilon)$ bounds a disk in the hyperbolic shift locus, so the monodromy must be trivial. Now shrink the parameter loop down to the point $z = 1$. A winding number argument shows that at some point during this shrinking, the image $f_\alpha^{\circ k}(0)$ must exactly hit the corresponding repelling point, and hence belong to the Julia set. This contradicts the hypothesis that (f_{α_0}) was isolated in $\mathcal{C} \cap \mathrm{Per}_1(1)$. But if this point is not isolated, then we see as above that it is indeed possible to approximate f_{α_0} by a map with 0 eventually mapping to a repelling periodic point.

Now as we vary $\lambda = 1$ to a value slightly below 1, the point in moduli space satisfying this critical orbit relation, say $f^{\circ k}(0) = f^{\circ l}(0)$ deforms continuously, and necessarily belongs to the connectedness locus. This proves (4–6).

Remark. In fact it is conjectured that $\mathcal{C} \cap \mathrm{Per}_1(1)$ is equal to the Hausdorff limit of $\mathcal{C} \cap \mathrm{Per}_1(\lambda)$ as $\lambda \nearrow 1$. However, our arguments will leave open the possibility of a "parabolic queer component" in $\mathcal{C} \cap \mathrm{Per}_1(1)$, whose points can be approximated arbitrarily closely by points in the hyperbolic shift locus.

Proof that $\mathcal{C} \cap \mathrm{Per}_1(1)$ is connected. For $|\lambda| < 1$, it is known that $\mathcal{C} \cap \mathrm{Per}_1(\lambda)$ is compact, connected and full, with connected boundary. (Compare [Goldberg and Keen 1990], [Milnor 1993].) If $\mathcal{C} \cap \mathrm{Per}_1(1)$ were not connected, then we could choose boundary points in two different components. By (4–6), these could be approximated by points in the boundary of say $\mathcal{C} \cap \mathrm{Per}_1(1-\varepsilon)$. Since this is true for arbitrarily small ε, it would follow from (4–5) that these points must actually belong to the same component. Thus $\mathcal{C} \cap \mathrm{Per}_1(1)$ is connected.

In order to prove that $\mathcal{C} \cap \mathrm{Per}_1(1)$ is full, or equivalently that the parabolic shift locus is connected, we will need a sharper form of the construction used to prove Theorem B.5.

Lemma 4.4. *If $(f) \in \mathcal{S}_{\mathrm{par}}$ has no critical orbit relation, then there exists an attracting petal for f which contains both critical values.*

By an *attracting petal* we mean a simply-connected open set P which eventually captures all orbits in the parabolic basin, and such that f maps the closure $\overline{P}$ homeomorphically, with $f(\overline{P}) \subset P \cup \{\hat{z}\}$ where $\hat{z}$ is the parabolic fixed point. By a *critical orbit relation*, we mean some relation of the form $f^{\circ k}(c_1) = f^{\circ l}(c_2)$.

We start with some petal P_0, with smooth boundary containing no points of the critical orbits. We may assume that P_0 contains no critical value. We construct $P_0 \subset P_1 \subset P_2 \subset \cdots$ inductively, defining P_{k+1} as the connected component of $f^{-1}(P_k)$ that contains P_k. We define the two integers $0 \le k_1 \le k_2$, by setting k_j equal to the smallest integer such that P_{k_j} contains j distinct critical values. Then P_{k_1} is itself a petal, but P_{k_1+1} is not, since it contains a critical point (using Lemma 3.4). The proof of Lemma 4.4 will be by induction on the difference $k_2 - k_1$. To start the induction, if $k_1 = k_2$, then P_{k_1} is the required petal, and we are done.

Suppose then that $k_1 < k_2$. Let v_1 be the critical value which is contained in P_{k_1}. By Lemma 3.4 we know that P_{k_1+1} is a simply connected open set which contains the corresponding critical point c_1 and is a branched n-fold covering of P_{k_1}. Let $x = f^{\circ(k_2-k_1)}(c_2)$ be the unique point in $P_{k_1+1} \setminus P_{k_1}$ which belongs to the second critical orbit. Choose some path γ within $\overline{P}_{k_1} \setminus \overline{P}_{k_1-1}$ which joins the critical value v_1 to the boundary of P_{k_1}, and which avoids the point $f(x) = f^{\circ(k_2-k_1)}(v_2)$. Then the preimage of γ under f is a union $\gamma_1 \cup \cdots \gamma_n$ of n paths, each joining the critical point c_1 to the boundary of P_{k_1+1}. These n paths cut the open set P_{k_1+1} into n regions, each of which maps diffeomorphically onto $P_{k_1} \setminus \gamma$. Exactly one of these n regions contains the point x, and exactly one of these n regions contains P_{k_1}.

Assertion. *It is possible to choose the path γ so that x and P_{k_1} belong to the same connected component of $P_{k_1+1} \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$.*

For example, in Figure 4, as drawn, this requirement fails; but if we modify the path γ in a neighborhood of $f(x)$ so that it passes above $f(x)$ rather than below, the requirement is satisfied. More generally, we can choose the path γ from v_1 so as to loop any number of times around an arc joining v_1 to $f(x)$ before terminating on ∂P_{k_1}. By choosing the number of loops appropriately, we can easily guarantee that x lies in the required component of $P_{k_1+1} \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$. Details are left to the reader.

The proof of Lemma 4.4 now proceeds as follows. Construct a new attracting petal P_0' which contains no critical value by removing a thin neighborhood of γ from P_{k_1}. Then the preferred component P_1' of $f^{-1}(P_0')$ will consist of the component of

$$P_{k_1+1} \setminus \big(\text{neighborhood of } (\gamma_1 \cup \cdots \cup \gamma_n)\big)$$

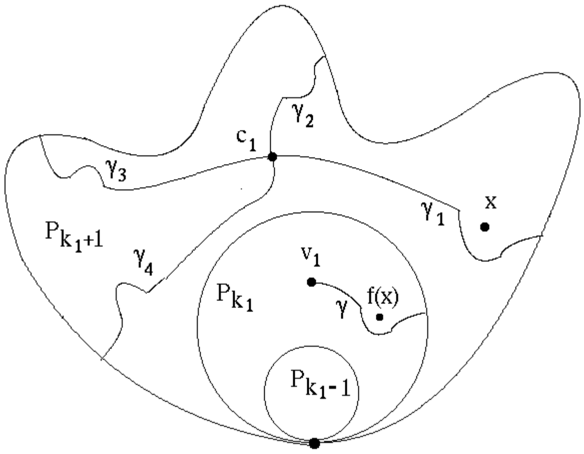


FIGURE 4. The three petals P_{k_1-1}, P_{k_1}, and P_{k_1+1}. The point $x = f^{\circ(k_2-k_1)}(c_2)$ belongs to the orbit of the second critical point. A path γ from v_1 to ∂P_{k_1} within $P_{k_1} \setminus (\overline{P}_{k_1-1} \cup \{f(x)\})$ lifts to n distinct paths γ_i from c_1 to ∂P_{k_1+1}. We must choose this path γ so that x and P_{k_1} belong to the same connected component of $P_{k_1+1} \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$.

that contains P_{k_1}. By the construction, both v_1 and $x = f^{\circ(k_2-k_1)}(c_2)$ belong to this set P_1'. Therefore, the new difference $k_2' - k_1'$ will be equal to $k_2 - k_1 - 1$. The conclusion of Lemma 4.4 now follows by induction. □

Proof of Theorem 4.2 (conclusion). We must show that the parabolic shift locus is connected. The proof will make use of a standard quasiconformal surgery argument, as suggested to me by D. Schleicher. After a small perturbation of f, we may assume that there are no critical orbit relations. Choose a petal P as in Lemma 4.4, and choose an embedded disk $\overline{\Delta} \subset P \setminus \overline{f(P)}$ which contains both critical values in its interior. Now choose a diffeomorphism from Δ to itself which is the identity near the boundary and which moves the critical value v_2 arbitrarily close to v_1. Pulling the standard conformal structure back under this diffeomorphism, we obtain a new conformal structure on $P \setminus \overline{f(P)}$. Now push this new structure forward under the various iterates of f, and also pull it backwards under the various iterates of f^{-1}. Since the Shishikura Condition is satisfied: *every orbit intersects Δ at most once,* it follows that we obtain a well defined measurable conformal structure, invariant under f. By the measurable Riemann Mapping Theorem, there is a quasiconformal mapping Φ which conjugates this new structure to the standard conformal structure. Now $g = \Phi \circ f \circ \Phi^{-1}$ is a new rational map, topologically conjugate to f. By the construction, the two critical values of g are close together in an embedded disk $\Phi(\Delta)$ which contains no critical point. By Corollary 1.8, it follows that the invariant $|A(g)|$ is arbitrarily close to infinity. Thus g belongs to the unbounded component of the parabolic shift locus. Since f and g clearly belong to the same connected component of $\mathcal{S}_{\mathrm{par}}$, this shows that $\mathcal{S}_{\mathrm{par}}$ is connected. □

Alternative proof of Theorem 4.2. I am grateful to M. Shishikura for suggesting a quite different argument, based on an explicit model for the parabolic shift locus which can be outlined as follows. Start with the map

$$f(z) = z^n + b \quad \text{with} \quad b = (n-1)/n^{n/(n-1)},$$

which has a parabolic fixed point at $z = (1/n)^{1/(n-1)}$. The corresponding parabolic basin $\mathcal{B}$ is connected and simply-connected, and contains a single critical point $0 \mapsto b$. Let $\mathcal{P}_0 \subset \mathcal{B}$ be the largest attracting petal such that the Fatou coordinate map carries $\mathcal{P}_0$ diffeomorphically onto a right half-plane. Thus the critical point 0 belongs to the boundary of $\mathcal{P}_0$, and the critical value b belongs to the boundary of the smaller petal $\mathcal{P}_{-1} = f(\mathcal{P}_0)$. Our model space

$$\mathcal{S}^*_{\mathrm{par}} = (\mathcal{B} \smallsetminus (\mathcal{P}_{-1} \cup \{b\}))/\alpha,$$

conformally isomorphic to a punctured disk, is obtained by removing the subset $\mathcal{P}_{-1} \cup \{b\}$ from the basin $\mathcal{B}$, and then gluing the two halves of $\partial\mathcal{P}_{-1} \smallsetminus \{b\}$ onto each other under the correspondence $\alpha : z \mapsto \bar{z}$. To each point v in this model space, we construct a multiply connected parabolic basin $\mathcal{B}'_v$ with two critical points as follows. The original basin $\mathcal{B}$ can be described as the union of open subsets

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots,$$

where each $\mathcal{P}_k$ is the interior of a region bounded by a Jordan curve, and where f maps each

$$\mathcal{P}_{k+1} = f^{-1}(\mathcal{P}_k)$$

onto $\mathcal{P}_k$ by an n-fold cyclic covering, branched only over b. Let k_0 be the smallest index such that $v \in \mathcal{P}_{k_0}$. Construct a new family

$$\mathcal{P}'_0 \subset \mathcal{P}'_1 \subset \mathcal{P}'_2 \subset \cdots$$

as follows. Let $\mathcal{P}'_k = \mathcal{P}_k$ for $k \le k_0$, but let $\mathcal{P}'_{k+1}$ be the n-fold cyclic covering of $\mathcal{P}'_k$, branched over both b and v for $k \ge k_0$. The covering should extend over the boundary, so that each boundary curve of $\mathcal{P}'_k$ for $k \ge k_0$ is covered by n distinct boundary curves for $\mathcal{P}'_{k+1}$. Then the inclusion $\mathcal{P}'_{k-1} \hookrightarrow \mathcal{P}'_k$ lifts inductively to an inclusion $\mathcal{P}'_k \hookrightarrow \mathcal{P}'_{k+1}$. Let $\mathcal{B}'_v$ be the union of the $\mathcal{P}'_k$. Note that there is a canonical parabolic map f_v from this Riemann surface $\mathcal{B}'_v$ to itself, carrying each $\mathcal{P}'_{k+1}$ onto $\mathcal{P}'_k$ by an n-fold branched covering.

In the special case where $v \in \partial\mathcal{P}_{-1} \smallsetminus \{b\}$, the two critical values b and v are on the boundary of the same petal $\mathcal{P}_{-1}$, and there is a canonical isomorphism $\mathcal{B}'_v \to \mathcal{B}'_{\bar{v}}$ which carries v to b and b to $\bar{v}$. Hence we identify $\mathcal{B}'_v$ with $\mathcal{B}'_{\bar{v}}$ in this special case.

Conversely, suppose that we start with a point (g) in the parabolic shift locus. Let $\mathcal{Q}_0$ be the largest petal for g such that the Fatou coordinate maps $\mathcal{Q}_0$ isomorphically onto a right half-plane. Then $\partial\mathcal{Q}_0$ contains at least one critical point c_0, and there is a unique conformal isomorphism $\varphi_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ which conjugates f to g and (extended over the boundary) carries 0 to c_0. Now let $\mathcal{Q}_k = g^{-k}\mathcal{Q}_0$, and let $k_0 \ge 0$ be the smallest index such that $\mathcal{Q}_{k_0}$ contains both critical values of g. Then φ_0 extends uniquely to a conformal isomorphism $\varphi_{k_0} : \mathcal{P}_{k_0} \to \mathcal{Q}_{k_0}$, still conjugating f to g. Let v be the unique point of $\mathcal{P}_{k_0}$ which maps to the second critical value under φ_{k_0}. In this way we obtain a holomorphic map $(g) \mapsto v$ from the parabolic shift locus $\mathcal{S}_{\mathrm{par}}$ to the model space $\mathcal{S}^*_{\mathrm{par}}$. (Furthermore, it is not hard to check that φ_{k_0} extends uniquely to a conformal isomorphism $\mathcal{B}'_v \to \widehat{\mathbb{C}} \smallsetminus J(g)$ which conjugates f_v to g.)

We must show that this correspondence $(g) \mapsto v$ is a proper map. That is, if (g_j) is any sequence of points in $\mathcal{S}_{\mathrm{par}}$ which has no accumulation point within $\mathcal{S}_{\mathrm{par}}$, then we must show that the corresponding sequence v_j has no accumulation point within the model space. First suppose that the (g_j) converge, within $\mathrm{Per}_1(1)$, to a point of the connectedness locus. If the corresponding sequence of points v_j converged to a limit within the model space, then the corresponding sequence f_{v_j} of model parabolic basins would possess annuli with moduli bounded away from zero which separate the two critical points from the Julia set, yielding a contradiction. Similarly, suppose that the (g_j) diverge within $\mathrm{Per}_1(1)$

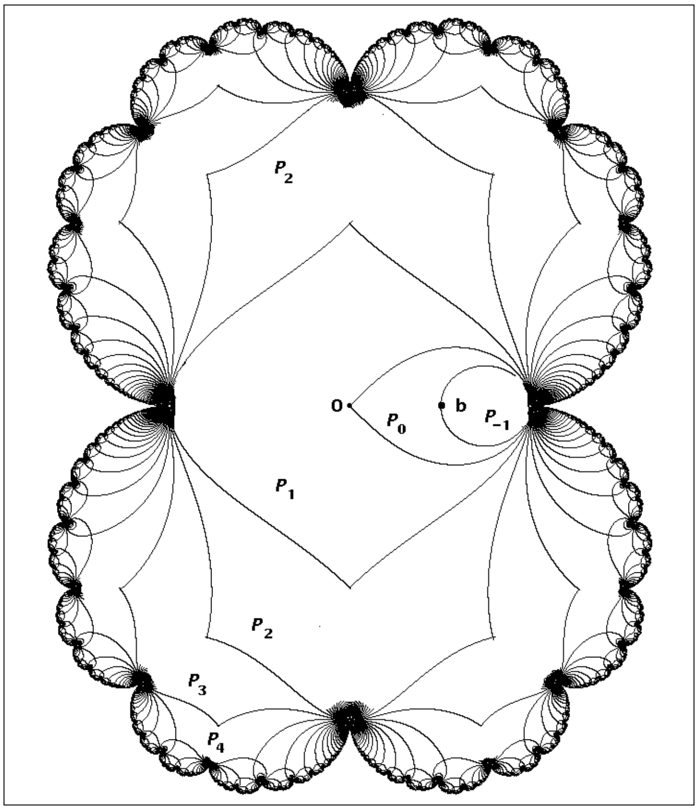


FIGURE 5. The regions $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots$ for the quadratic case $n = 2$. (In the region outside of $\mathcal{P}_4$, all of the iterated preimages of $\partial\mathcal{P}_0$ have been drawn in, so that this figure looks more complicated near the basin boundary.)

to the point at infinity. Using Corollary 1.8, we see that the distance between the two critical values of f_{v_j} must tend to zero. In other words, v_j must converge towards the puncture point b in the model space. Thus our correspondence $(g) \mapsto v$ from $\mathcal{S}_{\rm par}$ to $\mathcal{S}^*_{\rm par}$ is proper. In order to show that it is a conformal isomorphism, we need only check that it has degree one, which follows easily from the asymptotic formula (4–4). □

Remark. The argument shows that the model space obtained from Figure 5 is "inside out" with respect to Figures 1(d) and 3. The puncture point b in Figure 5 corresponds to the point at infinity in the earlier figures.

5. REAL MAPS

This section will study bicritical maps such that the invariants X and Y of Section 1 are both real. Recall that a homeomorphism $\alpha : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *antiholomorphic* if it has the form $\alpha(z) = \varphi(\bar{z})$, where φ is some Möbius automorphism of $\widehat{\mathbb{C}}$ and $\bar{z}$ is the complex conjugate. A is an *involution* if $\alpha \circ \alpha$ is the identity map.

Lemma 5.1. *The invariants $X(f)$ and $Y(f)$ are both real if and only if the bicritical map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ commutes with some antiholomorphic involution α. This α is unique if and only if (f) lies off the symmetry locus (or off the real part $\Sigma_{\mathbb{R}}$ of the symmetry locus).*

Proof. We temporarily introduce the notation $\mathbf{c}(z) = \bar{z}$ for complex conjugation. Then the rational map $g = \mathbf{c} \circ f \circ \mathbf{c}$ can be obtained from f by replacing all of its coefficients by their complex conjugates. Evidently $X(g) = \overline{X(f)}$ and $Y(g) = \overline{Y(f)}$. Therefore, if X and Y are real, then it follows that f is holomorphically conjugate to g, say $f \circ \varphi = \varphi \circ g$. Setting $\alpha = \varphi \circ \mathbf{c}$, this means that

$$f \circ \alpha = f \circ \varphi \circ \mathbf{c} = \varphi \circ g \circ \mathbf{c} = \varphi \circ \mathbf{c} \circ f = \alpha \circ f,$$

so that f commutes with α. It follows that f commutes with $\alpha \circ \alpha$, which is a holomorphic map from $\widehat{\mathbb{C}}$ to itself. If (f) lies off of the symmetry locus of Corollary 1.4, this proves that $\alpha \circ \alpha$ is the identity map, as required. Even for a generic f in the symmetry locus which commutes with a unique ι, since α must either fix or interchange the two critical points, it follows that $\alpha \circ \alpha$ must fix both critical points, and hence be the identity map. Finally, in the two exceptional cases, corresponding to $f(z) = z^n$ or $f(z) = 1/z^n$, the assertion is clearly satisfied. Conversely, if f commutes with $\alpha = \varphi \circ \mathbf{c}$, then composing the equation $f \circ \varphi \circ \mathbf{c} = \varphi \circ \mathbf{c} \circ f$ on the right with $\mathbf{c}$, we see that $f \circ \varphi = \varphi \circ g$. Therefore $X(f) = X(g)$ and $Y(f) = Y(g)$ are real, as asserted.

If there is a second antiholomorphic map β commuting with f, then $\alpha \circ \beta$ is a holomorphic map commuting with f, so that f belongs to the symmetry locus. Conversely, if (f) does belong to the symmetry locus, let ι be a holomorphic involution commuting with f. Then $\alpha \circ \iota \circ \alpha$ is also a holomorphic involution commuting with f. In the generic case where ι is unique, it follows that α commutes with ι, and that $\alpha \circ \iota$ is another antiholomorphic involution commuting with f. In other words, f can be given the structure of a "real map" in two essentially distinct ways. The discussion in the two exceptional cases where ι is not unique is straightforward, and will be left to the reader. □

We must now distinguish two different cases. The antiholomorphic involution α may have a circle of fixed points (for example the unit circle if $\alpha(z) = 1/\bar{z}$). In this case, a Möbius change of coordinates will reduce it to the standard form $\alpha(z) = \bar{z}$ with fixed point set $\mathbb{R} \cup \{\infty\}$, and it follows easily that the corresponding map f can be expressed as a rational map with real coefficients. On the other hand, it may happen that α has no fixed points at all. In this case, a Möbius change of coordinates reduces α to the standard *antipodal map* $\alpha(z) = -1/\bar{z}$. It follows from a theorem of Borsuk and Hopf that a map f which commutes with this antipodal map necessarily has odd degree. (Compare [Alexandroff and Hopf 1935].) Such an f gives rise to a conformal self-map with only one critical point on the non-orientable "Klein surface" which is obtained from $\widehat{\mathbb{C}}$ by identifying each z with $-1/\bar{z}$.

More precisely, we can distinguish six cases as follows. Suppose that f commutes with the antiholomorphic involution α. For the first five cases we suppose that α has a circle S^1 of fixed points. Note that f induces a map from this circle to itself.

Case +n. For any degree n, it may happen that f induces an n-fold covering map of degree $+n$ from the circle S^1 to itself. In this case the Julia set $J(f)$ either coincides with S^1, in which case $\langle f \rangle$ has just two Fatou components, each mapped onto itself by f, or else $J(f)$ is a Cantor subset of S^1, so that $\langle f \rangle$ belongs to the shift locus.

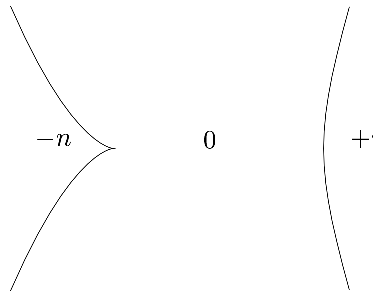


FIGURE 6. Symmetry locus in the real (X, Y)-plane for degree $n = 2$. The three complementary domains are labeled according to the degree of the associated map from the circle of real points to itself. The picture for higher even values of n would have another cusp, corresponding to $\langle z \mapsto z^n \rangle$, but would otherwise be similar.

Case −n. If f induces an n-fold covering of degree $-n$, then again the Julia set may be S^1 or a Cantor subset. Correspondingly, f either interchanges the two Fatou components, or belongs to the shift locus.

Case 0. If the degree n is even, then the only other possibility is that f maps S^1 onto a proper subset of itself by a map of degree zero. In this case, the forward orbits of the two critical points lie in the image $f(S^1)$, which is an interval $I \subset S^1$. (Compare [Milnor 1993].) Restricting f to this interval, we obtain a map which is either monotone, or unimodal, or bimodal of a rather restricted type since each point of I has at most two preimages in I. Thus the dynamics is largely controlled by the theory of smooth interval maps. In particular, any attracting or parabolic cycle must be contained in I, and if there is no such cycle then the Julia set must be the entire Riemann sphere.

We can pass between these three cases only by crossing the symmetry locus. (Compare Figure 6.)

For the remaining three cases, we assume that the degree n is odd.

Case +1. For any odd n, the circle S^1 may map to itself by a homeomorphism of degree $+1$ with two critical inflection points. The dynamics of the critical orbits is then governed by the theory of monotone degree one circle maps, with a rotation number which is well defined up to sign. (Compare [Bamón and Bobenrieth 1999].) If the rotation number is p/q, both critical orbits converge to (the same or different) parabolic or attracting cycles of period q, whereas if the rotation number is irrational the Julia set is the entire Riemann sphere.

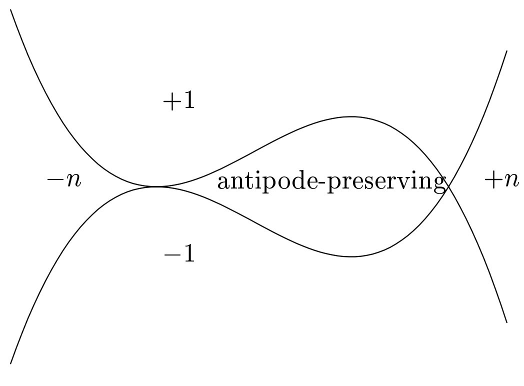


FIGURE 7. Symmetry locus in the real (X, Y)-plane for degree $n = 3$, with the five complementary domains appropriately labeled.

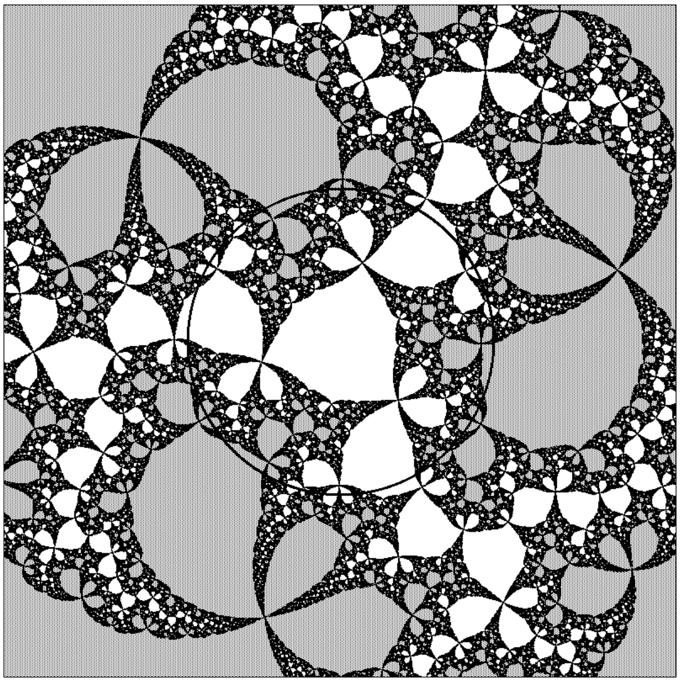


FIGURE 8. Dynamic plane for a degree $n = 3$ bicritical map which commutes with the antipodal map, with the unit circle drawn in. (The parameters are $X = -0.235$, $Y = 0.213$.) There are two attracting period 4 orbits. The antipodal map of $\widehat{\mathbb{C}}$, reflection in the unit circle composed with the 180° rotation, interchanges the two attracting basins, which are colored white and grey respectively. The Julia set, like that for an arbitrary degree n bicritical map, has n-fold rotational symmetry. It is conjectured that this particular map can be obtained as a mating of the form $(z^3+b) \perp\!\!\perp (z^3-\bar{b})$, with $b \approx 0.584-0.270\,i$. (Compare [Tan 1992].)

Case -1. Similarly S^1 may map to itself by a homeomorphism of degree -1. In this case, both critical orbits must converge to parabolic or attracting cycles of period one or two.

Antipode Preserving Case. If

$$f(-1/\bar{z}) = -1/\overline{f(z)},$$

then (f) necessarily belongs to the connectedness locus. For every hyperbolic component in this region of the (X, Y)-plane, there are either two antipodal attracting orbits of the same period p, or else a single attracting orbit of period $2p$ satisfying the identity $\alpha(x) = f^{\circ p}(x)$. In the latter case it is convenient to say that the hyperbolic component is of type $p + p$.

The division of the real (X, Y)-plane into five regions for a typical odd n is shown in Figure 7. The antipode-preserving region in this plane is shown in detail in Figure 9 for two values of n, and a typical associated Julia set is shown in Figure 8.

6. THE EXTENDED MODULI SPACE $\widehat{\mathcal{M}} = \mathcal{M} \cup L_\infty$

The next two sections will study the limiting behavior as the conjugacy class (f) becomes degenerate. We will say that a sequence of conjugacy classes (f_j) *diverges to infinity* in the moduli space $\mathcal{M}$ if this sequence eventually leaves any compact subset of $\mathcal{M}$. Using the results of Section 2, it is not hard to check that the sequence $\{(f_j)\}$ diverges to infinity within $\mathcal{M}$ if and only if

- the larger of the two invariants $|X(f_j)|$ and $|Y(f_j)|$ tends to infinity with j,

or if and only if

- the largest, $\max_i |\lambda_i(f_j)|$, of the fixed point multipliers tends to infinity with j.

It will be convenient to number the fixed points of each f_j so that the corresponding multipliers $\lambda_i(f_j)$ satisfy $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_{n+1}|$.

Lemma 6.1. *If $\{(f_j)\}$ diverges to infinity within $\mathcal{M}$, then all but two of the fixed point multipliers must tend to infinity, so that*

$$|\lambda_{n+1}| \ge \cdots \ge |\lambda_4| \ge |\lambda_3| \to \infty.$$

Furthermore, after passing to a suitable subsequence, exactly one of the following two statements must hold:

(1) *either $\lambda_2 \to \infty$ also, but $\lambda_1 \to 0$,*
(2) *or λ_2 remains bounded and λ_1 remains bounded away from zero, while the product $\lambda_1\lambda_2$ converges to $+1$.*

Proof. After passing to a subsequence, we can assume that X and Y and each of the λ_i tends to a well defined limit in $\mathbb{C} \cup \{\infty\}$. First suppose that $|Y| \to \infty$ while $|X|$ remains bounded. Then it follows from Theorem 2.6 that the elementary symmetric function σ_k remains bounded for $k \ne n$, but that $|\sigma_n| \to \infty$. Since σ_n is the sum of $n + 1$ terms, of which $\lambda_2\lambda_3 \cdots \lambda_{n+1}$ is the largest in absolute value, it certainly follows that $|\lambda_2\lambda_3 \cdots \lambda_{n+1}| \to \infty$. Since the product σ_{n+1} remains bounded, this implies that $\lambda_1 \to 0$. We must show that $|\lambda_2| \to \infty$. Otherwise, if say $\lambda_2, \ldots, \lambda_l$ remained bounded, with $l \ge 2$, then a straightforward argument would show that the $\binom{n+1}{l}$-fold sum σ_{n-l+1}, with dominant summand

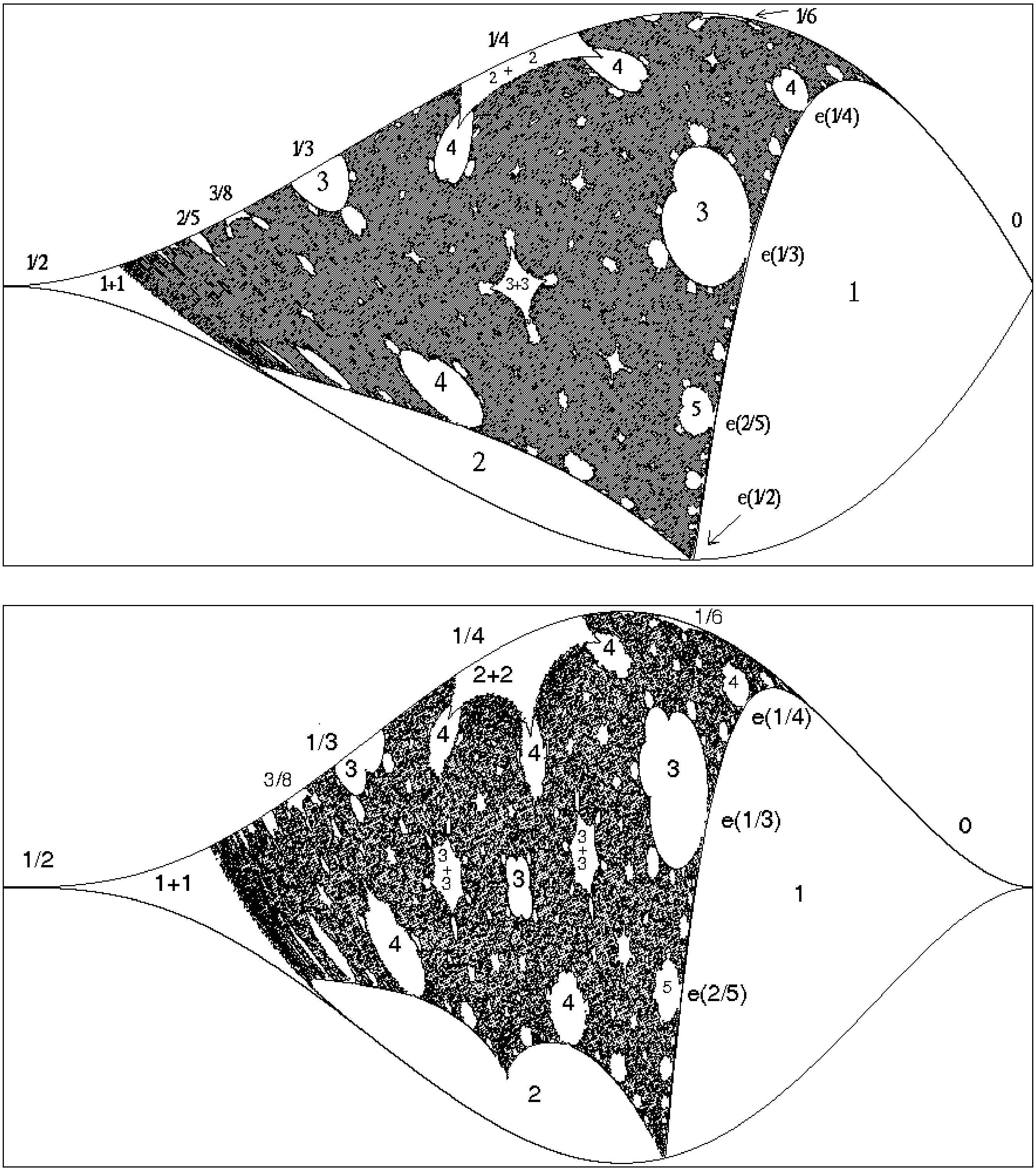


FIGURE 9. Antipode-preserving region of the (X, Y)-plane for degrees 3 and 5, with periods of the larger hyperbolic components labeled. (Note that the components of type $p+p$ have a tricorn-like geometry. Compare [Nakane and Schleicher 1995].) Along the top edge, these maps give rise to degree one circle homeomorphisms, with rotation number as indicated. Also, along the edge of the principal hyperbolic component, the multiplier of one of the two indifferent fixed points is indicated, using the notation $e(x) = e^{2\pi i x}$.

$\lambda_{l+1}\cdots\lambda_{n+1}$, also tended to infinity, yielding a contradiction. Thus $|\lambda_2|\to\infty$, and we are in case (1).

Now suppose that $|X|\to\infty$. In particular, we assume that $X\neq 0$, so that

$$\lambda_1\lambda_2\cdots\lambda_{n+1}=\sigma_{n+1}=n^{n+1}X^{n-1}\neq 0.$$

Let

$$\hat\sigma_k=\sigma_{n+1-k}/\sigma_{n+1}$$

be the k-th elementary symmetric function of the reciprocals $1/\lambda_1,\ldots,1/\lambda_{n+1}$. This is well defined and finite whenever $X\neq 0$. If $X\to\infty$, we see from Lemma 2.4 and Theorem 2.6 that

$$\begin{aligned}&\hat\sigma_k\to 0\quad\text{for } k\geq 3,\\&\hat\sigma_2\to 1.\end{aligned}\tag{6–1}$$

(However, the limiting behavior of $\hat\sigma_1$ depends not only on X but also on Y.) Suppose that exactly p of the λ_i tend to zero, while exactly q of them tend to finite nonzero limits. If $p\geq 1$, it is easy to check that $\hat\sigma_p\to\infty$ and also that $\hat\sigma_{p+q}\to\infty$. By (6–1), it follows that $p\leq 1$, and also that $q=0$ whenever $p=1$. Thus if $p=1$, we are again in case (1).

Finally, suppose that $p=0$. Then evidently $\hat\sigma_q$ tends to a finite nonzero limit, while $\sigma_k\to 0$ for $k>q$. Making use of (6–1), it follows immediately that $q=2$, and that we are in case (2). □

A convenient coordinate near the line at infinity is provided by the sum of reciprocals

$$\hat\sigma_1=\sum\frac{1}{\lambda_i}=\frac{\sigma_n}{\sigma_{n+1}}=\frac{\sigma_n}{n^{n+1}X^{n-1}}.$$

If only λ_1 and λ_2 remain finite, with $\lambda_1\to\lambda$ and $\lambda_2\to\lambda^{-1}$, note that this coordinate $\hat\sigma_1$ tends to $\lambda+\lambda^{-1}$. We can now define the *extended moduli space* $\widehat{\mathcal{M}}$ to be the disjoint union $\mathcal{M}\cup L_\infty$, where L_∞ is a complex line with coordinate $\hat\sigma_1\in\mathbb{C}$. To make this union into a complex manifold, we cover it with two coordinate patches, each biholomorphic to $\mathbb{C}^2$. The first coordinate patch is $\mathcal{M}$ itself, with coordinates X and σ_n. The second is $(\mathcal{M}\smallsetminus L_0)\cup L_\infty$ with coordinates $\hat X=1/X$ and $\hat\sigma_1$. In the overlap $\mathcal{M}\smallsetminus L_0$, these two coordinate systems are related by the biholomorphic map

$$\hat X=\frac{1}{X},\qquad \hat\sigma_1=\frac{\sigma_n}{n^{n+1}X^{n-1}}.$$

Lemma 6.2. *The union $\widehat{\mathcal{M}}$, with complex structure defined in this way, is a well defined Hausdorff complex manifold. Furthermore, the coordinate function X extends to a locally trivial holomorphic projection*

$$X:\widehat{\mathcal{M}}\longrightarrow\widehat{\mathbb{C}},$$

where each fiber L_X is conformally isomorphic to $\mathbb{C}$.

The proof is straightforward.

(Similarly, if $\widehat{\mathcal{M}}_{\mathbb{R}}\subset\widehat{\mathcal{M}}$ denotes the closure of the real part of moduli space, as discussed in Section 5, then we obtain a real line bundle

$$\mathbb{R}\hookrightarrow\widehat{\mathcal{M}}_{\mathbb{R}}\xrightarrow{X}\widehat{\mathbb{R}}=\mathbb{R}\cup\infty.$$

Topologically, $\widehat{\mathcal{M}}_{\mathbb{R}}$ is either a cylinder or a Möbius band according as n is odd or even.)

Remark 6.3. Although the fibers of the line bundle $\mathbb{C}\hookrightarrow\widehat{\mathcal{M}}\to\widehat{\mathbb{C}}$ have a sharp geometric and algebraic interpretation, it is not known whether they have any dynamic meaning. However, three particular fibers certainly do have dynamic interpretations. The fiber L_0 consists of maps with a superattracting fixed point; compare Figure 1(a). The fiber L_{-1} consists of maps for which one critical point maps immediately to the other. (For a study of this family, see [Bamón and Bobenrieth 1999].) Finally, the line at infinity, L_∞ is certainly quite unique. Computer pictures suggest that each fiber L_X may intersect $\overline{\mathcal{C}}$ in a set which is full and connected, so that $L_X\cap\mathcal{S}_{\text{hyp}}$ is conformally isomorphic to $\mathbb{C}\smallsetminus\overline{\mathbb{D}}$. I know of no reason why this should be true, although it would be compatible with the description of the fundamental group of $\mathcal{S}_{\text{hyp}}$. (Compare Remark 6.6 and Lemma 7.8.) I don't have a good algorithm for making pictures of $L_X\cap\mathcal{C}$. However, if we pass to the $(n+1)$-fold branched covering in which one fixed point is marked, then we can parametrize by its multiplier λ. (Compare Remark 2.10.) It is relatively easy to make pictures in the disk $|\lambda|<1$ where this marked point is attracting. (See Figure 10.) Note that each $(f)\in L_X$ with a unique attracting fixed point embeds uniquely in this disk, provided that $X\neq 0$. In particular, $L_X\cap\mathcal{S}_{\text{hyp}}$ embeds uniquely. However conjugacy classes with two attracting fixed points are represented twice.

Lemma 6.4. *The space of all holomorphic sections of the bundle $X:\widehat{\mathcal{M}}\to\widehat{C}$ is an n-dimensional vector*

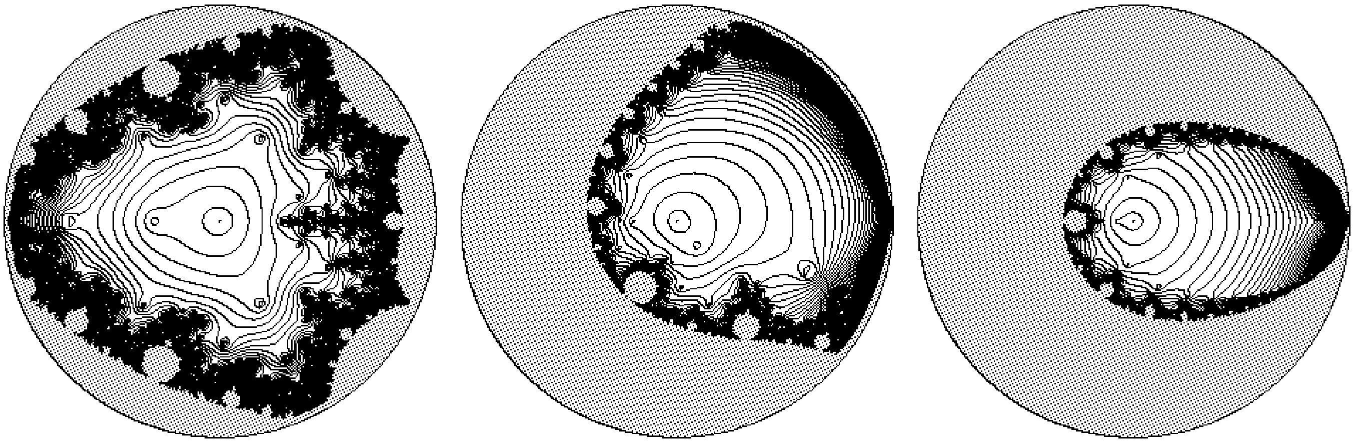


FIGURE 10. Pictures in the disk $|\lambda| < 1$ for X equal to $-0.2,\ 0.2i$ and 0.2 respectively, for degree $n = 2$. (Compare Remark 6.3.) The connectedness locus is shaded grey; while the curves in the shift locus indicate the number of iterations needed for both critical orbits to reach a small neighborhood of the attracting fixed point.

space, consisting of all polynomial functions of the form

$$\sigma_n = \sum_{j=0}^{n-1} c_j X^j, \qquad \hat\sigma_1 = \sum_{j=0}^{n-1} c_j\, \hat X^{n-1-j}/n^{n+1}.$$

It follows that a function $Y = Y(X)$ gives rise to a holomorphic section of this bundle if and only if the sum $Y(X) + 2X^n$ is a polynomial of degree $\le n-1$ in X.

The proof is straightforward. (Compare Lemma 2.4 and Theorem 2.6.)

Corollary 6.5. *For each $\lambda \ne 0$, the closure within $\widehat{\mathcal{M}}$ of the affine algebraic curve $\mathrm{Per}_1(\lambda) \subset \mathcal{M}$ is a smooth compact algebraic curve $\overline{\mathrm{Per}_1(\lambda)} \subset \widehat{\mathcal{M}}$, which can be described as the image of a smooth section of the complex line bundle $\widehat{\mathcal{M}} \to \widehat{\mathbb{C}}$. The intersection $\overline{\mathrm{Per}_1(\lambda)} \cap L_\infty$ of this curve with the line at infinity consists of the point with coordinates $\hat X = 0, \quad \hat\sigma_1 = \lambda + \lambda^{-1}$.*

Proof. This follows immediately, using Theorem 2.3 and Lemma 2.4. $\square$

Note that the coordinate $\hat\sigma_1$ on the line at infinity is real, and belongs to the interval $-2 \le \hat\sigma_1 \le 2$, if and only if $|\lambda| = 1$. (Compare Theorem 7.5.)

(As noted in Section 2, the curve $\mathrm{Per}_1(0)$ should be identified with the fiber L_0 of this fibration with multiplicity $n-1$. Similarly, it may be useful to identify $\mathrm{Per}_1(\infty)$ with the fiber L_∞ counted $n-1$ times.)

Other examples of smooth sections of this complex line bundle are the coordinate curve $\sigma_n = 0$ and the half symmetry locus

$$Y = -2X^{(n-1)/2}(X+1)^{(n+1)/2}$$

of Section 1 when n is odd (but not the locus $Y = 0$ or the other half symmetry locus).

Remark 6.6. We can actually compactify moduli space by adding one more point $\widehat\infty$, contained in a third coordinate neighborhood. However, the result is no longer a manifold when $n > 2$, but rather an orbifold: a neighborhood of $\widehat\infty$ is homeomorphic to a cone over a 3-dimensional lens space whose fundamental group is cyclic of order $n-1$. This third

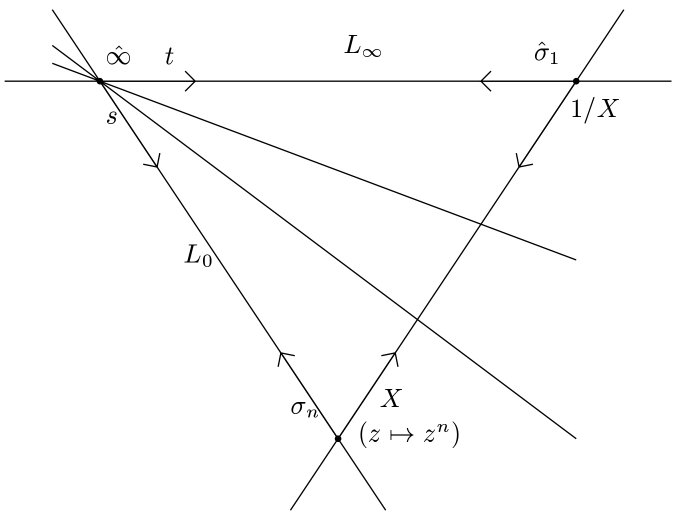


FIGURE 11. Schematic picture of the compactified moduli space $\widehat{\mathcal{M}} \cup \widehat\infty$, with a singular point at $\widehat\infty$, showing the three overlapping coordinate systems.

coordinate neighborhood W can be parametrized by coordinates s, t subject to the identifications $(s, t) = (\alpha s, \alpha t)$, where α ranges over all $(n-1)$-st roots of unity. They are related to the coordinates (X, σ_n) on the overlap where $\sigma_n \neq 0$ and $s \neq 0$, and to the coordinates $(\hat{X}, \hat{\sigma}_1)$ on the overlap where $\hat{\sigma}_1 \neq 0$ and $t \neq 0$, by the identities

$$X = \frac{t}{s}, \quad \sigma_n = \frac{1}{s^{n-1}},$$

and

$$\hat{X} = \frac{s}{t}, \quad \hat{\sigma}_1 = \frac{1}{n^{n+1}t^{n-1}}.$$

There is just one point $\widehat{\infty} \in W$ that does not belong to $\widehat{\mathcal{M}}$, namely the singular point $\widehat{\infty}$ with coordinates $s = t = 0$. It is not difficult to check that these coordinate transformations are compatible, so that $\widehat{\mathcal{M}} \cup \widehat{\infty}$ is a well defined compact Hausdorff space. A small neighborhood of infinity in $\widehat{\mathcal{M}}$ can be identified with a neighborhood of $\widehat{\infty}$ in W with this singular point removed. Evidently, for reasonable choice of neighborhood, the fundamental group will be cyclic of order $n-1$.

Intuitively, we can think of the fibers of our holomorphic line bundle as a pencil of lines though this exceptional point $\widehat{\infty}$. This pencil of lines sweeps out the compactified moduli space $\widehat{\mathcal{M}} \cup \widehat{\infty}$. Using this construction, we can reinterpret Lemma 6.1 as follows. If a sequence of points $(f_j) \in \mathcal{M}$ converges within $\widehat{\mathcal{M}} \cup \widehat{\infty}$ to the point $\widehat{\infty}$, then we are in Case (1) of Lemma 6.1, with all but one of the fixed point multipliers tending to infinity. On the other hand, if a sequence converges to some point of $L_\infty \subset \widehat{\mathcal{M}}$, then we are in Case (2) of Lemma 6.1, so that two of the multipliers converge to finite nonzero limits with product $+1$ while the others tend to infinity.

7. THE EXTENDED HYPERBOLIC SHIFT LOCUS $\widehat{\mathcal{S}}_{\mathrm{hyp}} \subset \widehat{\mathcal{M}}$

Theorem 7.1. *The connectedness locus $\mathcal{C}$ is contained in a compact subset of the extended moduli space $\widehat{\mathcal{M}}$. Hence the closure $\overline{\mathcal{C}} \subset \widehat{\mathcal{M}}$ is compact.*

It follows that the complement $\widehat{\mathcal{M}} \smallsetminus \overline{\mathcal{C}}$ is a neighborhood of infinity in $\widehat{\mathcal{M}}$. By definition, this complementary open neighborhood of infinity, consisting of $\mathcal{S}_{\mathrm{hyp}}$ together with the points of $L_\infty \smallsetminus (\overline{\mathcal{C}} \cap L_\infty)$, will be called the *extended hyperbolic shift locus* $\widehat{\mathcal{S}}_{\mathrm{hyp}}$. (See Theorem 7.4 for a precise description of $\overline{\mathcal{C}} \cap L_\infty$.)

The following statement is completely equivalent to Theorem 7.1.

Corollary 7.2. *There exists a constant k depending only on the degree n with the following property. If all but one of the fixed point multipliers of f are greater than k in absolute value, then (f) belongs to the hyperbolic shift locus.*

(Compare [Milnor 1993, § 8.8], which shows that the best value for k in the degree 2 case lies between 3 and 6.)

Proof of Theorem 7.1 and Corollary 7.2. We will first find a rough criterion for belonging to the hyperbolic shift locus depending on the invariants X, Y_1, Y_2, Y of Section 1, and then relate this to the topology of $\widehat{\mathcal{M}}$. Consider a sequence of maps f_j with marked critical points so that the conjugacy classes (f_j) tend to infinity in $\mathcal{M}$. Thus, passing to a subsequence if necessary, at least one of the invariants $X(f_j)$ and $Y(f_j)$ must tend to infinity. Since $Y_1 + Y_2 = Y$ and $Y_1Y_2 = (X+1)^{n+1}X^{n-1}$ by (1–3), it follows that Y_1 or Y_2 must tend to infinity. Interchanging the two critical points if necessary, we may assume that $Y_2(f_j) \to \infty$ as $j \to \infty$.

In particular, putting the f_j into the normal form (1–1), we may assume that the coefficients c and d are nonzero and hence, after a linear change of coordinate, we may assume that $c = d = 1$. In this way, taking the critical values to be $f(0) = v$ and $f(\infty) = v + \delta$, we obtain the normal form

$$f(z) = f_j(z) = \frac{(v+\delta)z^n + v}{z^n + 1} = v + \frac{\delta z^n}{z^n + 1}.$$

(We will write $v = v_j$ and $\delta = \delta_j$ whenever it is necessary for clarity.) Note that $f^{-1}(\infty)$ is the set of n-th roots of -1. The invariants

$$X = v/\delta, \quad Y_2 = 1/\delta^n$$

can be computed from (1–2). It follows that $\delta \to 0$, and that the critical value v satisfies the equation $v^n = X^n/Y_2$. It will be convenient to set $\varepsilon = \sqrt{|\delta|} > 0$. Thus $\varepsilon = \varepsilon_j$ also tends to zero as $j \to \infty$.

Case 1. Suppose that $v^n = X^n/Y_2$ is bounded away from -1, or equivalently that v is bounded away from the n-th roots of -1, as $j \to \infty$. Then we will prove that (f_j) belongs to the hyperbolic shift locus for large j. Let $N_\varepsilon(v)$ be the open neighborhood

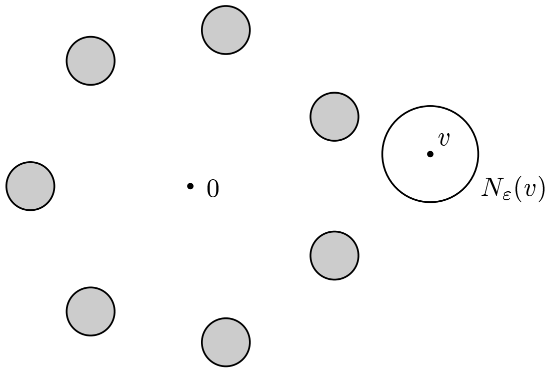


FIGURE 12. Dynamics for (f) near $\widehat{\infty}$. Here the critical points are at 0 and ∞, while the critical values as well as the attracting fixed point are in the disk $N_\varepsilon(v)$. The Julia set is contained in the n shaded disks $\overline{U}_i$, each of which maps diffeomorphically onto the complement $\widehat{\mathbb{C}} \smallsetminus \overline{N}_\varepsilon(v)$. As $(f) \to \widehat{\infty}$, each of these $n+1$ disks shrinks to a point.

of radius $\varepsilon = \sqrt{|\delta|}$ about v. Then the pre-image $W = f^{-1}N_\varepsilon(v)$ consists of all z with

$$|z^n + 1| > |\delta|/\varepsilon = \varepsilon.$$

If ε is small, this set can be described as the complement $W = \widehat{\mathbb{C}} \smallsetminus (\overline{U}_1 \cup \cdots \cup \overline{U}_n)$ of the union of closed neighborhoods $\overline{U}_1, \ldots, \overline{U}_n$ of the various n-th roots of -1, where each $\overline{U}_j$ has radius roughly ε/n. Thus W is a connected open set which contains both critical points. (Compare the figure, where W is the complement of the shaded region.) Since v is bounded away from the roots of -1, it follows that the image $f(W) = N_\varepsilon(v)$ is compactly contained in W provided that j is sufficiently large. Using the Poincaré metric for W, it follows that all orbits in W converge to a common attracting fixed point. Using Theorem B.5, it follows that (f_j) is contained in the hyperbolic shift locus.

Case 2. Suppose in fact that v^n converges to -1 (and hence that $|v| \to 1$), but that

$$|1 + v^n|/\varepsilon \to \infty \quad \text{as } j \to \infty,$$

so that $\varepsilon \ll |1 + v^n|$ for large j. Note that the distance between v and the closest n-th root of -1 is approximately $|1 + v^n|/n$ for large j. It now follows, just as in Case 1, that $f(W)$ is compactly contained in W, and hence that (f_j) belongs to the hyperbolic shift locus, provided that j is large.

Remark 7.3. More explicitly, in both Cases 1 and 2, it follows for large j that the Julia set J is the disjoint union of compact subsets $J_k = J \cap \overline{U}_k$, where f maps each $\overline{U}_k$ diffeomorphically onto the strictly larger set $\widehat{\mathbb{C}} \smallsetminus N_\varepsilon(v)$. Hence f is hyperbolic, that every point of J is uniquely determined by its itinerary with respect to the partition $J = J_1 \cup \ldots \cup J_n$. In fact, as $j \to \infty$, it follows that the Julia set becomes more and more hyperbolic, in the sense that the multiplier of any periodic orbit in J tends to infinity.

Case 3. We now return to the proof of Theorem 7.1 and Corollary 7.2. Suppose finally that the ratio $|1 + v^n|/\varepsilon$ remains bounded as $j \to \infty$, although $\varepsilon \to 0$ hence $1 + v^n \to 0$. (By Cases 1 and 2, this must be the case if each (f_j) belongs to the connectedness locus.) Then we will show that the (f_j) remain within some compact subset of $\widehat{\mathcal{M}}$. Setting

$$1 + v^n = 1 + X^n/Y_2 = \eta,$$

we see that $|\eta|$ is less than some constant time ε, so that $|\eta^2|$ is less than a constant times $\varepsilon^2 = |\delta| \sim 1/|X|$. We can now solve for

$$\begin{aligned} Y_2 &= -X^n(1 + \eta + \eta^2 + \cdots) \\ &= -X^n - X^n\eta + O(X^{n-1}), \\ Y_1 &= (X+1)^{n+1}X^{n-1}/Y_2 = -(X+1)^{n+1}(1-\eta)/X \\ &= -X^n + X^n\eta + O(X^{n-1}), \end{aligned}$$

and hence

$$Y = Y_1 + Y_2 = -2X^n + O(X^{n-1}).$$

Since X tends to infinity, it is appropriate to pass to the coordinates $\hat{X} = 1/X$ and $\hat{\sigma}_1 = \sum 1/\lambda_k$ of Section 6. Recall from Theorem 2.6 that

$$\begin{aligned} \hat{\sigma}_1 = \frac{\sigma_n}{\sigma_{n+1}} &= \frac{n^n(Y + P_n(X))}{n^{n+1}P_{n+1}(X)} \\ &= \frac{Y + 2X^n + O(X^{n-1})}{nX^{n-1}}. \end{aligned}$$

It follows that $\hat{\sigma}_1$ remains bounded as $j \to \infty$ and $X \to \infty$, so that the (f_j) remain within a compact subset of the extended moduli space $\widehat{\mathcal{M}}$.

Theorem 7.1 now follows immediately, and Corollary 7.2 is proved as follows. Recall that λ_2 is defined to be the fixed point multiplier with next-to-smallest absolute value. Suppose, for every positive integer j, that we could find a conjugacy class $(f_j) \in \mathcal{M} \cap \overline{\mathcal{C}}$ with $|\lambda_2(f_j)| > j$. The resulting sequence cannot be in Case 1 or Case 2, and therefore must be in Case 3, with $X(f_j) \to \infty$ and $\hat{\sigma}_1(f_j)$ bounded. But this contradicts the hypothesis that $|\lambda_2(f_j)| \to \infty$. □

We can describe the set $\overline{\mathcal{C}} \subset \widehat{\mathcal{M}}$ and its complement $\widehat{\mathcal{S}}_{\rm hyp}$ more precisely as follows.

Theorem 7.4. *The closure of the connected locus $\mathcal{C}$ within the extended moduli space $\widehat{\mathcal{M}} = \mathcal{M} \cup L_\infty$ is a compact set which consists of $\mathcal{C} \cup \mathrm{Per}_1(1) \subset \mathcal{M}$, together with a closed line segment consisting of all points in the line $L_\infty \cong \mathbb{C}$ such that the coordinate $\hat{\sigma}_1$ is real with $-2 \le \hat{\sigma}_1 \le 2$.*

Equivalently, we can say that $\overline{\mathcal{C}} \cap L_\infty$ consists of all points of L_∞ with $\hat{\sigma}_1$ of the form

$$\hat{\sigma}_1 = 2\cos\theta = \lambda + \lambda^{-1}, \quad \text{where } \lambda = e^{i\theta},$$

so that λ ranges over the unit circle. It follows that the complement

$$\widehat{\mathcal{S}}_{\rm hyp} = \widehat{\mathcal{M}} \smallsetminus \overline{\mathcal{C}}$$

consists of $\mathcal{S}_{\rm hyp} \subset \mathcal{M}$, together with all points of L_∞ with $\hat{\sigma}_1 \in \mathbb{C} \smallsetminus [-2, 2]$ or in other words all points for which

$$\hat{\sigma}_1 = \lambda + \lambda^{-1} \quad \text{with } 0 < |\lambda| < 1.$$

The proof of Theorem 7.4 begins as follows. To see that $\overline{\mathcal{C}}$ contains all points of L_∞ with $\hat{\sigma}_1 = e^{i\theta} + e^{-i\theta}$, note that each such point can be described as the intersection

$$L_\infty \cap \overline{\mathrm{Per}_1}(e^{i\theta}).$$

But clearly $\mathrm{Per}_1(e^{i\theta})$ is disjoint from the hyperbolic shift locus, and hence contained in $\overline{\mathcal{C}}$.

The proof in the other direction will depend on a study of dynamic behavior for conjugacy classes $(f) \in \mathcal{M} \subset \widehat{\mathcal{M}}$ which are sufficiently close to the line $L_\infty \subset \widehat{\mathcal{M}}$. We must first find a normal form for such classes. Suppose the multipliers at the $n+1$ fixed points are approximately $\lambda, \lambda^{-1}, \infty, \ldots, \infty$. Here we must exclude the case $\lambda = 0$, and also the case $\lambda = 1$ which turns out to be particularly difficult. We will use the normal form (1–1) with $ad - bc = 1$ and with $a + b = c + d = u$, so that there is a fixed point of multiplier $\lambda = n/u^2$ at $z = 1$. Suppose that there is another fixed point at $z = \kappa$. (Here we must assume that $\kappa^n \ne 1$, since $\kappa^n = 1$ would imply that $f(\kappa) = f(1) = 1$. In practice, we will only be interested in the limiting case as $\kappa \to 1$.) Solving the fixed point equation $a\kappa^n + b = (c\kappa^n + d)\kappa$ (together with the equations $ad - bc = 1$ and $a + b = c + d = u$), we find the unique solution

$$a = -P + \kappa Q, \quad b = \kappa^n P - \kappa Q,$$
$$c = \ -P + Q, \quad d = \kappa^n P - Q,$$

where we have temporarily introduced the abbreviations

$$P = \frac{u}{\kappa^n - 1}, \qquad Q = \frac{1}{u(\kappa - 1)}.$$

It follows easily that

$$X = bc = -(\kappa^n P - \kappa Q)(P - Q)$$

is given by the asymptotic formula

$$X \sim -\frac{(\lambda - 1)^2}{n\lambda}\,\frac{1}{(\kappa - 1)^2} \quad \text{as } \kappa \to 1.$$

Thus, given $\lambda \ne 0, 1$, we can realize any large value of the invariant X by suitable choice of the parameter $\kappa \approx 1$. (Note by Lemma 2.1 that X and λ together determine the remaining invariant Y.) The multiplier at κ, call it λ', can be computed by the formula

$$\lambda' = \frac{n\kappa^{n-1}}{(c\kappa^n + d)^2} = n\kappa^{n-1}\Big(\frac{u(\kappa - 1)}{\kappa^n - 1}\Big)^2 \sim u^2/n = 1/\lambda$$

as $\kappa \to 1$. For $\kappa \approx 1$, the two critical values $f(\infty) = a/c$ and $f(0) = b/d$ are close to 1 and κ, and extremely close to each other. In fact the difference is given by

$$\frac{a}{c} - \frac{b}{d} = \frac{1}{cd} \sim \frac{1}{X} = O(\kappa - 1)^2,$$

and we can compute

$$\frac{a}{c} = t \cdot 1 + (1 - t)\cdot\kappa \quad \text{with } t = \frac{1}{1 - Q/P} \sim \frac{1}{1 - \lambda},$$

and similarly $b/d = t' \cdot 1 + (1 - t')\cdot\kappa$ with $t' \sim 1/(1 - \lambda)$.

It will be convenient to conjugate f by the Möbius involution

$$\varphi(z) = \varphi^{-1}(z) = \kappa\,\frac{z - 1}{z - \kappa}.$$

which interchanges the critical point 0 with the fixed point 1, and interchanges the critical point ∞ with the fixed point κ. Thus the conjugate map

$$g = \varphi \circ f \circ \varphi$$

has critical points at $\varphi(0) = 1$ and at $\varphi(\infty) = \kappa$, and has a fixed point of multiplier λ at $\varphi(1) = 0$

and a fixed point of multiplier $\approx \lambda^{-1}$ at $\varphi(\kappa) = \infty$. The critical values $g(1)$ and $g(\kappa)$ are both close to λ. In fact computation shows that

$$g(\kappa) = \kappa Q/P = \frac{\kappa + \kappa^2 + \cdots + \kappa^n}{u^2} = \lambda + O(\kappa - 1)$$

as $\kappa \to 1$, and similarly

$$g(1) = g(\kappa)/\kappa^n = \lambda + O(\kappa - 1).$$

Theorem 7.5. *Suppose that we fix $\lambda \neq 0, 1$. Then as $\kappa \to 1$ the rational function $g = g_\kappa$ converges uniformly to the linear function $w \mapsto \lambda w$ throughout any compact subset of $\widehat{\mathbb{C}} \smallsetminus \{1\}$.*

(Compare [Milnor 1993, § 4]. Here "uniform convergence" refers to convergence with respect to the spherical metric on the target space $\widehat{\mathbb{C}}$.) Thus, as $\kappa \to 1$ the two critical points of g crash together, and g converges to a linear map, except in a small neighborhood of 1.

The proof will be based on the following. Let $\mathbb{D}_r$ be the disk $\{z \in \mathbb{C} : |z| < r\}$. If U is an open neighborhood of 0 in $\mathbb{C}$, it will be convenient to define the *inradius*

$$s(U_i) = \max\{s : \mathbb{D}_s \subset U\} = \operatorname{dist}(0, \mathbb{C} \smallsetminus U)$$

as the maximum radius of a disk centered at the origin and contained in U.

Lemma 7.6. *Given simply connected open sets $U_i \subset \mathbb{C}$ with inradius $s(U_i)$ tending to infinity, there exist radii $r_i \to \infty$ and conformal isomorphisms $h_i : \mathbb{D}_{r_i} \xrightarrow{\cong} U_i$ so that $\{h_i\}$ converges uniformly to the identity map on any compact subset of $\mathbb{C}$.*

Proof. Let u range over univalent maps $u : \mathbb{D} \to \mathbb{C}$ on the unit disk with $u(0) = 0$ and $u'(0) = 1$. Note that the space of all such u is compact, and that the inradius of the image always satisfies

$$s\big(u(\mathbb{D})\big) \leq 1. \tag{7–1}$$

(See [Carleson and Gamelin 1993, § I.1], for example.) It follows that the second derivative of u is uniformly bounded on any compact subset of $\mathbb{D}$, say

$$|u''(w)| \leq 2k \quad \text{for } |w| \leq \tfrac{1}{2}.$$

Hence

$$|u(w) - w| \leq k|w^2| \quad \text{for } |w| \leq \tfrac{1}{2} \tag{7–2}$$

and for all such maps u, where k is a uniform constant. Let $\rho_i : \mathbb{D} \xrightarrow{\cong} U_i$ be the Riemann map, satisfying $\rho_i(0) = 0$ and $\rho_i'(0) = r_i > 0$, and define $h_i : \mathbb{D}_{r_i} \to U_i$ by $h_i(z) = \rho_i(z/r_i)$. Applying (7–1) and (7–2) to the map $u(w) = \rho_i(w)/r_i = h_i(r_i w)/r_i$, we find easily that $s(U_i) \leq r_i$ so that $r_i \to \infty$, and that

$$|h_i(z) - z| \leq k|z^2|/r_i \quad \text{for } |z| \leq r_i/2.$$

Thus $\{h_i\}$ converges uniformly to the identity for z restricted to any compact subset of the plane. □

Proof of Theorem 7.5. Let $\Gamma = \Gamma_\kappa$ be a circle arc joining the two critical values $g(1)$ and $g(\kappa)$. Then the preimage $g^{-1}(\Gamma)$ is a union of n circle arcs joining the two critical points 1 and κ, where two adjacent arcs span an angle of $2\pi/n$ at either common endpoint. Furthermore, each of the n connected components of $\widehat{\mathbb{C}} \smallsetminus g^{-1}(\Gamma)$ maps biholomorphically onto $\widehat{\mathbb{C}} \smallsetminus \Gamma$. In fact the corresponding statement is true for any bicritical map, since any such map can be expressed as a composition of the n-th power map for which it is clearly true, together with Möbius transformations which carry circles to circles. (Compare the normal form (1–1).)

Both the diameter of Γ and the diameter of $g^{-1}(\Gamma)$ depend on the precise choice of circle arc Γ. However, with a little care we can choose $\Gamma = \Gamma_\kappa$ so that both of these diameters tend to zero as the distance $|\kappa - 1|$ between the two critical points tends to zero. It then follows that there is one largest component $U = U_\kappa$ of $\widehat{\mathbb{C}} \smallsetminus g^{-1}(\Gamma)$, and that the diameter of the complement $\widehat{\mathbb{C}} \smallsetminus U$ tends to zero as $\kappa \to 1$. After conjugating by a rotation of the Riemann sphere which interchanges 1 and ∞, we can use the Lemma to construct conformal isomorphisms $h_\kappa : \Delta_\kappa \to U_\kappa$, where $\Delta_\kappa \subset \widehat{\mathbb{C}} \smallsetminus \{1\}$ is the complement of a small round disk about 1, so that h_κ converges uniformly to the identity map on any compact subset of $\widehat{\mathbb{C}} \smallsetminus \{1\}$. Further, without loss of generality, we may assume that h_κ fixes the points 0 and ∞.

Similarly we can choose conformal isomorphisms $h'_\kappa : \widehat{\mathbb{C}} \smallsetminus \Gamma_\kappa \to \Delta'_\kappa$ fixing 0 and ∞, where Δ' is the complement of a small round disk about $g_\kappa(1) \approx \lambda$, so that h'_κ converges uniformly to the identity on any compact subset of $\widehat{\mathbb{C}} \smallsetminus \{1/\lambda\}$. Now the composition $h'_\kappa \circ g_\kappa \circ h_\kappa$ maps the round disk Δ_κ conformally onto the round disk Δ'_κ, and hence extends to a Möbius automorphism of the Riemann sphere. Since it fixes 0 and ∞, it must have the form

$$h'_\kappa \circ g_\kappa \circ h_\kappa(w) = \lambda_\kappa w,$$

where $\lambda_\kappa \to \lambda$ as $\kappa \to 1$. Now since h_κ and h'_κ converge uniformly to the identity except near 1 and $g_\kappa(1)$ respectively, it follows that $g_\kappa(w)$ converges to λw except near 1. □

Corollary 7.7. *For $0 < |\lambda| < 1$ and for $|X|$ greater than some constant depending continuously on λ, the conjugacy class (f) belongs to the hyperbolic shift locus.*

Proof. Suppose that $|\lambda| < 1-\varepsilon < 1$. Then $|g(w)| \approx |\lambda w| \le |w|(1-\varepsilon)$ for κ close to 1, or equivalently for $|X|$ large, provided that w is bounded away from 1. Thus every w in the disk $D_{1-\varepsilon}$ belongs to the attracting basin of 0 for $|X|$ large. Since the two critical values of g are close to λ, with $|\lambda| < 1-\varepsilon$, they both belong to this basin. By Theorem B.5, it follows that the conjugacy class $(f) = (g)$ belongs to $\mathcal{S}_{\text{hyp}}$. □

Evidently Theorem 7.4 follows immediately.

Lemma 7.8. *This open set $\mathcal{S}_{\text{hyp}} \subset \widehat{\mathcal{M}}$ is connected. Its fundamental group is cyclic of order $n-1$.*

The proof can be outlined as follows. Note first that there is a holomorphic map

$$\pi : \mathcal{S}_{\text{hyp}} \longrightarrow \mathbb{D}$$

to the open unit disk, which maps each (f) to the multiplier at its unique attracting fixed point. (The extension of π to the line at infinity is straightforward: If $\hat\sigma_1 = \lambda + \lambda^{-1}$ with $|\lambda| < 1$, then π takes the value λ.) For any $\lambda \in \mathbb{D} \smallsetminus \{0\}$, the fiber $\pi^{-1}(\lambda)$ is isomorphic to the open disk, with a point on L_∞ as center. (Compare the discussion in [Goldberg and Keen 1990] and [Milnor 1993].) In this way we obtain a topological fibration

$$\mathbb{D} \to \pi^{-1}(\mathbb{D} \smallsetminus \{0\}) \to \mathbb{D} \smallsetminus \{0\}.$$

However, the fiber $\pi^{-1}(0)$ is different in two ways. First it is missing a point at infinity, and hence is a punctured disk. Second, there is no local cross-section near zero. Instead we have something like a Seifert fibration. In fact, since $\sigma_{n+1}/n^{n+1} = X^{n-1}$ by Corollary 2, it follows that

$$\lambda \cong \hat\sigma_1^{-1} = n^{n+1} X^{n-1}/\sigma_n$$

for λ close to zero and A close to $+1$ in the shift locus. Thus a small loop around the line $A = 1$ in the shift locus corresponds to a loop in the λ-plane which winds $n-1$ times around the origin. Thus $n-1$ times a generator for the fundamental group of $\pi^{-1}(\mathbb{D} \smallsetminus \{0\}$, which is free cyclic, maps to zero in the fundamental group of $\mathcal{S}_{\text{hyp}}$.

8. THE CURVES $\overline{\text{Per}_p}(\lambda) \subset \widehat{\mathcal{M}}$

Roughly speaking, the curve $\text{Per}_p(\lambda)$ is the set of all conjugacy classes with an orbit of period $p \ge 1$ and multiplier $\lambda \in \mathbb{C}$. However, we must be somewhat careful to give a definition which yields a well behaved algebraic curve which depends continuously on λ, even in exceptional cases. (One difficulty arises when the multiplier is a root of unity, since a sequence of orbits of period pq with multiplier converging to $+1$ may converge to an orbit of period p with a q-th root of unity as multiplier. Another difficulty arises for $\lambda = 0$, since the curves $\text{Per}_p(\lambda)$ converge towards an $(n-1)$-fold branched covering of $\text{Per}_p(0)$ as $\lambda \to 0$.)

We first count periodic orbits. As in [Milnor 1993; 2000], define positive integers $\nu_n(p)$ by the formula

$$n^p = \sum_{q|p} \nu_n(q) \quad\Longleftrightarrow\quad \nu_n(p) = \sum_{q|p} \mu(p/q)\, n^q,$$

to be summed over all divisors $1 \le q \le p$, where the Möbius function $\mu(p/q)$ is defined to be $(-1)^m$ if $p/q = l_1 \cdots l_m$ is a product of m distinct primes l_j, and $\mu(p/q) = 0$ otherwise. The first few values are as follows.

p	1	2	3	4	5	6
$\nu_n(p)$	n	n^2-n	n^3-n	n^4-n^2	n^5-n	$n^6-n^3-n^2+n$

Since $f^{\circ p}$ is a rational map of degree n^p, with n^p+1 fixed points in the generic case, it follows easily that a generic f has $\nu_n(p)$ points of period p for $p \ge 2$. (However, for $p = 1$ it has $\nu_n(1) + 1 = n+1$ fixed points.)

We next construct a commutative diagram

$$\begin{array}{ccc} \mathbf{Per}_p & \to & \mathbf{Bicrit} \\ \downarrow & & \downarrow \\ \text{Per}_p & \to & \mathcal{M} \end{array}$$

where both horizontal arrows represent branched coverings having degree $\nu_n(p)$ for $p \geq 2$, but having degree $\nu_n(1) + 1$ for $p = 1$. (To simplify the notation, the subscript n on **Bicrit** and $\mathcal{M}$ has been suppressed.) Start with the variety

$$\mathbf{V}_p \subset \mathbf{Bicrit} \times \widehat{\mathbb{C}}$$

consisting of all pairs (f, z) where f is a bicritical map of degree n and z is a point of $\widehat{\mathbb{C}}$ satisfying $f^{\circ p}(z) = z$. Here p can be any positive integer. Note that $\mathbf{V}_q \subset \mathbf{V}_p$ for every divisor q of p, Let $\mathbf{Per}_p \subset \mathbf{Bicrit}_n \times \widehat{\mathbb{C}}$ be the Zariski closure of the set

$$\mathbf{V}_p \smallsetminus \bigcup_{\substack{q|p \\ q<p}} \mathbf{V}_q$$

consisting of points in $\mathbf{V}_p$ which do not belong to $\mathbf{V}_q$ for any $q < p$. Clearly $\mathbf{V}_p$ can be expressed as the union of $\mathbf{Per}_q$ as q ranges over all divisors of p including p itself.

By definition, two points (f, z) and (g, w) of $\mathbf{Per}_p$ are *conjugate* if there is a Möbius automorphism φ of $\widehat{\mathbb{C}}$ so that $g = \varphi \circ f \circ \varphi^{-1}$ and $w = \varphi(z)$. The orbifold consisting of all conjugacy classes $((f, z))$ of points of $\mathbf{Per}_p$ will be denoted by Per_p. Evidently the correspondence $((f, z)) \mapsto (f)$ yields the required branched covering $\mathrm{Per}_p \to \mathcal{M}$.

Remark. These varieties $\mathbf{Per}_p$ and Per_p are actually irreducible. The analogous statement for unicritical polynomials was proved by Bousch [1992] in the quadratic case and by Lau and Schleicher [1994] for all degrees. Irreducibility of these varieties for bicritical rational maps follows, since any irreducible component of Per_p must intersect the locus of polynomial maps.

Definition. The *multiplier map*

$$\lambda : \mathrm{Per}_p \to \mathbb{C}$$

carries each $((f, z))$ to the derivative of $f^{\circ p}$ at z. Let $F_{\lambda_0} \subset \mathrm{Per}_p$ be the fiber, consisting of all $((f, z))$ with $\lambda((f, z)) = \lambda_0$, and let $\mathrm{Per}_p(\lambda_0)$ be its image under the projection to $\mathcal{M}$. (Intuitively, a point of $\mathrm{Per}_p(\lambda)$ is a conjugacy class of maps (f) which posses a period p orbit of multiplier λ, while a point of F_λ consists of such an (f) together with a specific choice of period p point for a representative map f.)

Remark. Each k modulo p in the cyclic group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ acts on the varieties $\mathbf{Per}_p \to \mathrm{Per}_p$ by the correspondence $(f, z) \mapsto (f, f^{\circ k}(z))$. Thus we obtain quotient orbifolds and a larger commutative diagram

$$\begin{array}{ccccc}
\mathbf{Per}_p & \to & \mathbf{Per}_p/\mathbb{Z}_p & \to & \mathbf{Bicrit} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Per}_p & \to & \mathrm{Per}_p/\mathbb{Z}_p & \to & \mathcal{M} \\
& & \downarrow \lambda & & \\
& & \mathbb{C} & &
\end{array}$$

where now the horizontal arrows represent branched coverings of degree p on the left and $\nu_n(p)/p$ on the right. Starting with any $\lambda_0 \in \mathbb{C}$ we can form the preimage in $\mathrm{Per}_p/\mathbb{Z}_p$ and then project to the subset $\mathrm{Per}_p(\lambda_0) \subset \mathcal{M}$.

Theorem 8.1. *Let $\overline{\mathrm{Per}}_p(\lambda)$ be the closure of $\mathrm{Per}_p(\lambda)$ within the extended moduli space $\widehat{\mathcal{M}}$. If $\lambda \neq 0$, then $\overline{\mathrm{Per}}_p(\lambda)$ is a compact, not necessarily irreducible, algebraic curve in the extended moduli space $\widehat{\mathcal{M}}$. The projection map $((f, z)) \mapsto X(f)$ carries $\overline{\mathrm{Per}}_p(\lambda)$ onto $\widehat{\mathbb{C}}$ with degree $\nu_n(p)/n$. Equivalently, the number of intersections of $\overline{\mathrm{Per}}_p(\lambda)$ with any fiber L_X of the fibration $X : \widehat{\mathcal{M}} \to \widehat{\mathbb{C}}$, counted with multiplicity, is independent of λ and X, being equal to $\nu_n(p)/n$. The same statements are true for $\lambda = 0$ and $p \geq 2$ provided that the point set $\mathrm{Per}_p(0)$ is counted with multiplicity $n - 1$.*

In other words, for $p \geq 2$, it is asserted that the set $\mathrm{Per}_p(0)$ intersects a generic fiber L_X in

$$\frac{\nu_n(p)}{n(n-1)}$$

distinct points, each of which must be counted with multiplicity $n-1$ in order to get the correct count of $\nu_n(p)/n$. (Here is one intuitive explanation for this multiplicity: For $(f) \in \mathrm{Per}_p(0)$, suppose that we perturb f within the much larger space consisting of all rational maps of degree n. Generically, the periodic critical point will split up into $n-1$ nearby critical points, any one of which can be periodic. Thus the locus $\mathrm{Per}_p(0) \subset \mathcal{M}$ splits up locally, in this larger context, into $n - 1$ nearby sheets.)

Outline Proof of Theorem 8.1. We need only consider the case $p \geq 2$, since the period one case has been discussed in Sections 2 and 6. Since $\mathrm{Per}_p(\lambda)$ is an algebraic curve in $\mathcal{M} \cong \mathbb{C}^2$, it can be defined by a single polynomial equation in the coordinates X and σ_n. Substituting $\hat{X} = 1/X$ and $\hat{\sigma}_1 = \sigma_n/(n^{n+1}X^{n-1})$, and multiplying through by an appropriate power of $\hat{X}$, we obtain a corresponding polynomial equation relating $\hat{X}$ and $\hat{\sigma}_1$. This shows that $\overline{\mathrm{Per}}_p(\lambda)$ is an algebraic curve in the variety $\widehat{\mathcal{M}}$. If $|\lambda| > 1$ so that the associated periodic orbit is contained in the Julia set, then it follows from Remark 7.3 that this curve is contained in a compact subset of $\widehat{\mathcal{M}}$, and hence is itself compact. On the other hand, if $|\lambda| \le 1$ with period $p \ge 2$, then $\mathrm{Per}_p(\lambda)$ is contained in the connectedness locus, which has compact closure within $\widehat{\mathcal{M}}$.

To compute the degree of the projection map

$$\overline{\mathrm{Per}}_p(\lambda) \to \widehat{\mathbb{C}},$$

we first look at the exceptional special case $\lambda = 0$ and count the number of intersections of $\mathrm{Per}_p(0)$ with the fiber L_0. If $f_b(z) = z^n + b$ with invariants $X = 0$ and $Y = b^{n-1}$, then the equation $f_b^{\circ p}(0) = 0$ has degree n^{p-1} in the unknown b, so there are n^{p-1} solutions b, counted with multiplicity. Subtracting off the numbers of solutions for proper divisors of p, we see that there are $\nu_n(p)/n$ choices for b. Since we assume that $p \ge 2$, it follows that $b \ne 0$. Hence there are $n-1$ choices of b for every choice of $Y = b^{n-1}$. Thus the projection $X : \overline{\mathrm{Per}}_p(0) \to \widehat{\mathbb{C}}$ has degree $\nu_n(p)/(n(n-1))$ for $p \ge 2$. The first few values are as follows:

p	2	3	4	5
$\dfrac{\nu_n(p)}{n(n-1)}$	1	$n+1$	$n(n+1)$	$(n+1)(n^2+1)$
p		6		
$\dfrac{\nu_n(p)}{n(n-1)}$	$(n+1)(n^3+n-1)$			

To check what happens as we perturb the multiplier λ away from zero, we can simply apply the following result. See [Lau and Schleicher 1994, § 2.2].

Proposition (Lau and Schleicher). *If W is any hyperbolic component in the b-parameter plane for the family of maps $z \mapsto z^n + b$, then the multiplier map $W \to \mathbb{D}$ is an $(n-1)$-fold branched covering, ramified only at the unique inverse image of zero.*

Thus, as we perturb λ away from zero, each intersection of $\mathrm{Per}_p(0)$ with L_0 splits into $n-1$ distinct intersection, and we obtain the required count of $\nu_n(p)/n$ points in $\mathrm{Per}_p(\lambda) \cap L_0$ for $\lambda \ne 0$. □

Example 8.2. For the special case $p = 2$, it follows that $\overline{\mathrm{Per}}_2(0)$ is the image of a smooth section of the fibration $X : \widehat{\mathcal{M}} \to \widehat{\mathbb{C}}$. We can compute this section explicitly as follows. Using the normal form (1–1) with $ad - bc = 1$, recall that

$$X = bc, \quad X + 1 = ad, \quad Y_1 = a^{n+1}b^{n-1},$$

and that $f(0) = b/d$. Thus the critical point 0 has period exactly two if and only if

$$f(b/d) = \frac{ab^n + bd^n}{cb^n + d^{n+1}}$$

is zero, but $b \ne 0$. This yields the equation $ab^{n-1} + d^n = 0$. Multiplying by a^n, we obtain $Y_1 + (X+1)^n = 0$. If $Y_1 \ne 0$, it follows that

$$Y_2 = (X+1)^{n+1}X^{n-1}/Y_1 = -(X+1)x^{n-1},$$

and therefore that

$$Y = Y_1 + Y_2 = -(X+1)^n - (X+1)X^{n-1}.$$

On the other hand, if $Y_1 = 0$ then it is easy to check that $X + 1 = Y_2 = Y = 0$ also, so that the equation is still satisfied. Similarly, if the other critical point ∞ is periodic, then interchanging the roles of Y_1 and Y_2 we obtain the same equation.

Remark 8.3. It seems likely that the curve $\overline{\mathrm{Per}}_p(\lambda)$ is usually irreducible, however there are certainly exceptions. As an example, if n is odd then it was noted in Section 1 that the curve $\overline{\mathrm{Per}}_2(n^2)$ is reducible, since it contains the half symmetry locus Σ_- as an irreducible component. (This period two curve is strictly larger that Σ_- since it has degree $n - 1 \ge 2$ over $\widehat{\mathbb{C}}$ while Σ_- has degree one.)

Here is another example. If $1 \le q < p$ is a proper divisor of p, and if ξ is a primitive (p/q)-th root of unity, then it is not hard to show that

$$\mathrm{Per}_q(\xi) \subset \mathrm{Per}_p(1)$$

since, under a small perturbation of f, any period q orbit of multiplier ξ will split off a period p orbit with multiplier close to 1. In general, it follows that $\mathrm{Per}_p(1)$ is reducible for $p \ge 2$. The only exception occurs in the degree two case, where $\mathrm{Per}_2(1)$ actually coincides with the curve $\mathrm{Per}_1(-1)$. (Thus a quadratic rational map cannot have any period two orbit with multiplier 1.)

We can describe the intersection of $\overline{\mathrm{Per}}_p(\eta)$ with the line at infinity rather precisely as follows.

Theorem 8.4. *For $p \geq 2$ the intersection $\overline{\mathrm{Per}}_p(\eta) \cap L_\infty$ depends only on the period p and not on the multiplier η. This intersection is a finite set consisting of points with coordinate $\hat{\sigma}_1$ of the form $\lambda + \lambda^{-1}$ where λ is a q-th root of unity for some $q \leq p$.*

Proof. First consider the case $\eta = 0$, and suppose that $\lambda^q \neq 1$ for $1 \leq q \leq p$. In other word, we suppose that the complex numbers

$$1,\, \lambda,\, \lambda^2,\, \ldots,\, \lambda^p$$

are all distinct. Let $(f) \in \mathcal{M}$ be a conjugacy class with $\hat{X}$ close to zero and with $\hat{\sigma}_1$ close to $\lambda + \lambda^{-1}$. Using Theorem 7.5 we can find a normal form for (f) so that both critical points are close to 1, with the first $p+1$ points of both critical orbits close to

$$1 \mapsto \lambda \mapsto \lambda^2 \mapsto \cdots \mapsto \lambda^p \neq 1.$$

Thus, if the pair $(\hat{X}, \hat{\sigma}_1)$ is sufficiently close to $(0, \lambda + \lambda^{-1})$, then it follows that neither critical orbit can have period p, hence $(f) \notin \mathrm{Per}_p(0)$, as required.

(This argument does not apply if $\lambda^q = 1$ for some $q < p$. For then the q-th forward image of the critical point is close to 1, lying in a small region whose image under f covers the entire Riemann sphere. Hence the further orbit of the critical points cannot be predicted without more information.)

Now consider $\mathrm{Per}_p(\eta)$ with variable η. Consider the coordinate $\hat{\sigma}_1$ for the various points of $\mathrm{Per}_p(\eta) \cap L_\infty$. The elementary symmetric functions of these $\nu_n(p)/n$ (not usually distinct) coordinate values can be expressed as holomorphic functions of the parameter η. But if $|\eta| < 1$and $p \geq 2$ then $\mathrm{Per}_p(\eta)$ is contained in the connectedness locus, so that

$$\overline{\mathrm{Per}}_p(\eta) \cap L_\infty \subset \overline{\mathcal{C}} \cap L_\infty \cong [-2, 2] \subset \mathbb{R}.$$

In other words, every one of the $\nu_n(p)/n$ values of $\hat{\sigma}_1$ is real, hence the elementary symmetric functions of these coordinate values are also real. But a holomorphic function from $\eta \in \mathbb{C}$ to $\mathbb{C}$ which takes only real values throughout the open disk $|\eta| < 1$ must be constant. This proves the the intersection is independent of η for all $\eta \in \mathbb{C}$. □

Remark 8.5. More precisely, for $p \geq 2$ it can be conjectured that the point of L_∞ with coordinate $\hat{\sigma}_1 = \lambda + \lambda^{-1}$ belongs to $\overline{\mathrm{Per}}_p(\eta)$ if and only if λ is a primitive q-th root of unity for some $2 \leq q \leq p$, or in other words if and only if

$$\hat{\sigma}_1 = 2\cos\frac{2\pi r}{q}$$

for some integers $0 < r < q \leq p$. Thus it is believed that the point of L_∞ with coordinate $\hat{\sigma}_1 = 2$ does not lie on any $\overline{\mathrm{Per}}_p(\eta)$ with $p \geq 2$. (Compare [Stimson 1993] and [Rees $\geq$ 2000], which give a more precise description of the curves $\overline{\mathrm{Per}}_p(0)$ near L_∞ in the degree two case, and see also [Milnor 1993].)

Following is an intuitive argument which attempts to justify this statement, and also to compute the multiplicities of the various points of $\mathrm{Per}_p(0) \cap L_\infty$. I suspect that it could be made into a rigorous proof, but certainly have not done so. We count solutions to the equation $g(z) = k$, where $k \in \mathbb{C}$ is some given constant and where g is the bicritical map of Theorem 7.5, with $|X|$ very large so that $g(z) \approx \lambda z$ outside of a small neighborhood of 1. If k is bounded away from λ, then there is a unique solution $z \approx k/\lambda$ which is bounded away from 1. Hence the remaining $n-1$ solutions must all lie in some small neighborhood of 1. We express these facts symbolically, in the limit as $|X| \to \infty$, by writing

$$k/\lambda \longrightarrow k$$

but

$$1 \xrightarrow{n-1} k \quad \text{for } k \neq \lambda.$$

In the special case $k = \lambda$, all n of the solutions must be very close to 1, so we write

$$1 \xrightarrow{n} \lambda.$$

The numbers on the right in the table are computed by muliplying these factors of $n-1$ or n by the number of values of r/q which are listed on the left. If $N(p, q)$ denotes the number of solutions of period dividing p which are computed in this way, corresponding to the case where λ is a primitive q-th root of unity, $q \geq 2$, then it is not hard to check that these numbers can be computed recursively as follows: $N(p, q) = 0$ if $p < q$; $N(p, q) = \varphi(q)$ if $p = q$, where $\varphi(q)$, the Euler φ-function, is the number of primitive q-th roots of unity; and

$$N(p, q) = nN(p-q, q) + \sum_{j=1}^{q-1}(n-1)N(p-j, q)$$

p	r/q	limiting orbit of $g(1) \approx \lambda$	number	total
2	$\frac{1}{2}$	$\lambda \longrightarrow 1$	1	
				1
3	$\frac{1}{3}, \frac{2}{3}$	$\lambda \longrightarrow \lambda^2 \longrightarrow 1$	2×1	
	$\frac{1}{2}$	$\lambda \longrightarrow 1 \overset{n-1}{\longrightarrow} 1$	$n-1$	
				$n+1$
4	$\frac{1}{4}, \frac{3}{4}$	$\lambda \longrightarrow \lambda^2 \longrightarrow \lambda^3 \longrightarrow 1$	2×1	
	$\frac{1}{3}, \frac{2}{3}$	$\lambda \longrightarrow \lambda^2 \longrightarrow 1 \overset{n-1}{\longrightarrow} 1$	$2\times(n-1)$	
	$\frac{1}{2}$	$\lambda \longrightarrow 1 \overset{n}{\longrightarrow} \lambda \longrightarrow 1$	n	
	$\frac{1}{2}$	$\lambda \longrightarrow 1 \overset{n-1}{\longrightarrow} 1 \overset{n-1}{\longrightarrow} 1$	$(n-1)^2$	
				n^2+n+1
5	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\lambda \longrightarrow \lambda^2 \longrightarrow \lambda^3 \longrightarrow \lambda^4 \longrightarrow 1$	4×1	
	$\frac{1}{4}, \frac{3}{4}$	$\lambda \longrightarrow \lambda^2 \longrightarrow \lambda^3 \longrightarrow 1 \overset{n-1}{\longrightarrow} 1$	$2\times(n-1)$	
	$\frac{1}{3}, \frac{2}{3}$	$\lambda \longrightarrow \lambda^2 \longrightarrow 1 \overset{n-1}{\longrightarrow} \lambda^2 \longrightarrow 1$	$2\times(n-1)$	
	$\frac{1}{3}, \frac{2}{3}$	$\lambda \longrightarrow \lambda^2 \longrightarrow 1 \overset{n-1}{\longrightarrow} 1 \overset{n-1}{\longrightarrow} 1$	$2\times(n-1)^2$	
	$\frac{1}{2}$	$\lambda \longrightarrow 1 \overset{n}{\longrightarrow} \lambda \longrightarrow 1 \overset{n-1}{\longrightarrow} 1$	$n(n-1)$	
	$\frac{1}{2}$	$\lambda \longrightarrow 1 \overset{n-1}{\longrightarrow} 1 \overset{n}{\longrightarrow} \lambda \longrightarrow 1$	$n(n-1)$	
	$\frac{1}{2}$	$\lambda \longrightarrow 1 \overset{n-1}{\longrightarrow} 1 \overset{n-1}{\longrightarrow} 1 \overset{n-1}{\longrightarrow} 1$	$(n-1)^3$	
				n^3+n^2+n+1

TABLE 1. Numbers of points $(g) \in L_X$ such that g admits a periodic critical orbit of period dividing p, with $|X|$ very large, grouped by the conjectured limiting behavior of this orbit as $X \to \infty$. By definition, $\lambda = e^{2\pi i r/q}$ in each row. The sum on the right must be equal to $1 + n + n^2 + \cdots + n^{p-2}$ for all values of p.

if $p > q$. If $N(p,q)$ is defined by this recursion, then it seems empirically that the sum over q is given by

$$\sum_{q\geq 2} N(p,q) = 1 + n + n^2 + \cdots + n^{p-2}.$$

For any given p, this is easy to check by computer; but I don't know a general proof.

On the other hand, for every $p \geq 2$, the actual number of solutions of period exactly p is known to be $\nu_n(p)/(n(n-1))$. There are no solutions of period 1, hence the number of solutions with period d dividing p must be equal to

$$\sum_{\substack{d|p \\ d\neq 1}} \frac{\nu_n(d)}{n(n-1)} = \frac{n^p - n}{n(n-1)} = 1 + n + n^2 + \cdots + n^{p-2}.$$

This agrees precisely with the intuitive computation, which supports the conjecture that the computed values $N(p,q)$ are indeed correct, and that there are no solutions at all corresponding to the case $\lambda = 1$.

It is interesting to note that, for each p, the contribution from $q = 2$ is by far the largest. Indeed, for large n, the contribution of $(n-1)^{p-2}$, corresponding to a critical orbit for g which tends asymptotically to

$$1 \mapsto -1 \mapsto 1 \mapsto 1 \mapsto \cdots \mapsto 1,$$

contributes much more than all of the other asymptotic behaviors combined. The associated point of $\mathrm{Per}_p(0) \cap L_\infty$ has coordinate $\hat{\sigma}_1 = \lambda + \lambda^{-1} = -2$.

APPENDIX A. NO HERMAN RINGS

The following statement is a straightforward consequence of results of Shishikura.

Theorem A.1. *A rational map f with only two critical points cannot have any Herman rings.*

The proof we give is closely modeled on Shishikura's argument (compare [Shishikura 1987, § 8 and remark on page 4]), although it avoids the use of surgery constructions.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$ which possesses a cycle of Herman rings $H = H_1 \cup \cdots \cup H_p$ of period $p \geq 1$. That is, assume that the H_i are disjoint annuli with $\partial H_i \subset J(f)$, where i ranges over the group of integers modulo p, and assume that f maps each H_i diffeomorphically onto H_{i+1}. Choosing some base point $b \in H$, the closure of the forward orbit of b is a union $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_p$ of smooth simple closed curves, where $\Gamma_i \subset H_i$ and $f(\Gamma_i) = \Gamma_{i+1}$. Since $f(\Gamma) = \Gamma$, we have

$$\Gamma \subset f^{-1}(\Gamma) \subset f^{-2}(\Gamma) \subset \cdots .$$

(If Γ is chosen to be disjoint from the postcritical set, then each $f^{-n}(\Gamma)$ will be a union of at most pd^n disjoint simple closed curves.) The connected components of $\widehat{\mathbb{C}} \smallsetminus f^{-n}(\Gamma)$ are open sets which will be called *Shishikura puzzle pieces* of *level* n. Evidently f maps each puzzle piece of level $n > 0$ onto a puzzle piece of level $n-1$ by a possibly branched covering map.

Each Γ_i separates the Riemann sphere into two disks. In order to label these disks, choose orientations for the loops Γ_i compatible with the mapping. Then for each Γ_i we can speak of the disk D_i^{L} to the *left* of Γ_i with boundary Γ_i, and the disk D_i^{R} to the *right* of Γ_i with boundary Γ_i . Fixing some level n, note that each Γ_i lies on the boundary of exactly two puzzle pieces. Let $L_i = L_i^{(n)} \subset D_i^{\mathrm{L}}$ be the adjacent piece to the left of Γ_i and $R_i = R_i^{(n)} \subset D_i^{\mathrm{R}}$ the adjacent piece to the right. Define the *level n left neighborhood* $L^{(n)}$ of Γ to be the union $L_1^{(n)} \cup \cdots \cup L_p^{(n)}$, and define the *right neighborhood* $R^{(n)} = R_1^{(n)} \cup \cdots \cup R_p^{(n)}$ similarly. Then we have the following result. (See [Shishikura 1987, § 7].)

Theorem A.2. *The left neighborhood $L^{(n)}$ of Γ contains at least one critical point of f, and similarly the right neighborhood $R^{(n)}$ contains at least one critical point.*

Proof. We may assume that $n \geq 1$. If $L_1 \cup \cdots \cup L_p$ contains no critical point, then we will show that f must map each L_i onto L_{i+1}, and hence that $f^{\circ p}$ must map L_i onto itself. This would imply that each L_i is contained in the Fatou set $\mathbb{C} \smallsetminus J(f)$. But that is impossible, since L_i intersects the boundary of the ring H_i, which is contained in the Julia set. This contradiction will prove the Theorem.

For each left hand puzzle piece $L_i^{(n)}$ we have the following diagram.

$$L_i^{(n)} \xrightarrow{f} L_{i+1}^{(n-1)} \supset L_{i+1}^{(n)}.$$

If there is no critical point in $L_i^{(n)}$, then f maps $L_i^{(n)}$ onto $L_{i+1}^{(n-1)}$ by a covering map (which may be one-to-one). Hence, if we pass to universal covering spaces, this diagram takes the form

$$\tilde{L}_i^{(n)} \cong \tilde{L}_{i+1}^{(n-1)} \longleftarrow \tilde{L}_{i+1}^{(n)}.$$

In other words, we can choose a single valued holomorphic branch of f^{-1} which maps $\tilde{L}_{i+1}^{(n)}$ into $\tilde{L}_i^{(n)}$. Each puzzle piece $L_i^{(n)}$ is a hyperbolic Riemann surface, and hence has a Poincaré metric. Fixing n, for any smooth curve segment $\alpha : [0,1] \to L_i^{(n)}$, let $l_i(\alpha)$ be the Poincaré arclength. Then one of the following two possibilities must hold:

Case 1. If $f(L_i^{(n)})$ is precisely equal to $L_{i+1}^{(n)}$, then this branch of f^{-1} is a Poincaré isometry from $\tilde{L}_{i+1}^{(n)}$ onto $\tilde{L}_i^{(n)}$. In this case f preserves Poincaré arclength, so that

$$l_{i+1}(f \circ \alpha) = l_i(\alpha)$$

for every smooth curve in $L_i^{(n)}$.

Case 2. If $f(L_i^{(n)})$ is strictly larger than $L_{i+1}^{(n)}$, then this branch of f^{-1} is strictly distance reducing. Thus f must strictly increase Poincaré arclength, in the sense that

$$l_{i+1}(f \circ \alpha) > l_i(\alpha)$$

for any non-constant curve in $L_i^{(n)}$ which maps into $L_{i+1}^{(n)}$.

Now choose some orbit closure $\Gamma' \subset H \cap L^{(n)}$, and let $\alpha : \mathbb{R}/\mathbb{Z} \to \Gamma' \cap H_0$ be a smooth parametrization of one connected component. Then $f^{\circ i} \circ \alpha$ is a smooth parametrization of $\Gamma' \cap H_i$. Comparing the discussion above, we have

$$\begin{aligned} l_0(\alpha) &\leq l_1(f \circ \alpha) \leq l_2(f^{\circ 2} \circ \alpha) \\ &\leq \cdots \leq l_p(f^{\circ p} \circ \alpha) = l_0(\alpha), \end{aligned}$$

since $f^{\circ p} \circ \alpha$ and α parametrize the same loop. Thus equality must hold throughout, and Case 2 can never occur. This proves that f must map every $L_i^{(n)}$ onto $L_{i+1}^{(n)}$. As noted above, this leads to a contradiction. Therefore $L^{(n)}$ must contain at least one critical point, which completes the proof of Theorem A.2. $\square$

Remark A.3. *If $n \geq p-1$, then $L^{(n)}$ is disjoint from $R^{(n)}$.* This statement is clear when $p = 1$, since $L_i^{(n)} \cap R_i^{(n)} = \varnothing$. If $L^{(p-1)} \cap R^{(p-1)} \neq \varnothing$ with $p > 1$, then $L_{i_0}^{(p-1)}$ would be precisely equal to $R_{i_0+\delta}^{(p-1)}$ for some i_0 and some $\delta \not\equiv 0$. This would imply that $L_{i_0+1}^{(p-2)} = R_{i_0+1+\delta}^{(p-2)}$. Continuing inductively it would follow that $L_i^{(0)} = R_{i+\delta}^{(0)}$ for every i. But this would imply that the entire disk D_i^{R} to the right of Γ_i is contained in $D_{i+\delta}^{\mathrm{R}}$. Therefore

$$D_0^{\mathrm{R}} \subset D_\delta^{\mathrm{R}} \subset D_{2\delta}^{\mathrm{R}} \subset \cdots \subset D_{p\delta}^{\mathrm{R}} = D_0^{\mathrm{R}},$$

which is impossible.

Now we specialize to the case of a rational map of degree d with only two critical points. Note the following basic observation. If a simple closed curve $\Gamma_i \subset \widehat{\mathbb{C}}$ bounds a closed disk $\overline{D}$ which is disjoint from the two critical values, then $f^{-1}(\overline{D})$ is the union of d disjoint topological disks, each of which contains no critical point and maps homeomorphically onto $\overline{D}$. On the other hand, if Γ_i separates the two critical values, then $f^{-1}(\Gamma_i)$ is a single simple closed curve, which maps onto Γ_i by a d-fold covering map. Evidently this second case can never occur for the loops Γ_i associated with a cycle of Herman rings.

It will be convenient to put one of the two critical points at infinity. Then one of the two components of the complement of any Γ_i is bounded, and maps diffeomorphically onto its image under f . I will call this component the *inside* D_i^{in} of Γ_i . The other component D_i^{out} is unbounded, and contains both critical points.

Call Γ_k a *separating* loop if it separates the critical points from the critical values. *Then the inside D_i^{in} of each Γ_i maps diffeomorphically onto:*

the outside D_{i+1}^{out} if Γ_{i+1} is separating,

the inside D_{i+1}^{in} if Γ_{i+1} is non-separating.

Call Γ_k *minimal* if the union Γ is disjoint from the open disk D_k^{in}.

Lemma A.4. *There exists a (necessarily unique) Γ_k which is both separating and minimal.*

Proof [Shishikura 1987, § 8]. There certainly exists at least one minimal Γ_i. If every Γ_i were minimal and non-separating, then the inside of every Γ_i would map diffeomorphically onto the inside of Γ_{i+1} and we would have a cycle of Siegel disks rather than Herman rings. Therefore, either there is some Γ_i which is minimal and separating, as required, or else there is some Γ_i which is not minimal. In the latter case, we can choose some non-minimal Γ_i so that Γ_{i+1} is minimal. Then D_i^{in}, which contains other Γ_j, cannot map diffeomorphically onto D_{i+1}^{in} which does not. Hence in this case Γ_{i+1} must be a minimal separating loop, as required. $\square$

Proof of Theorem A.1. Let Γ_k be the loop of Lemma A.4. Since Γ_k is minimal, the closure $\overline{D}_k^{\mathrm{out}}$ contains the union $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_p$. Since Γ_k is separating, the preimage $f^{-1}(\overline{D}_k^{\mathrm{out}})$ is a union of d disjoint bounded closed disks. Evidently this union contains $f^{-1}(\Gamma)$ and has boundary $f^{-1}(\Gamma_k)$. It follows that the complementary region $f^{-1}(D_k^{\mathrm{in}})$ is the unique unbounded puzzle piece of level one. This puzzle piece contains both critical points, and has no Γ_i other than Γ_{k-1} on its boundary. Therefore the level one puzzle piece to the inner side of Γ_{k-1} is bounded and contains no critical point, while for any $i \neq k-1$ both of the pieces $L_i^{(1)}$ and $R_i^{(1)}$ are bounded and contain no critical point. Thus $L^{(1)}$ and $R^{(1)}$ cannot both contain critical points, which contradicts Theorem A.2 and completes the proof that a bicritical map has no Herman ring. $\square$

APPENDIX B. TOTALLY DISCONNECTED JULIA SETS

We will first prove the following result, with no restriction on the number of critical points. Let f be a rational function of degree $n \geq 2$.

Theorem B.1. *The Julia set J of f is totally disconnected and contains no critical point if and only if all of the critical values of f lie in a single Fatou component.*

In the hyperbolic case, this was proved by Rees [1990]. I attempted to extend the argument to the parabolic case in [Milnor 1993], but the details were not quite right. The proof given here is rather different and perhaps easier. It will be based on the following ideas.

Definition. A map $g : K \to K'$ between metric spaces is called *locally distance increasing* if every point of K has a neighborhood V so that

$$\operatorname{dist}(g(x), g(y)) > \operatorname{dist}(x, y) \tag{B–1}$$

for all $x \neq y$ in V. If K is compact, an equivalent condition is that for some $\varepsilon > 0$ we have (B–1) whenever $0 < \operatorname{dist}(x, y) < \varepsilon$.

Lemma B.2. *Let $g : K \to K$ be a locally distance increasing map from a compact metric space to itself. If g is injective on each connected component of K, then the space K must be totally disconnected.*

Proof. First note that *there exists a constant $\varepsilon' > 0$ so that* (B–1) *holds whenever x and y are distinct points belonging to the same connected component of K and satisfying* $\operatorname{dist}\big(g(x), g(y)\big) < \varepsilon'$. For otherwise, with ε as above, there would exist pairs (x_i, y_i) belonging to the same component K_i of K so that $\operatorname{dist}\big(g(x_i), g(y_i)\big)$ converges to zero as $i \to \infty$, but with $\operatorname{dist}(x_i, y_i) \geq \varepsilon$. After passing to a subsequence, we may assume that the points x_i converge to some point x and that the y_i converge to some y, where $\operatorname{dist}(x, y) \geq \varepsilon$ and where $g(x) = g(y)$. But this is impossible since x and y belong to the same connected component of K. In fact, the set L of all accumulation points of K_i as $i \to \infty$ is a connected subset of K containing both x and y.

The *diameter* $\operatorname{diam} K \geq 0$ of a non-vacuous compact metric space is defined to be the maximum distance between two of its points. More generally, for any integer $m \geq 1$ define $\operatorname{diam}_m K$ to be the largest number δ so that there exist $m + 1$ points $x_0, x_1, \ldots, x_m$ in K that are *δ-separated* in the sense that $\operatorname{dist}(x_i, x_j) \geq \delta$ for $i \neq j$. (Think of $m + 1$ repelling points, which try to get as far as possible from each other.) Thus

$$\operatorname{diam} K = \operatorname{diam}_1 K \geq \operatorname{diam}_2 K \geq \operatorname{diam}_3 K \geq \cdots \geq 0.$$

Note that $\operatorname{diam}_m K = 0$ if and only if K is a finite set with at most m elements.

The proof of Lemma B.2 now proceeds as follows. Choose m large enough so that K can be covered by subsets $X_1, \ldots, X_m$ of diameter less than ε'. It then follows that $\operatorname{diam}_m K < \varepsilon'$, since a collection of ε'-separated points can have at most one point in each X_i. Let $\delta_{\max}$ be the supremum of the numbers $\operatorname{diam}_m K_\alpha$ as K_α ranges over all connected components of K. We want to prove that $\delta_{\max} = 0$. Otherwise, if $\delta_{\max} > 0$, we will obtain a contradiction by constructing a "largest" component K_μ, with $\operatorname{diam}_m K_\mu$ equal to $\delta_{\max}$, and then showing that $\operatorname{diam}_m \big(f(K_\mu)\big) > \delta_{\max}$.

Let K_i be a sequence of components of K such that the numbers $\delta_i = \operatorname{diam}_m K_i$ converge to the supremum $\delta_{\max}$, and choose points $x_0(i)$, $x_1(i)$, ..., $x_m(i) \in K_i$ that are δ_i-separated. After passing to a subsequence, we may assume that each sequence $x_j(1)$, $x_j(2)$, $x_j(3)$, ... converges to a limit $x_j \in K$. Let L be the set of all accumulation points of $\{K_i\}$ as $i \to \infty$. Then L is connected, and hence is contained in some connected component K_μ. Furthermore $\operatorname{diam}_m K_\mu = \delta_{\max}$ since L contains $m+1$ points x_j which are $\delta_{\max}$-separated.

Now, assuming that $\delta_{\max} > 0$, we will obtain a contradiction by showing that $\operatorname{diam}_m \big(f(K_\mu)\big)$ must be strictly larger than $\delta_{\max}$. In fact, for each $0 \leq i < j \leq m$ we have either

$$\operatorname{dist}\big(f(x_i), f(x_j)\big) > \operatorname{dist}(x_i, x_j) \geq \delta_{\max},$$

or else

$$\operatorname{dist}\big(f(x_i), f(x_j)\big) \geq \varepsilon' > \operatorname{diam}_m K \geq \delta_{\max}.$$

This contradiction proves that $\delta_{\max}$ must be zero. Hence every connected component K_α is finite, and hence consists of a single point. □

Proof of Theorem B.1. One direction is clear, since a totally disconnected set cannot separate the Riemann sphere. For the proof in the other direction, suppose that all critical values lie in a single Fatou component U. Choose a smoothly embedded closed disk $\Delta^* \subset U$ which contains all of the critical values in its interior. Let $\Delta = \widehat{\mathbb{C}} \smallsetminus \operatorname{interior}(\Delta^*)$ be the complementary closed disk which contains the Julia set. Since Δ contains no critical values, its pre-image $f^{-1}(\Delta)$ splits as a disjoint union

$$f^{-1}(\Delta) = \Delta_1 \cup \cdots \cup \Delta_n$$

of smoothly embedded closed disks, each of which maps homeomorphically onto Δ under f. Now every connected component J_α of the Julia set must be contained in one of the Δ_j, and hence must map homeomorphically under f. Thus, to prove Theorem B.1, we need only show that f restricted to the Julia set is locally distance increasing with respect to a suitably chosen metric. In fact we will prove the following. (For related statements, see [Douady and Hubbard 1985a; Tan and Yin 1996; Tan 1997].)

Lemma B.3. *A rational map f, restricted to its Julia set J, is locally distance increasing with respect to a suitably chosen metric if and only if f has no critical points in J.*

Proof. If there is a critical point c in the Julia set J, then $f|_J$ is not even one-to-one near c, so it certainly cannot be locally distance increasing. Suppose then that there are no critical points in J. Let P be the closure of the *postcritical set* of f (the union of all forward orbits of critical values). If we exclude trivial cases where P has only two elements, then each connected component of the complement $U = \widehat{\mathbb{C}} \smallsetminus P$ has a well defined Poincaré metric. The associated Poincaré distance function will be denoted by $\mathrm{dist}_U(x, y)$. Let $U_0 = f^{-1}(U) \subset U$. Then f maps U_0 onto U by a covering map. Since U_0 is a proper subset of U, it follows that

$$\mathrm{dist}_U(x, y) < \mathrm{dist}_{U_0}(x, y) = \mathrm{dist}_U(f(x), f(y))$$

for any two points $x \neq y$ in U_0 which are sufficiently close to each other. Thus f is locally distance increasing on U_0 with respect to the metric dist_U.

Since there are no critical points in J, it follows from the Sullivan classification of Fatou components that every critical orbit must converge to an attracting or parabolic cycle. Therefore the intersection $\Pi = P \cap J$ can be described as the set of all parabolic periodic points. If there are no parabolic points, then $J \subset U_0$, and it follows that f is locally distance increasing on J with respect to the metric dist_U. However, if f has parabolic points, then the Poincaré metric becomes infinite at such points. We will modify this metric near these points so as to obtain a better behaved metric.

It will be convenient to choose coordinates so that the Julia set J is contained in the finite plane $\mathbb{C}$. The Poincaré metric on $U = \widehat{\mathbb{C}} \smallsetminus P$ has the form $\rho(z)|dz|$ throughout $U \cap \mathbb{C}$, where ρ extends to a function

$$\rho : \mathbb{C} \to (0, \infty]$$

which is continuous everywhere, but takes the value $+\infty$ on the points of $P \cap \mathbb{C}$. In particular, $\rho(z)$ tends to infinity as z tends to any point in the set $\Pi = P \cap J$ of parabolic points. First suppose that these parabolic points are all fixed under f, with multiplier $+1$. Let $M > 0$ be a constant which is large enough so that

$$M > \rho(z)/|f'(z)| \tag{B–2}$$

for every z in the finite set $f^{-1}(\Pi) \smallsetminus \Pi$. (This ratio $\rho(z)/|f'(z)|$ is always finite at the points of $f^{-1}(\Pi) \smallsetminus \Pi$, since such points are non-critical and outside P.) Let $N_\varepsilon(\Pi)$ be the open neighborhood of Euclidean radius ε about the parabolic set Π. We define a new Riemannian metric $\eta(z)|dz|$ on the open set $U' = N_\varepsilon(\Pi) \cup (\mathbb{C} \smallsetminus P)$ by setting

$$\eta(z) = \begin{cases} M & \text{for } z \in N_\varepsilon(\Pi), \\ \rho(z) & \text{for } z \notin N_\varepsilon(\Pi). \end{cases}$$

This function η has a jump discontinuity on the circle of radius ε about each parabolic point; but this will not cause any difficulty. As usual, we use this Riemannian metric to define a distance $\mathrm{dist}_\eta(x, y)$ between points of U' as the infimum of lengths

$$\mathrm{length}_\eta(\Gamma) = \int_\Gamma \eta(z)|dz|,$$

where Γ ranges over all piecewise smooth paths joining the two points within U'.

If ε is sufficiently small, then we will prove that $f|_J$ is locally distance increasing with respect to this metric dist_η.

Since $\eta(\Pi) = \infty$, we can certainly choose ε small enough so that $\rho(z) > M$ throughout the neighborhood $N_{2\varepsilon}(\Pi)$. Evidently this will guarantee that any Euclidean straight line segment lying within $N_\varepsilon(\Pi)$ is the unique η-geodesic of minimal η-length joining its endpoints. To study behavior of f near a parabolic point z_0, we proceed as follows. Assuming for convenience that $z_0 = 0$, we can write the local power series expansion as

$$f(z) = z\bigl(1 + az^m + (\text{higher terms})\bigr),$$

where $m \geq 1$ and $a \neq 0$. Using the Leau–Fatou Flower Theorem, we obtain the estimate

$$\frac{az^m}{|az^m|} \to 1$$

as $z \to 0$ within the Julia set J. Hence

$$f(z) = z\bigl(1 + |az^m| + o(z^m)\bigr),$$
$$f'(z) = 1 + (m+1)|az^m| + o(z^m)$$

as $z \to 0$ within J. It follows easily that $f|_J$ is distance increasing throughout some neighborhood of each parabolic point. Furthermore, if ε' is sufficiently small, it follows that

$$|f(z)| > |z|, \quad |f'(z)| > 1 \quad \text{for } 0 < |z| < \varepsilon'. \qquad \text{(B–3)}$$

To prove that f is locally distance increasing at points of $J \smallsetminus \Pi$ we will prove the infinitesimal form of the required inequality. That is we will prove that

$$\eta(f(z))\,|f'(z)| > \eta(z) \qquad \text{(B–4)}$$

throughout $J \smallsetminus \Pi$, and hence that

$$\int_\Gamma \eta(f(z))\,|df(z)| = \int_\Gamma \eta(f(z))\,|f'(z)dz| > \int_\Gamma \eta(z)\,|dz|$$

for any path Γ in some neighborhood of $J \smallsetminus \Pi$. There are four cases to consider: If both z and $f(z)$ belong to $N_\varepsilon(\Pi)$, then (B–4) follows from (B–3). If neither z nor $f(z)$ belongs to $N_\varepsilon(\Pi)$, then it follows from the corresponding property of the Poincaré metric. If z belongs to $N_\varepsilon(\Pi)$ but $f(z)$ does not, then it follows since

$$\eta(f(z))|f'(z)| = \rho(f(z))\,|f'(z)| > \rho(z) > M = \eta(z).$$

Finally, if $z \in f^{-1}N_\varepsilon(\Pi) \smallsetminus N_\varepsilon(\Pi)$, then we proceed as follows. A compactness argument shows that $f^{-1}N_\varepsilon(\Pi)$ shrinks down to $f^{-1}(\Pi)$ as $\varepsilon \searrow 0$. Thus, given ε' we can find ε so that every point z of $f^{-1}N_\varepsilon(\Pi)$ has Euclidean distance at most ε' from some point $\hat{z} \in f^{-1}(\Pi)$. If $\hat{z} \in \Pi$, then since $f(z) \in \mathbb{N}_\varepsilon(\Pi)$ it follows from (B–3) that $z \in N_\varepsilon(\Pi)$ also, contradicting the hypothesis that $z \notin N_\varepsilon(\Pi)$. On the other hand, if $\hat{z} \neq \Pi$ then it follows from (B–2) that $M\,|f'(\hat{z})| > \eta(\hat{z})$. If ε is sufficiently small, it evidently follows that $M\,|f'(z)| > \eta(z)$ also.

This proves B.3 in the special case where every parabolic point is a fixed point of multiplier $+1$. To handle the general case, choose a positive integer k so that every parabolic point of $f^{\circ k}$ is a fixed point of multiplier $+1$, and let $\mathrm{dist}_\eta(x,y)$ be a metric with the required property for $f^{\circ k}$. Then f itself will be locally distance increasing on J with respect to the metric

$$\mathrm{dist}'(x,y) = \sum_{j=0}^{k-1} \mathrm{dist}_\eta\big(f^{\circ j}(x), f^{\circ j}(y)\big).$$

This proves B.3, and completes the proof of Theorem B.1. □

Let f be rational of degree $n \geq 2$ with Julia set J.

Lemma B.4. *Suppose that there exists a closed disk $\Delta^* \subset \widehat{\mathbb{C}}$ that*

(a) *contains all of the critical values of f in its interior,*
(b) *satisfies $f(\Delta^*) \subset \Delta^*$, and*
(c) *eventually absorbs every orbit in the Fatou set.*

Then $f|_J$ is topologically conjugate to the one-sided shift on n symbols.

Proof. It follows from (b) that the interior of such a disk Δ^* is contained in the Fatou set. Therefore, by (a) and Theorem B.1, the Julia set J is totally disconnected. As in the proof of Lemma B.2, the complementary closed disk Δ contains J, and $f^{-1}(\Delta)$ splits as a disjoint union $\Delta_1 \cup \cdots \cup \Delta_n$, where each Δ_i maps homeomorphically onto Δ. But now we have the additional information that each Δ_i is a subset of Δ. It follows inductively that each finite intersection

$$\Delta_{i_0\, i_1 \cdots i_k} = \Delta_{i_0} \cap f^{-1}(\Delta_{i_1}) \cap \cdots \cap f^{-k}(\Delta_{i_k})$$

is a closed topological disk. In fact

$$f : \Delta_{i_0\, i_1 \ldots i_k} \xrightarrow{\simeq} \Delta_{i_1\, i_2 \ldots i_k} \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \Delta_{i_{k-1}\, i_k} \xrightarrow{\simeq} \Delta_{i_k} \xrightarrow{\simeq} \Delta.$$

Now given an infinite sequence $I = (i_0, i_1, i_2, \ldots)$, it follows that the intersection J_I of the nested sequence

$$\Delta_{i_0} \supset \Delta_{i_0\, i_1} \supset \Delta_{i_0\, i_1\, i_2} \supset \cdots$$

is compact, connected, and non-vacuous. This intersection is contained in the Julia set, since by condition (c) every orbit outside of the Julia set eventually leaves the disk Δ. Hence by Theorem B.1 each J_I is a single point.

Equivalently, to each point z of J we can assign a sequence $I(z) = (i_0, i_1, i_2, \ldots)$ of integers between 1 and n by the condition that $z_j \in D_{i_j}$ where

$$f : z = z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$$

is the orbit of z. Evidently J_I consists of the unique point which has symbol sequence equal to I. Since f maps J_I to $J_{\sigma(I)}$, where

$$\sigma(i_0, i_1, \ldots) = (i_1, i_2, \ldots)$$

is the shift operator, this proves that the correspondence $z \mapsto I(z)$ is a topological conjugacy from $f|_J$ onto the one-sided shift on n symbols. $\square$

We will only try to apply this lemma in the following very special case.

Theorem B.5. *If f has only two critical values, both in the same Fatou component, then $f|_J$ is topologically conjugate to a one-sided shift.*

Note. The hypothesis that f has only two critical values implies that f has only two critical points (making use of the fact that the twice punctured sphere has free cyclic fundamental group). By contrast, a Chebyshev polynomial of degree $n \geq 3$ has only three distinct critical values on the Riemann sphere but has n distinct critical points.

Proof of Theorem B.5. Let U_0 be a simply connected open set bounded by a smooth Jordan curve, which satisfies $f(U_0) \subset U_0$, and which eventually absorbs every orbit in the Fatou set. Thus in the hyperbolic case U_0 will be a neighborhood of the attracting point, while in the parabolic case U_0 will be a carefully chosen attracting petal. We can assume that the boundary of U_0 is disjoint from the two critical orbits. Now define a sequence of connected open sets with smooth boundary

$$U_0 \subset U_1 \subset U_2 \subset \cdots$$

by setting U_{k+1} equal to the connected component of $f^{-1}(U_k)$ which contains U_k. If U_k is simply connected and contains at most one critical value, then it is easy to show that U_{k+1} is also simply connected. Thus we can continue this construction until we obtain some U_m which contains both critical values. The closure $\overline{U}_m$ will then be the required disk Δ^*, so that we can apply Lemma B.4. $\square$

APPENDIX C. CROSS-RATIO FORMULAS

In Remark 1.6 we mentioned that the invariant X of Lemma 1.1 can be defined as a cross-ratio. This appendix will give corresponding formulas for the invariants Y_1 and Y_2. I will use the nonstandard notation

$$\begin{array}{c|c} p & q \\ \hline r & s \end{array} = \frac{(p-q)(r-s)}{(p-r)(q-s)} \in \widehat{\mathbb{C}}$$

(product of row differences, divided by product of column differences). This cross-ratio symbol can be characterized as follows. It is well defined unless three of the four points $\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}$ in $\widehat{\mathbb{C}}$ coincide, and it takes the value:

$$\begin{cases} 0 & \text{if two entries in the same row are equal,} \\ \infty & \text{if two entries in the same column are equal,} \\ 1 & \text{if two diagonally opposite entries are equal.} \end{cases}$$

Furthermore, if three of the four variables p, q, r, s are fixed and distinct, then it represents a Möbius transformation of the remaining variable.

As an example, since a Möbius transformation which fixes three points is the identity, it follows that

$$\begin{array}{c|c} x & 0 \\ \hline \infty & 1 \end{array} = x.$$

Furthermore, it follows easily that

$$\begin{array}{c|c} \varphi(p) & \varphi(q) \\ \hline \varphi(r) & \varphi(s) \end{array} = \begin{array}{c|c} p & q \\ \hline r & s \end{array}$$

for any Möbius transformation φ. This symbol is independent of the order of rows or columns, that is it satisfies the symmetry relations

$$\begin{array}{c|c} p & q \\ \hline r & s \end{array} = \begin{array}{c|c} r & s \\ \hline p & q \end{array} = \begin{array}{c|c} s & r \\ \hline q & p \end{array} = \begin{array}{c|c} q & p \\ \hline s & r \end{array}.$$

Now let c_1 and c_2 be the critical points of the bicritical map f, let $v_i = f(c_i)$ be the critical values, and let $p_i \in f^{-1}(c_i)$ be any one of the n preimages of c_i. For example, using the usual normal form

$$f(z) = (az^n + b)/(cz^n + d),$$

we can take

$$\begin{array}{lll} c_1 = \infty, & v_1 = a/c, & p_1 \in \sqrt[n]{-d/c} \\ c_2 = 0, & v_2 = b/d, & p_2 \in \sqrt[n]{-b/a}. \end{array}$$

Using these notations, we can write Remark 1.6 in the form

$$\begin{array}{c|c} c_1 & v_1 \\ \hline c_2 & v_2 \end{array} = \frac{-v_2}{v_1 - v_2} = \frac{-b/d}{a/c - b/d} = \frac{-bc}{ad - bc} = -X.$$

Now consider the cross-ratio

$$\begin{array}{c|c} c_1 & v_2 \\ \hline p_1 & c_2 \end{array} = \frac{p_1}{v_2} = \frac{\sqrt[n]{-d/c}}{b/d}.$$

This is well defined only up to multiplication by an n-the root of unity. However its n-th power

$$\left(\begin{array}{c|c} c_1 & v_2 \\ \hline p_1 & c_2 \end{array} \right)^n = \frac{-d/c}{b^n/d^n} = \frac{-d^{n+1}}{b^n c}$$

is always well defined as an element of $\widehat{\mathbb{C}}$. If $c \neq 0$ then we can multiply numerator and denominator by c^{n-1}, thus reducing this expression to the form

$$\left(\begin{array}{c|c} c_1 & v_2 \\ \hline p_1 & c_2 \end{array}\right)^n = \frac{-c^{n-1}d^{n+1}}{b^n c^n} = \frac{-c^{n-1}d^{n+1}}{(ad-bc)^n}\left(\frac{bc}{ad-bc}\right)^{-n} = \frac{-Y_2}{X^n}.$$

This proves that we can compute Y_2 by a cross-ratio whenever $X \neq 0$. Still assuming that $c \neq 0$, we can also consider the expression

$$\left(\begin{array}{c|c} c_1 & v_1 \\ \hline p_1 & c_2 \end{array}\right)^n = (p_1/v_1)^n = \frac{-d/c}{a^n/c^n} = \frac{c^{n-1}d^{n+1}}{a^n d^n} = \frac{-Y_2}{(X+1)^n}.$$

Note that the left side of this equation is indeterminate if and only if $c = 0$, or equivalently if and only if $c_1 = v_1 = p_1$. But whenever this happens, it follows that $Y_2 = 0$. Since X and $X+1$ cannot both be zero, this shows that we can compute Y_2 in all cases. Interchanging the roles of the two critical points, we find similar expressions for Y_1.

APPENDIX D. ENTIRE AND MEROMORPHIC MAPS

Douady has pointed out to me that there is an analogous theory of entire transcendental or meromorphic maps that have only two singular values.

Lemma D.1. *Suppose that $f : U \to \widehat{\mathbb{C}}$ is a holomorphic map with only two singular values, say v_1 and v_2, where U is a connected open subset of $\widehat{\mathbb{C}}$. If f has infinite degree, then U must be the complement of a single point $s \in \widehat{\mathbb{C}}$, and f must map $\widehat{\mathbb{C}} \smallsetminus \{s\}$ onto $\widehat{\mathbb{C}} \smallsetminus \{v_1, v_2\}$ by a free cyclic covering map.*

Here, by a *singular value* $v \in \widehat{\mathbb{C}}$ we mean either a critical value, or an asymptotic value for the map f. (By definition, z is an *asymptotic value* if it can be described as the limit $z = \lim_{t\to 1} f(p(t))$ where $p : [0,1) \to U$ is a path which eventually leaves any compact subset of U.) The Lemma asserts that the f in question cannot have any critical values, so we could equally well describe it as a map with two asymptotic values.

For a quite different characterization of this family of maps, see [Devaney and Keen 1988a; 1989].

Proof of Lemma D.1. It is not difficult to see that f must carry the complement $U \smallsetminus f^{-1}\{v_1, v_2\}$ onto $\widehat{\mathbb{C}} \smallsetminus \{v_1, v_2\}$ by a covering map. Since the fundamental group of the image space is cyclic, this can only be a cyclic covering. If it has infinite degree, then it must be a universal covering. Hence $U \smallsetminus f^{-1}\{v_1, v_2\}$ must be conformally equivalent to the universal covering of $\widehat{\mathbb{C}} \smallsetminus \{v_1, v_2\}$, or in other words to $\mathbb{C}$ itself. Since U cannot be the entire Riemann sphere, this proves that $U = \widehat{\mathbb{C}} \smallsetminus \{s\}$ for some unique point s, and that

$$f : U \to \widehat{\mathbb{C}} \smallsetminus \{v_1, v_2\}$$

is a free cyclic covering map. Note that f has an essential singularity at s. □

If we put the two singular values at zero and infinity, then one example of a universal covering of $\widehat{\mathbb{C}} \smallsetminus \{v_1, v_2\} = \mathbb{C} \smallsetminus \{0\}$ is given by the exponential map

$$\exp : \mathbb{C} \to \mathbb{C} \smallsetminus \{0\}.$$

Hence f can be described as the composition $f = \exp \circ \mu$, where μ is some conformal isomorphism from $\widehat{\mathbb{C}} \smallsetminus \{s\}$ onto $\mathbb{C}$. Evidently we can express any such μ as a Möbius transformation

$$\mu(z) = \frac{az+b}{cz+d} \quad \text{with } ad - bc \neq 0,$$

so that

$$f(z) = \exp \circ \mu(z) = \exp\left(\frac{az+b}{cz+d}\right), \tag{D–5}$$

with essential singularity at $s = -d/c \in \widehat{\mathbb{C}}$.

In practice, it will be convenient to conjugate $f = \exp \circ \mu$ by μ, so as to obtain a normal form

$$g = \mu \circ f \circ \mu^{-1} = \mu \circ \exp .$$

Thus

$$g(w) = \frac{ae^w + b}{ce^w + d}, \tag{D–6}$$

or equivalently $g(2w) = \frac{ae^w + be^{-w}}{ce^w + de^{-w}}$. The essential singularity of g is at infinity, and its singular values are $v_1 = a/c$ and $v_2 = b/d$. Note the identity

$$g(w + 2\pi i) = g(w).$$

In fact we can describe g as a composition

$$\mathbb{C} \xrightarrow{\text{projection}} \mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\cong} \widehat{\mathbb{C}} \smallsetminus \{a/c, b/d\}.$$

Since this description is rather rigid, we are free to change the coordinate w only by a translation, or a sign change followed by translation. If we conjugate g by a translation, then the resulting map $w \mapsto g(w+t)-t$ has the form (D–6) with matrix of coefficients

$$\begin{bmatrix} e^t(a-tc) & b-td \\ e^tc & d \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} e^{t/2}(a-tc) & e^{-t/2}(b-td) \\ e^{t/2}c & e^{-t/2}d \end{bmatrix}$$

if we prefer to work only with unimodular matrices). Thus the additive group of translations acts on the manifold $\mathrm{PGL}(2,\mathbb{C})$ by a correspondence which we can write as

$$\Phi_t : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}. \tag{D–7}$$

The difference

$$v_1 - v_2 = \frac{a}{c} - \frac{b}{d} = \frac{ad-bc}{cd}$$

remains invariant under this transformation, since v_1 and v_2 are both translated by the same constant $-t$. Define X to be the reciprocal

$$X = \frac{1}{v_1 - v_2} = \frac{cd}{ad-bc} \in \mathbb{C}. \tag{D–8}$$

This number X is a conformal conjugacy invariant. For if we replace $g(w)$ by $-g(-w)$ then the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of coefficients will be replaced by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -b & -a \\ d & c \end{bmatrix}, \tag{D–9}$$

so that (v_1, v_2) is replaced by $(-v_2, -v_1)$, and again the expression (D–8) remains invariant. Since any cooordinate change which carries (D–6) to another expression of the same form is a composition of translations and sign changes, it follows that X is indeed a conformal conjugacy invariant.

We will prove several statements which are reminiscent of the descriptions of moduli space in Sections 1 and 2. For each fixed X, we need one further complex number to give a complete conjugacy class invariant. Furthermore, given $X(f)$ and given $\lambda \neq 0$, there is a unique conjugacy class of maps (f) which have a fixed point of multiplier λ. However, unlike the finite degree situation of Section 1, it is necessary to treat the cases $X = 0$ and $X \neq 0$ separately in order to obtain reasonable moduli spaces.

Entire Transcendental Case. If $X = 0$, or equivalently if the essential singularity occurs at one of the two singular values, then putting $s = v_1$ at infinity and putting v_2 at zero, the normal form (D–6) reduces to

$$g(w) = ke^w \quad \text{for some constant } k \in \mathbb{C} \smallsetminus \{0\}.$$

Here $k = g'(0)$ is a conjugacy class invariant. Thus we obtain the *exponential family*, which has been studied by several authors. (See [Devaney 1991], for example.) We could also ue the normal form

$$f(z) = g(kz)/k = e^{kz},$$

again with $s = \infty$ and zero as singular values, and again with invariant $k = f'(0)$. If we specify that g has a fixed point $w_0 = g(w_0)$ of multiplier $g'(w_0) = \lambda$, then since $g'(w_0) = g(w_0) = w_0$ it follows that $w_0 = \lambda$, and hence that the invariant $k = w_0 e^{-w_0}$ is uniquely determined by λ.

Meromorphic Case. If $X \neq 0$, or equivalently if the three distinguished points s, v_1, v_2 are pairwise distinct, then $g : \mathbb{C} \to \widehat{\mathbb{C}} \smallsetminus \{v_1, v_2\}$ is a meromorphic map. (For a survey of meromorphic dynamics, see [Bergweiler 1993].) In analogy with Section 1 we will construct a conjugacy invariant $Y = Y_1 + Y_2$. Let

$$Y_1 = \frac{c}{d}\exp\frac{a}{c}, \qquad Y_2 = \frac{d}{c}\exp\frac{-b}{d},$$

with product

$$Y_1 Y_2 = \exp\left(\frac{a}{c} - \frac{b}{d}\right) = \exp\left(\frac{ad-bc}{cd}\right) = e^{1/X}. \tag{D–10}$$

Then it is not difficult to check that both Y_1 and Y_2 are invariant under the action (D–7), and that Y_1 and Y_2 are interchanged by the transformation (D–9). Hence the sum $Y = Y_1 + Y_2$ is a conjugacy class invariant. The two quantities Y_1 and Y_2 individually will be described as *half-invariants*, since only the unordered pair $\{Y_1, Y_2\}$ is actually a conjugacy invariant.

The choice of $X \neq 0$ and Y uniquely determines the conjugacy class. In fact we can compute the unordered pair $\{Y_1, Y_2\}$ by solving a quadratic equation. Given Y_1, we can assume, by using the action (D–7), that $c = d$ so that $Y_1 = e^{a/c}$. Thus we can compute the ratio a/c, up to a summand in $2\pi i\mathbb{Z}$, and then compute the ratio b/d from the identity $a/c - b/d = 1/X$. Thus the function $g(w) = (ae^w + b)/(ce^w + d)$ is uniquely determined, up to a summand in $2\pi i\mathbb{Z}$ which can be removed by a corresponding translation of the coordinate w. This proves the following.

Lemma D.2. *The moduli space for meromorphic maps with two singular values is biholomorphic to* $(\mathbb{C} \smallsetminus \{0\}) \times \mathbb{C}$, *with coordinates* X *and* Y.

Cross-Ratios. Just as in Appendix C, we can express the Y_i as cross-ratios. For example, using the normal form (D–5), one easily checks that:

$$\begin{array}{c|c} v_1 & s \\ \hline f(v_1) & v_2 \end{array} = -Y_1.$$

However, it does not seem possible to compute the invariant X as a cross-ratio.

Bad Topology. It might seem natural to combine the moduli space for the exponential family, together with the moduli space for our meromorphic family, into a larger moduli space, isomorphic to a union $(\mathbb{C} \smallsetminus \{0\}) \cup (\mathbb{C} \smallsetminus \{0\}) \times \mathbb{C})$. However this does not yield any useful result since the natural topology on the union is not Hausdorff. To see this, consider the two maps $z \mapsto e^z$ and $z \mapsto 2e^z$ in the exponential family. Although these are not conjugate, I will show that given any neighborhood N_1 of the first and any neighborhood N_2 of the second within the space of maps with two singular values, some map in N_1 is conjugate to a map in N_2. Consider the analytic function $\xi \mapsto 2\xi^2 e^{1/\xi}$, which has an essential singularity at the origin. Since this function omits the values 0 and ∞, it follows from Picard's Theorem that we can choose a sequence of values of ξ tending to zero which satisfy the equation $2\xi^2 e^{1/\xi} = 1$. For each such ξ, consider the two meromorphic maps

$$\frac{e^w}{\xi e^w + 1}, \quad \frac{2e^w}{2\xi e^w + 1}.$$

A brief computation shows that both have invariants $X = \xi$ and $Y = \xi e^{1/\xi} + 1/\xi$. Hence the two are conjugate, although these sequences tend to non-conjugate limits as $\xi \to 0$.

Comparison with Bicritical Maps. Our invariants for meromorphic maps look rather different from the invariants of Section 1, but in fact there is a definite relationship. As in [Devaney et al. 1986], we can approximate the exponential map by the unicritical polynomials $E_n(w) = (1 + w/n)^n$. Hence we can approximate the meromorphic function g of (D–6) by the bicritical maps

$$g_n(w) = \frac{aE_n(w) + b}{cE_n(w) + d}$$

throughout any compact subset of $\mathbb{C}$. A straightforward computation then shows that

$$X(g) = \lim_{n\to\infty} X(g_n)/n$$

and that

$$Y_i(g) = \lim_{n\to\infty} Y_i(g_n)/X(g_n)^n.$$

The Symmetry Locus. There is a non-trivial Möbius automorphism which commutes with g if and only if $Y_1(g) = Y_2(g)$. In view of (D–10), this means that

$$Y_1 = Y_2 = (Y/2) = \pm e^{1/2X}.$$

If we choose the plus sign, then the most general example is conjugate to

$$g(w) = k\frac{e^w - 1}{e^w + 1} = k\ \tanh(w/2),$$

with a fixed point of multiplier $k/2$ at the center of symmetry. Here $Y_1 = Y_2 = e^k$ and $X = 1/2k$. (For a discussion of the tangent family, see [Devaney and Keen 1988b], for example.) If we choose the minus sign, then we can take

$$g(w) = k\frac{e^w + 1}{e^w - 1} = k\ \coth(w/2).$$

This is just the image of $k\ \tanh(w/2)$ under the involution $g \mapsto J_g \circ g$ of Remark 1.5.

Fixed Points and the Curves $\mathrm{Per}_1(\lambda)$**.** Suppose that g has a fixed point $w_0 = g(w_0)$ with multiplier $\lambda = g'(w_0)$. We will prove that the conjugacy class $\langle g \rangle$ is uniquely determined by $X(g)$ together with λ. We again use the normal form (D–6), but now we translate coordinates so that the fixed point is at the origin. Then the fixed point equation $0 = g(0)$ reduces to the equation $a + b = 0$. If $ad - bc = 1$, we have $X = cd$, and we see easily that $g'(0) = 1/(c + d)^2$ so that $c + d = \pm 1/\sqrt{\lambda}$ is uniquely determined up to sign.

Hence we can solve for the unordered pair $\{c, d\}$ up to sign. We can then solve for a and $b = -a$ since $ad - bc = a(c + d) = 1$. Thus the conjugacy class is uniquely determined. Multiplying all coefficients by a common constant, this solution can be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2\lambda & -2\lambda \\ 1+r & 1-r \end{bmatrix}$$

with determinant $ad - bc = 4\lambda$, where $r^2 = 1 - 4\lambda X$ is distinct from 1. From this, one can easily write down a precise but somewhat complicated formula for the invariant Y as a function of λ and X. This function is holomorphic, since it can be expressed as the sum of a power series, convergent for $|r| < 1$, in which only even powers of r appear.

Real forms. A meromorphic map with two singular values commutes with some antiholomorphic involution α if and only if the invariants X and Y are both real. (Compare Section 5.) If $Y^2 > 4\,e^{1/X}$, then the half-invariants Y_1, Y_2 are also real, and the singular values are fixed by α. In this case, using the normal form (D–6) with real coefficients, the map g carries $\mathbb{R}$ diffeomorphically onto an open interval in $\mathbb{R} \cup \{\infty\}$, bounded by the two singular values. On the other hand, if $Y^2 < 4\,e^{1/X}$, then the half-invariants Y_1, Y_2 are complex conjugate, and the singular values are interchanged by α. In this case we can use a normal form

$$z \;\mapsto\; \frac{a \tan z + b}{c \tan z + d}$$

with real coefficients. These maps carry $\mathbb{R}$ onto $\mathbb{R} \cup \infty$ by a composition

$$\mathbb{R} \xrightarrow{\text{projection}} \mathbb{R}/\pi\mathbb{Z} \xrightarrow{\cong} \mathbb{R} \cup \{\infty\},$$

with degree $\pm\infty$. On the symmetry locus, with $Y^2 = 4\,e^{1/X}$, there are two possible choices for α and we can use either normal form.

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