Integral Geometry and Real Zeros of Thue–Morse Polynomials

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Dedicated to the memory of Gian-Carlo Rota

CONTENTS

We study the average number of intersecting points of a given curve with random hyperplanes in an n-dimensional Euclidean space. As noticed by A. Edelman and E. Kostlan, this problem is closely linked to finding the average number of real zeros of random polynomials. They showed that a real polynomial of degree n has on average $\frac{2}{\pi}\log n + O(1)$ real zeros (M. Kac's theorem).

This result leads us to the following problem: given a real sequence $(\alpha_k)_{k\in\mathbb{N}}$, study the average

$$\frac{1}{N}\sum_{n=0}^{N-1}\rho(f_n),$$

where $\rho(f_n)$ is the number of real zeros of $f_n(X) = \alpha_0+\alpha_1 X+\cdots+\alpha_n X^n$. We give theoretical results for the Thue–Morse polynomials and numerical evidence for other polynomials.

1. FINITE CURVES

Let $\mathcal{E}_n$ be real n-dimensional Euclidean affine space with a given orthonormal base. We identify points $x\in\mathcal{E}_n$ with the column vector ${}^t(x_1,\dots,x_n)$ whose entries are the coordinates of x.

A hyperplane $h\subset\mathcal{E}_n$ which does not contain the origin is represented by its Cartesian equation

$$\sum_{i=1}^{n} h_i x_i = 1,$$

or $hx=1$ for short, where $h=(h_1,h_2,\dots,h_n)$ is an element of the dual space $\mathcal{E}_n^*$. Both $\mathcal{E}_n$ and $\mathcal{E}_n^*$ are endowed with the Euclidean norm.

A change of orthonormal base in $\mathcal{E}_n$ induces a change of coordinates from the initial base to the new one:

$$x\mapsto x' = Ax + x^0$$

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Experimental Mathematics **9**:3, page 339

where A is an $n \times n$ orthogonal matrix. The hyperplane h then becomes $h' = hA^{-1}/(1 + hA^{-1}x^0)$ and it is easy to see that

$$\|h'\|^{-n-1} \prod_{i=1}^{n} dh'_i = \|h\|^{-n-1} \prod_{i=1}^{n} dh_i,$$

so that

$$dh = \|h\|^{-n-1} \prod_{i=1}^{n} dh_i$$

is the natural invariant measure on the space $\mathcal{E}_n^*$. Perhaps the most elementary way to see this classical result is to notice that any change of orthogonal base is a product of a rotation around the origin followed by n translations parallel to a coordinate axis. For each of these component base changes, the above invariance holds.

Let $\gamma \subset \mathcal{E}_n$ be a finite rectifiable curve, that is, a rectifiable curve of finite length $|\gamma|$. Let $\Omega(\gamma)$ be the family of hyperplanes h that intersect γ, and let $|\gamma \cap h|$ be the number of intersection points of h and γ. Santaló [1976, p. 245] establishes that

$$\int |\gamma \cap h| \, dh = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} |\gamma|, \tag{1–1}$$

where Γ is the usual Euler gamma function. See also [Klain and Rota 1997].

2. BREADTH OF A BOUNDED SET

Let K be a bounded convex set in $\mathcal{E}_n$. For $x \in \mathcal{E}_n$ call K_x the length of the orthogonal projection of K on the axis Ox, where O is the origin. Lengths are positive. The mean breadth of K is

$$\int_{\mathbb{S}_n} K_x \, dx \Big/ \int_{\mathbb{S}_n} dx,$$

where $\mathbb{S}_n$ is the unit sphere $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.

If L is any bounded set in $\mathcal{E}_n$, we define the mean breadth $B(L)$ as the mean breadth of the convex hull of L; in particular, the mean breadth of γ is well defined. Using Santaló's result again, we easily find that the h-measure of those hyperplanes h that intersect γ is

$$\int_{h \cap \gamma \neq \varnothing} dh = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} B(\gamma). \tag{2–1}$$

The quotient of equalities (1–1) and (2–1) thus gives us the average number $N(\gamma)$ of intersecting points of γ with a random hyperplane given that these hyperplanes meet γ:

$$N(\gamma) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{|\gamma|}{B(\gamma)}. \tag{2–2}$$

Let δ be the diameter of the set γ. Then clearly $B(\gamma) \leqslant \delta$. On the other hand the convex hull of γ contains a segment σ of length δ. Since

$$B(\gamma) \geqslant B(\sigma) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \delta,$$

where the equality is easily verified, we conclude that

$$\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{|\gamma|}{\delta} \leqslant N(\gamma) \leqslant \frac{|\gamma|}{\delta}. \tag{2–3}$$

3. EXTENSION TO FINITE DOODLES

Formulas (2–2) and (2–3) can be extended to generalized curves called "doodles". A *doodle* γ is a connected set which is a finite union of rectifiable curves $\gamma_1, \gamma_2, \ldots, \gamma_k$. The length of γ is naturally defined as

$$|\gamma| = \sum_{i=1}^{k} |\gamma_i|.$$

To any doodle γ we can associate a closed curve $\tilde{\gamma}$ that goes over γ twice. According to formula (2–2),

$$N(\tilde{\gamma}) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{2|\gamma|}{B(\gamma)},$$

since $|\tilde{\gamma}| = 2|\gamma|$ and $B(\tilde{\gamma}) = B(\gamma)$. But $N(\tilde{\gamma}) = 2N(\gamma)$ since each intersection point of a hyperplane h with $\tilde{\gamma}$ counts twice. Dividing by 2 leads to formula (2–2) for doodles.

This result should be compared to those of Favard [1932], Sulanke [1966] and Laurent-Gengoux (private communication).

4. DIMENSION

Several earlier articles have discussed the dimension of curves in the plane [Mendès France and Tenenbaum 1981; Dekking and Mendès France 1981; Mendès France 1991]. The extension to curves in $\mathcal{E}_n$ is straightforward.

Indeed, let γ be an unbounded, locally rectifiable curve in $\mathcal{E}_n$ such that bounded subsets of $\mathcal{E}_n$ contain only finite portions of γ. So for example in $\mathcal{E}_2$ curves like the one given in polar coordinates by $\theta = \sin 1/\rho$ are excluded: the unit ball centered at the origin contains infinitely many branches of the curve.

With no real loss of generality we assume that γ has an endpoint (or rather a starting point!). Let γ_s denote the beginning portion of γ of length s. Let $\varepsilon > 0$ be given. Consider the so-called Minkowski ε-sausage

$$\gamma(s,\varepsilon) = \{x \in \mathcal{E}_n \mid \operatorname{dist}(x,\gamma_s) \leqslant \varepsilon\}.$$

Let $|\gamma(s,\varepsilon)|$ be its volume. If $\delta(s)$ is the diameter of γ_s, define the dimension $d = \dim(\gamma)$ by the formula

$$\dim(\gamma) = \liminf_{s\to\infty} \frac{\log|\gamma(s,\varepsilon)|}{\log\delta(s)}. \tag{4–1}$$

Despite appearances, $\dim(\gamma)$ does not depend on ε: this is easily seen [Mendès France 1991, page 329]. Clearly $1 \leqslant d \leqslant n$, and it can be shown that for all $\alpha \in [1,n]$ there exists a curve γ for which $\dim(\gamma) = \alpha$.

The volume of $\gamma(s,\varepsilon)$ is at most

$$|\gamma(s,\varepsilon)| \leqslant \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\varepsilon^{n-1}s + O(\varepsilon^n),$$

so that

$$\dim(\gamma) \leqslant \liminf_{s\to\infty} \frac{\log s}{\log\delta(s)}. \tag{4–2}$$

Under certain circumstances equality holds: for example, when there exists a real number $A > 0$ such that unit balls contain portions of γ of total length less than A.

Let $\varepsilon > 0$ be given. Formula (4–2) shows that there exists a value $s(\varepsilon)$ such that

$$\frac{\log\delta(s)}{\log s} \leqslant \frac{1}{d} + \frac{\varepsilon}{2}$$

for all $s > s(\varepsilon)$, so that

$$\delta(s) \leqslant s^{1/d+\varepsilon/2}.$$

Inequality (2–3) therefore implies a lower bound for $N(\gamma_s) =: N(s)$:

$$N(s) \geqslant \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\, s^{1-1/d-\varepsilon/2} \geqslant s^{1-1/d-\varepsilon}. \tag{4–3}$$

On the other hand, if γ is a curve for which formula (4–2) is an equality, then

$$N(s) \leqslant s^{1-1/d+\varepsilon} \tag{4–4}$$

for infinitely many large s.

Inequality (4–3) implies that if $d > 1$ the average number of intersections of γ_s with a random hyperplane tends to infinity as s grows. In particular, if γ is an unbounded algebraic curve then $N(s)$ is less than the degree of γ: this forces $d = 1$. Unbounded algebraic curves are one-dimensional.

5. REAL ZEROS OF REAL POLYNOMIALS

Let γ be an unbounded curve in $\mathcal{E}_n$ as before. Let

$$x_1 = \varphi_1(s), \quad x_2 = \varphi_2(s), \quad \dots, \quad x_n = \varphi_n(s)$$

be its parametrical representation: the φ_i's are real functions of bounded variation on every finite interval and s denotes the length coordinate. The previous results show that the average number of real zeros of the equation

$$\sum_{i=1}^{n} h_i\varphi_i(t) - 1 = 0, \quad \text{for } 0 \leqslant t \leqslant s,$$

is

$$N(s) \geqslant s^{1-1/d-\varepsilon}$$

for all large s.

The most interesting case is when $\varphi_i(t) = t^i$, with $|t| \leqslant T \sim s^{1/n}$, for $i = 1, 2, \dots, n$. Then $N(s)$ would be the average number of real zeros in $[-T, +T]$ of the polynomial

$$P(t) \equiv h_n t^n + h_{n-1}t^{n-1} + \dots + h_1 t - 1.$$

Unfortunately, as noticed in Section 4, the dimension is 1 so all we get is the trivial result $N(s) > 0$.

Edelman and Kostlan [1995], reflecting on a result of M. Kac [1943; 1949; 1959], realized that it is possible — and even easy — to obtain the average number of real zeros of the polynomial

$$h_0 + h_1 t + \dots + h_n t^n$$

using results from integral geometry. Indeed, consider the curve

$$(x_0, \dots, x_n) = (1, t, t^2, \dots, t^n), \quad \text{for } t \in \mathbb{R},$$

or rather the curve γ obtained by projecting it centrally onto the unit sphere $x_0^2 + \dots + x_n^2 = 1$. Intersecting γ by a random hyperplane h

$$h_0x_0 + h_1x_1 + \dots + h_nx_n = 0,$$

where $h = (h_0, h_1, \dots, h_n)$ is uniformly distributed on the surface of the unit sphere $h_0^2 + \dots + h_n^2 = 1$, shows that the average number of intersecting points — that is, zeros of the polynomial $h_0 + h_1 t + \cdots + h_n t^n$ — is equal to $|\gamma|/\pi$. The length $|\gamma|$ is easily computed:

$$\frac{1}{\pi}|\gamma| = \frac{1}{\pi}\int_{-\infty}^{+\infty}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}\,dt.$$

This is Kac's formula, obtained by Edelman and Kostlan's analysis. Kac obtained an equivalence when n increases to infinity and the second authors were able to give a much more precise result which finally shows that the average number of real zeros of the polynomial is

$$\frac{2}{\pi}\log n + 0.625\ldots + \frac{2}{\pi n} + O\Big(\frac{1}{n^2}\Big).$$

The main difference between their approach and ours holds in projecting curves on the surface of the unit n-sphere so that curves of infinite length in $\mathcal{E}_{n+1}$ may well have a finite length on $\mathbb{S}_n$. Their normalizing factor $1/\pi$ is a measure of the set of hyperplanes $h \in \mathcal{E}^{n+1}$ which pass through the origin. Some of the hyperplanes do not intersect γ.

Erdős and Offord [1956] discussed a similar problem, namely to compute the average number of real zeros of a n degree random polynomial with coefficients ± 1. They find that the average is again $\frac{2}{\pi}\log n$. The method is very different and it would be desirable to be able to find a geometrical approach to their analysis.

6. DETERMINISTIC RESULTS

The deterministic counterpart of the Kac–Erdős–Offord theorems could be as follows. Given an infinite sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of ± 1, say the Erdős–Offord case, we form the polynomials

$$f_n(X) = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n.$$

Writing $\rho(f_n)$ for the number of real zeros of f_n counted with multiplicity, we want to study the average

$$\frac{1}{N}\sum_{n=0}^{N-1}\rho(f_n),$$

and to compare it with $\frac{2}{\pi}\log N$ as N goes to infinity. If the two quantities are equivalent we can consider the sequence $(\alpha_n)_{n\in\mathbb{N}}$ to be "random" in some sense. We conjecture that almost all sequences $\alpha \in \{-1,1\}^{\mathbb{N}}$ behave like that. We studied two deterministic $(+,-)$ sequences. The first is the Thue–Morse sequence, which is the only nontrivial example for which we can give a precise result. It is also a special case of more interesting sequences (see addendum at the end of this paper for details). Even though the Thue–Morse sequence is far from mimicking randomness, it still retains some features associated with randomness: namely, the spectral measure is continuous (yet not absolutely continuous as would be the case for a random sequence). Theorems 6.1 and 6.2 below show that the Thue–Morse sequence is actually far from random.

Before stating them, we consider another $(+,-)$ sequence which appears to behave à la Kac–Erdős–Offord. Put $\beta_0 = \beta_1 = 1$ and $\beta_n \equiv p_n \bmod 4$ for $n \geqslant 2$, where p_n is the n-th prime number. To this day it seems completely out of reach to prove any relevant theorem concerning the zeros of the related polynomials. So we leave it as a conjecture to establish that the average is equivalent to $\frac{2}{\pi}\log N$. See our numerical evidence obtained with the help of the command polsturm of PARI-GP up to $N = 600$. Figure 1 shows

$$\frac{\pi}{2N\log N}\sum_{n=0}^{N-1}\rho(f_n)$$

as a function of N when the coefficients of f_n follow a random sequence and the β sequence mentioned above.

In the remainder of the paper, $(\varepsilon_i)_{i\in\mathbb{N}}$ will represent the Thue–Morse sequence defined by $\varepsilon_i = (-1)^{\nu(i)}$ where $\nu(i)$ is the sum of the binary digits of i. Then $f_n(X)$ will be the Thue–Morse polynomial of degree n, that is, $f_n(X) = \varepsilon_0 + \varepsilon_1 X + \cdots + \varepsilon_n X^n$. For this very special sequence we are able to prove that

$$\lim_{N\to\infty}\frac{1}{N\log N}\sum_{n=0}^{N-1}\rho(f_n) = 0.$$

In fact, we shall show:

Theorem 6.1. *If $n \in \mathbb{N}$ is even, f_n has at most two real roots. More precisely*:

1. *If $n \equiv 0 \bmod 4$ and $\varepsilon_n = 1$ then f_n has no real root.*
2. *If $n \equiv 2 \bmod 4$ and $\varepsilon_n = 1$ then f_n has two negative roots.*

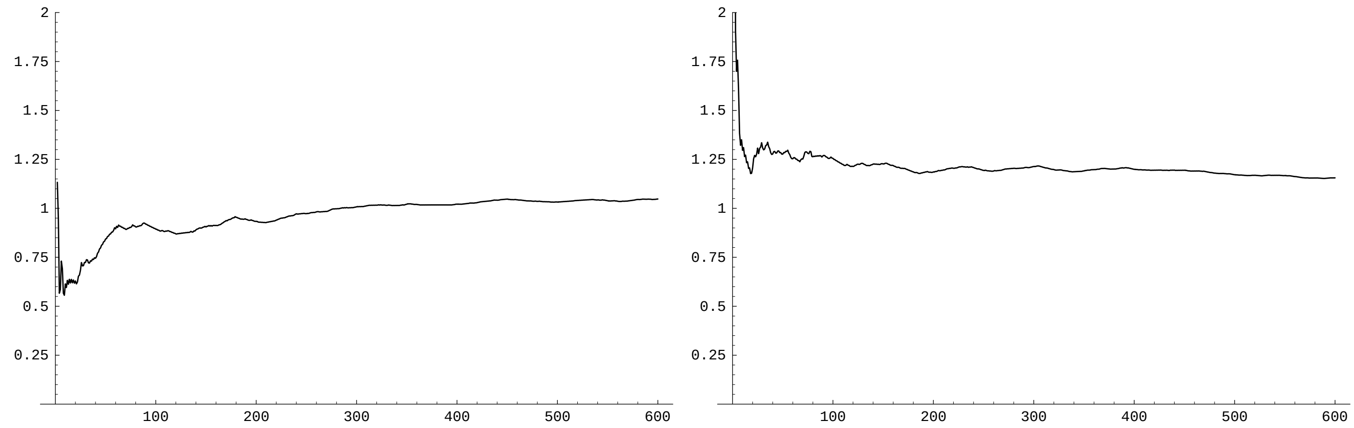

FIGURE 1. Plot of $\pi/(2N\log N)\ \sum_{n=0}^{N-1} \rho(f_n)$ as a function of N, for a random sequence (left) and the β sequence of the text (right).

3. *If* $\varepsilon_n = -1$ *then* f_n *has two real roots, one positive and the other negative.*
4. *If* $n \in \mathbb{N}$ *is odd, let* k *be the* 2*-adic valuation of* $n+1$. *Then if* $\varepsilon_n = (-1)^k$, f_n *has* $2k-1$ *real roots and if* $\varepsilon_n = (-1)^{k+1}$, f_n *has* $2k+1$ *real roots.*

Theorem 6.2. *The mean*

$$\frac{1}{N}\sum_{n=0}^{N-1}\rho(f_n)$$

tends to $\frac{11}{4}$ *as* N *tends to infinity.*

We start with elementary results.

7. ELEMENTARY PROPERTIES

First we remark that $\varepsilon_{2i} = \varepsilon_i$ and $\varepsilon_{2i+1} = -\varepsilon_{2i}$ and that the generating series of $(\varepsilon_i)_{i\in\mathbb{N}}$ is

$$\sum_{i=0}^{\infty}\varepsilon_i X^i = \prod_{l=0}^{\infty}\left(1 - X^{2^l}\right). \qquad (7\text{–}1)$$

Now we define the Thue–Morse word $\mathcal{E}$ on the alphabet $\{+,-\}$ by iterating the morphism φ defined by $\varphi(+) = +-$ and $\varphi(-) = -+$ so that

$$\mathcal{E} = \lim_{n\to\infty}\varphi^n(+).$$

The link between the Thue–Morse word and the Thue–Morse sequence is well known. The letter of order i of $\mathcal{E}$ is $+$ or $-$ depending on whether $\varepsilon_i = 1$ or $\varepsilon_i = -1$. We introduce another useful morphism, namely ψ satisfying $\psi(+) = +--+$ and $\psi(-) = -++-$. Since $\psi = \varphi^2$, we have also

$$\mathcal{E} = \lim_{n\to\infty}\psi^n(+).$$

At last, if $i \leqslant j$ we write ${}_i\mathcal{E}_j$ for the factor of $\mathcal{E}$ whose first (resp. last) letter is the letter of rank i (resp. j) of $\mathcal{E}$. For example ${}_0\mathcal{E}_3 = +--+$. In the same way, the definitions of ψ and φ ensure that

$${}_{4n}\mathcal{E}_{4n+3} = \psi(+) \text{ or } \psi(-). \qquad (7\text{–}2)$$

and that $+++$ and $---$ are never factors of $\mathcal{E}$.

We now get back to polynomials. We write $\mathcal{R}(P)$ for the set of real roots of a polynomial P. Note that in general $|\mathcal{R}(P)| \neq \rho(P)$ since $\rho(P)$ counts the roots with multiplicity. If we consider a polynomial P with coefficients ± 1 it is well known that $\mathcal{R}(P) \subset \left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$. In addition, for Thue–Morse polynomials, we have:

Lemma 7.1. $\mathcal{R}(f_n) \cap [-0.95, 0.95] = \varnothing$ *for* $n > 255$.

Proof. Indeed,

$$|f_n(x) - f_{255}(x)| \leqslant 0.95^{256}\frac{1}{1-0.95}.$$

Now (7–1) shows that $f_{255}(x) = \prod_{l=0}^{7}(1 - x^{2^l})$ so that for $x \in [-0.95, 0.95]$ we have

$$f_{255}(x) \geqslant f_{255}(0.95) = 0.000132\ldots,$$

whereas $0.95^{256}\frac{1}{1-0.95} \leqslant 0.00004$. $\square$

We now investigate odd degrees.

8. THUE–MORSE POLYNOMIALS OF ODD DEGREE

Lemma 8.1. *If* $n \equiv -1 \bmod 2^k$, *namely if* $n = v2^k + 2^k - 1$ *with* $v \geqslant 0$, *then*

$$f_n(X) = f_{2^k-1}(X)f_v(X^{2^k}).$$

Proof. We remark that every integer $i \in [0, n]$ can be written uniquely as the sum $i = 2^k q + r$ with $0 \leqslant q \leqslant v$ and $0 \leqslant r \leqslant 2^k - 1$. Since $\varepsilon_i = \varepsilon_r \varepsilon_q$, we have

$$\begin{aligned} f_n(X) &= \sum_{i=0}^{n} \varepsilon_i X^i = \sum_{r=0}^{2^k-1} \sum_{q=0}^{v} \varepsilon_r X^r \varepsilon_q X^{q2^k} \\ &= f_{2^k-1}(X) f_v(X^{2^k}). \end{aligned}$$ □

Now we specify the roots of f_{2^k-1}.

Lemma 8.2. *Let k be a positive integer. The only real roots of f_{2^k-1} are -1 and 1 respectively of order $k-1$ and k.*

Proof. This relies on the factorization

$$f_{2^k-1}(X) = (1-X)^k (1+X)^{k-1} \prod_{l=1}^{k-1} \big(1 + X^{2^l}\big)^{k-1-l}.$$

derived from (7–1). □

Lemma 8.3. *For $k \geqslant 2$, f_{2^k-1} is strictly increasing on $\big[-1, -\frac{1}{3}\big[$ and strictly decreasing on $[0, 1]$.*

Proof. The equation in the proof of the preceding lemma asserts that $f_{2^k-1}(x) > 0$ for x in $]-1, 1[$. Moreover

$$\frac{f'_{2^k-1}(x)}{f_{2^k-1}(x)} = \sum_{l=0}^{k-1} \frac{-2^l x^{2^l-1}}{1 - x^{2^l}}.$$

We deduce from this that $f'_{2^k-1}(x) < 0$ for $x \in [0, 1[$ and that $f'_{2^k-1}(x) > 0$ on $\big[-1, -\frac{1}{3}\big[$. □

Now we prove that the derivative of Thue–Morse polynomials can be bounded, in some cases, on an neighbourhood of -1 and 1. We set $I_n^- = \big[-1, -1 + \frac{3}{2n}\big]$ and $I_n^+ = \big[1 - \frac{3}{2n}, 1\big]$.

Lemma 8.4. *If $n \geqslant 3$ is an odd integer, we have $|f'_n(x)| \leqslant 4$ on I_n^+. If $n \equiv 3 \bmod 4$ then $|f'_n(x)| \leqslant 10$ on I_n^-.*

Proof. For the first point, suppose that $n \equiv 3 \bmod 4$. Then,

$$f_n(x) = f_3(x) g(x^4) = (1 - x^2) h(x).$$

It is easy to see that $g(x)$ and so also

$$h(x) = (1 + x) g(x^4) \tag{8–1}$$

have coefficients ± 1. The equality (8–1) then implies that $|h(x)| \leqslant (n+1)/2$ on $[0, 1]$ and that

$$|h'(x)| \leqslant \sum_{i=0}^{\frac{n-3}{4}} (4i + 4i + 1) = \frac{n^2 - n - 2}{4}.$$

Since $f'_n(x) = (1-x)^2 h'(x) - 2(1-x) h(x)$, we have

$$\begin{aligned} |f'_n(x)| &\leqslant \Big(\frac{3}{2n}\Big)^2 \frac{n^2 - n - 2}{4} + 2 \cdot \frac{3}{2n} \cdot \frac{n+1}{2} \\ &= \frac{33n^2 + 15n - 18}{16n^2} < 3 \end{aligned}$$

for all $n > 0$.

If $n \equiv 1 \bmod 4$ we write

$$f'_n(x) = f'_{n-2}(x) \pm \big((n-1)x^{n-2} - n x^{n-1}\big).$$

Obviously, for $n \geqslant 3$, $\big|(n-1)x^{n-2} - n x^{n-1}\big| \leqslant 1$ on $[0, 1]$. The result is proved on I_n^+ since $|f'_{n-2}(x)| < 3$, by the previous point.

For $x \in I_n^-$ we begin by the case $n \equiv 7 \bmod 8$, we write $f_n(x) = (1+x)^2 k(x)$ and as previously we get $|k(x)| \leqslant 2n + 2$ and $|k'(x)| \leqslant n^2 - n - 2$ which assert that $|f'_n(x)| < 4$ for all $n > 0$.

If $n \equiv 3 \bmod 8$, relation (7–2) ensures that

$$\big|f'_n(x)\big| \leqslant \big|f_{n-4}(x)\big| + \big|(n-3)x^{n-4} f_3(x)\big| + \big|x^{n-3} f'_3(x)\big|.$$

The summands are bounded by 4, 2, and 4. □

9. THUE–MORSE POLYNOMIALS OF EVEN DEGREE

If n is a positive integer we have

$$\begin{aligned} f_{2n}(X) &= f_n(X^2) - X f_{n-1}(X^2), \\ f_{2n}(X) &= (1 - X) f_n(X^2) + \varepsilon_n X^{2n+1}. \end{aligned}$$

We continue with a result on the monotonicity of f'_n near -1 and 1.

Lemma 9.1. *Let n be an even positive integer. Then f'_n does not vanish on $I_n^- \cup I_n^+$. More exactly, on I_n^- we have*

$$\operatorname{sgn}(f'_n) = \begin{cases} \varepsilon_n & \text{if } n \equiv 2 \bmod 4, \\ -\varepsilon_n & \text{if } n \equiv 0 \bmod 4, \end{cases}$$

and on I_n^+ we have $\operatorname{sgn}(f'_n) = \varepsilon_n$.

Proof. For $x \in I_n^+$, we write $f'_n(x) = f'_{n-1}(x) + \varepsilon_n n x^{n-1}$. Then for n large, Lemma 8.4 ensures that $n x^{n-1}$ is greater than $|f'_n(x)|$ on I_n^+. For small n we check the result directly. On I_n^-, the starting relations are $f'_n(x) = f'_{n+1}(x) - \varepsilon_{n+1}(n+1)x^n$ or

$f_n'(x) = f_{n-1}'(x) + \varepsilon_n n x^{n-1}$ and we conclude with the same arguments. $\square$

Concerning the real roots of f_n, two criteria must be taken into account, namely the remainder of n modulo 4 and the coefficient of the highest term ε_n of f_n. So we define four sets: $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, and $\mathcal{B}_1$. The letter $\mathcal{A}$ stands for the condition $n \equiv 0 \bmod 4$, and $\mathcal{B}$ for $n \equiv 2 \bmod 4$. The indices 0 and 1 mean respectively that $\varepsilon_n = 1$ and $\varepsilon_n = -1$. For example,

$$\mathcal{A}_1 = \{f_n \mid n \equiv 0 \bmod 4 \text{ and } \varepsilon_n = -1\}.$$

Figure 2 shows the behaviour of polynomials in each set.

Therefore we shall show that for n even, f_n has at most 2 real roots. The proof of this fact requires different methods depending on the part of $[-2, 2]$ that we consider.

The next lemma enables us to reduce the domain where we can expect f_n to vanish.

Lemma 9.2. *Let f_n be a Thue–Morse polynomial of even degree, then $\mathcal{R}(f_n) \subset [-2, 1]$. In addition, if $n \equiv 0 \bmod 4$ then $\mathcal{R}(f_n) \subset [-1, 1]$.*

Proof. As $n - 1$ is odd, Lemma 8.1 ensures that

$$\left|f_n(x)\right| \geqslant \left||x^n| - |f_1(x) f_v(x^2)|\right|. \tag{9–1}$$

Now if $x > 1$ then $|x^n| > \frac{|x-1|\,|x^n-1|}{|x^2-1|}$. This and (9–1) imply that f_n has no real root greater than 1.

If $n \equiv 0 \bmod 4$, Lemma 8.1 shows that $\left|f_n(x)\right| \geqslant \left||x^n| - |f_3(x) f_v(x^4)|\right|$. Moreover if $|x| > 1$ it is immediate that

$$|x^n| > \left|\frac{(x-1)(x^n-1)}{x^2+1}\right| = \left|\frac{f_3(x)(x^n-1)}{x^4-1}\right|.$$

Therefore $|x^n| - |f_3(x) f_v(x^4)| > 0$ if $|x| > 1$, which completes the proof. $\square$

Lemma 9.3. *Let $f_n \in \mathcal{A}_0 \cup \mathcal{B}_0$, then f_n is positive on $[0, 1]$. In particular, $\mathcal{R}(f_n) \cap [0, 1] = \varnothing$.*

Proof. We consider the set $\mathcal{C}_0 = \mathcal{A}_0 \cup \mathcal{B}_0$, ordered by the degree.

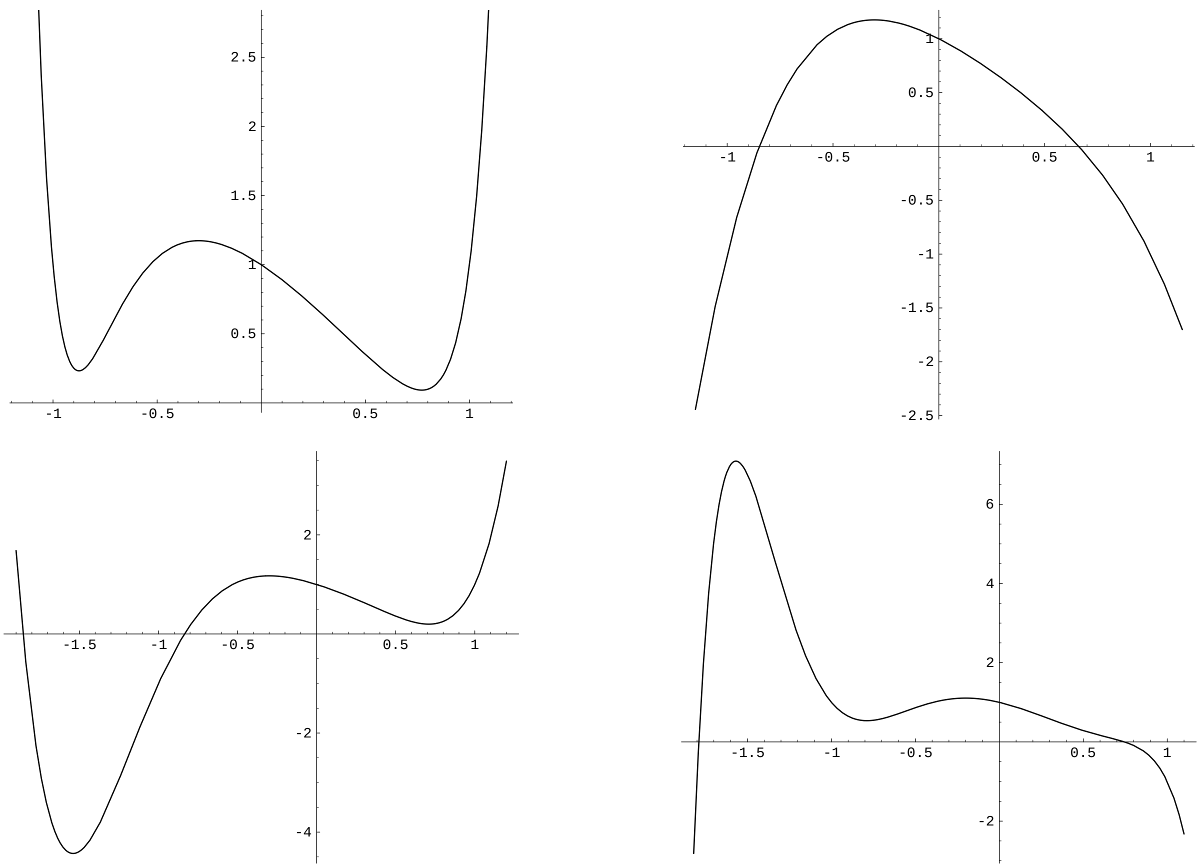

FIGURE 2. Typical representatives of $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$.

We prove the lemma by induction on the rank of the elements of $\mathcal{C}_0$. This result is true for the first polynomials of $\mathcal{C}_0$, namely, f_6, f_{10} and f_{12}.

Suppose the result is true for all polynomials in $\mathcal{C}_0$ whose rank is at most $r \geqslant 3$. Let d be the degree of the polynomial of rank r and f_n the polynomial of $\mathcal{C}_0$ whose rank is $r+1$. Examining $\varphi^3(+)$ and $\varphi^3(-)$ we deduce that

$$n - d \leqslant 6. \tag{9–2}$$

From (9) we know that

$$f_n(x) = (1-x) f_{n/2}(x^2) + x^{n+1}.$$

If $f_n \in \mathcal{A}_0$ then $f_{n/2} \in \mathcal{C}_0$ and by the hypothesis of induction, $f_{n/2}(x) > 0$ on $[0,1]$. Therefore f_n is positive on $[0,1]$, and the condition is fulfilled for a polynomial of rank $r+1$.

If $f_n \in \mathcal{B}_0$, we search the Thue–Morse polynomial of smallest degree m greater than n such that $f_m \in \mathcal{A}_0$. The idea is to show that the difference $f_n(x) - f_m(x)$ is positive. We check that m must satisfy $m = n+2$ or $m = n+6$ and therefore that $f_n(x) - f_m(x)$ must equal $x^{n+1} - x^{n+2}$ or

$$x^{n+1} + x^{n+2} - x^{n+3} - x^{n+4} + x^{n+5} - x^{n+6}.$$

Moreover, from (9–2), we have $m - d \leqslant 12$. Since $12 \leqslant d$,

$$\frac{m}{2} \leqslant d.$$

So we can write

$$f_m(x) = (1-x) f_{m/2}(x^2) + \varepsilon_m x^{m+1}$$

and conclude that f_m does not vanish on $[0,1]$.

To complete the proof, we verify that $x^{n+1} - x^{n+2}$ and $x^{n+1} + x^{n+2} - x^{n+3} - x^{n+4} + x^{n+5} - x^{n+6}$ are positive on $]0,1[$. Since $f_n(0) = f_n(1) = 1$, the hypothesis is fulfilled in this case too. The polynomial f_n of rank $r+1$ does not vanish on $[0,1]$. □

Lemma 9.4. *Let $f_n \in \mathcal{A}_0 \cup \mathcal{B}_1$, then f_n is positive on $[-1,0]$. In particular, $\mathcal{R}(f_n) \cap [-1,0] = \varnothing$.*

Proof. We consider the set

$$\mathcal{A}_0^- = \{f_n \in \mathcal{A}_0 \mid \varepsilon_{n-1} = -1\},$$

ordered by the degree. So the terms of f_n of degree n to $n-4$ are known. They correspond to the factor ${}_{n-4}\mathcal{E}_n = \psi(-)+$.

First we show by a descending induction that every polynomial in $\mathcal{A}_0^-$ is positive on $\left[-1, -\frac{3}{4}\right]$.

For this, we display polynomials in $\mathcal{A}_0^-$ which are positive on $\left[-1, -\frac{3}{4}\right]$ and whose degree is arbitrarily large. If $n_k = 2^{k+2} + 2^{k+1}$ with k odd, then $f_{n_k} \in \mathcal{A}_0$, $\varepsilon_{n_k-1} = -1$ and we are able to prove that f_{n_k} is positive on $\left[-1, -\frac{3}{4}\right]$.

Now we focus our attention on the second step of the induction. We suppose that f_n, the polynomial of rank r in $\mathcal{A}_0^-$ is positive on $\left[-1, -\frac{3}{4}\right]$ and we want to deduce that $f_m \in \mathcal{A}_0^-$ whose rank is $r-1$ is also positive on $\left[-1, -\frac{3}{4}\right]$.

In fact the terms of higher degree of f_n are well known. Examining $\mathcal{E}$ we find four cases:

$$\begin{aligned}
{}_{n-12}\mathcal{E}_n &= \psi(-+-)+,\\
{}_{n-16}\mathcal{E}_n &= \psi(-++-)+,\\
{}_{n-16}\mathcal{E}_n &= \psi(-+--)+,\\
{}_{n-20}\mathcal{E}_n &= \psi(-++--)+.
\end{aligned}$$

We obtain respectively $m = n-8$, $m = n-12$, $m = n-12$, $m = n-16$, so that

$$f_m(x) = f_n(x) + x^{m+1} P(x),$$

where $P(x)$ must be a polynomial among four determined polynomials. For example, if $m = n-8$ then $P(x) = 1 + x - x^2 + x^3 - x^4 - x^5 + x^6 - x^7$. We point out that $m+1$ is odd and that $P(x)$ is negative on $\left[-1, -\frac{3}{4}\right]$ in all the cases. Therefore the hypothesis $f_n(x) > 0$ proves that $f_m(x) > 0$ on $\left[-1, -\frac{3}{4}\right]$. Every polynomial in $\mathcal{A}_0^-$ is then positive on $\left[-1, -\frac{3}{4}\right]$.

If $f_n \in \mathcal{A}_0 \backslash \mathcal{A}_0^-$, we see that ${}_{n-4}\mathcal{E}_{n+11} = \psi(++-+)$ or ${}_{n-4}\mathcal{E}_{n+15} = \psi(++--+)$. Then

$$f_n(x) = f_m(x) + x^{m+1} P(x)$$

with $f_m(x) \in \mathcal{A}_0^-$, $m = n+8$ or $m = n+12$ and $P(x)$ negative on $\left[-1, -\frac{3}{4}\right]$ in all the cases. It follows that $f_n(x) > 0$ on $\left[-1, -\frac{3}{4}\right]$ for every polynomial $f_n \in \mathcal{A}_0$.

Finally, if $f_n \in \mathcal{B}_1$ then $f_{n-2} \in \mathcal{A}_0$ and $f_n(x) - f_{n-2}(x)$ is positive on $\left]-1, -\frac{3}{4}\right]$ which ensures that every polynomial in $\mathcal{A}_0 \cup \mathcal{B}_1$ is positive on $\left[-1, -\frac{3}{4}\right]$ since $f_n(-1) = -1$.

Let f_n be in $\mathcal{A}_0 \cup \mathcal{B}_1$. If $n \leqslant 255$ a direct study with the command polsturm of PARI-GP shows that f_n does not vanish on $[-1,0]$. If $n > 255$, Lemma 7.1 proves that f_n does not vanish on $\left[-\frac{3}{4}, 0\right] \subset [-0.95, 0]$ so that $\mathcal{R}(f_n) \cap [-1,0] = \varnothing$. □

The two following lemmas display intervals containing a root.

Lemma 9.5. *Let $f_n \in \mathcal{A}_1 \cup \mathcal{B}_1$, then f_n has a unique root in* $[0, 1]$.

Proof. Let $\mathcal{C}_1 = \mathcal{A}_1 \cup \mathcal{B}_1$ be ordered by the degree. If $f_n \in \mathcal{C}_1$ we show that $f_n'(x) < 0$ on $\left[\frac{1}{2}, 1\right]$ by a descending induction on the rank of the elements of $\mathcal{C}_1$.

First of all, for $k \geqslant 2$, f_{2^k} belongs to $\mathcal{A}_1$. Moreover, Lemma 8.3 asserts that $f_{2^k}' < 0$ on $[0, 1]$. So we can start the induction at degrees arbitrarily large.

If f_n, the polynomial of rank r of $\mathcal{C}_1$, is strictly decreasing on $\left[\frac{1}{2}, 1\right]$, then we prove that f_m, the polynomial of rank $r-1$ of $\mathcal{C}_1$, satisfies the same property.

If $f_n \in \mathcal{A}_1$, the tail of f_n corresponds to the factor

$$_{n-4}\mathcal{E}_n = \psi(+)- \text{ or } {}_{n-4}\mathcal{E}_n = \psi(-)-$$

so that $m = n-2$ or $m = n-4$.

If $f_n \in \mathcal{B}_1$, the tail of f_n must be

$$_{n-6}\mathcal{E}_n = \psi(-)+-- \quad \text{or } {}_{n-6}\mathcal{E}_n = \psi(+)+--$$

and $m = n-6$ or $m = n-4$.

So

$$f_m(x) = f_n(x) + x^{m+1}Q(x)$$

with 3 possibilities for $Q(x)$. In each case, it is easy to check that $x^{m+1}Q(x)$ is decreasing on $\left[\frac{1}{2}, 1-\frac{3}{2m}\right]$, because $Q(x)$ and $xQ'(x)$ are increasing on this interval. On I_m^+, Lemma 9.1 asserts that $f_m'(x) < 0$.

So each polynomial of $f_m \in \mathcal{C}_1$ is strictly decreasing on $\left[\frac{1}{2}, 1\right]$. Since f_m satisfies

$$f_m(0)f_m(-1) = -1$$

and does not vanish on $\left[0, \frac{1}{2}\right]$, we have proved the result. □

Lemma 9.6. *Let $f_n \in \mathcal{A}_1 \cup \mathcal{B}_0$. Then f_n has a unique root in* $[-1, 0]$.

Proof. For $n \leqslant 255$ a direct study shows that every $f_n \in \mathcal{A}_1 \cup \mathcal{B}_0$ has a unique root in $[-1, 0]$. We suppose until the end of the proof that $n > 255$.

Let

$$\mathcal{A}_1^+ = \{f_n \in \mathcal{A}_1 \mid \varepsilon_{n-1} = 1\}$$

be ordered by the degree. We show with the help of a descending induction that every element of $\mathcal{A}_1^+$ with a degree greater than 255 is strictly increasing on $[-1, -0.92]$.

First, if $k \geqslant 2$ is an even integer then f_{2^k} belongs to $\mathcal{A}_1^+$ and $f_{2^k}'(x) > 0$ on $[-1, -0.92]$, by Lemma 8.3.

The second step of the induction consists in proving that f_m, the polynomial of $\mathcal{A}_1^+$ of rank $r-1$, is increasing on $[-1, -0.92]$ assuming that f_n the polynomial of rank r is also increasing on this interval. The different possible tails of f_n are

$$_{n-12}\mathcal{E}_n = \psi(+-+)-,$$
$$_{n-16}\mathcal{E}_n = \psi(+--+)-,$$
$$_{n-16}\mathcal{E}_n = \psi(+-++)-,$$
$$_{n-20}\mathcal{E}_n = \psi(+--++)-,$$

so that

$$f_m(x) = f_n(x) + x^{m+1}R(x),$$

where $R(x)$ can take four different values. In each case, we check that $x^{m+1}R(x)$ is strictly increasing on $[-1 + 3/2m, \ -0.92]$.

As Lemma 9.1 proves that $f_m'(x) > 0$ on I_m^-, the monotonicity of $f_m(x)$ on $[-1, -0.92]$ is established for all the polynomials in $\mathcal{A}_1^+$ of degree > 255.

If $f_n \in \mathcal{A}_1 \setminus \mathcal{A}_1^+$ we point out that $_{n-4}\mathcal{E}_{n+11} = \psi(--+-)$ or $_{n-4}\mathcal{E}_{n+15} = \psi(--++-)$. So

$$f_n(x) = f_m(x) + x^{m+1}R(x)$$

with $f_m(x) \in \mathcal{A}_1^+$, $m = n+8$ or $m = n+12$ and $x^{m+1}R(x)$ strictly increasing on $[-1+3/2m, \ -0.92]$. Thus as previously, $f_n'(x) > 0$ on $[-1, -0.92]$.

Finally, $f_n \in \mathcal{A}_1$ is equivalent to $f_{n+2} \in \mathcal{B}_0$. Now $f_{n+2}(x) - f_n(x)$ is increasing on

$$[-1 + 3/(2n+4), \ -0.92]\,,$$

and we conclude on I_{n+2}^- by Lemma 9.1.

So every polynomial $f_n \in \mathcal{A}_1 \cup \mathcal{B}_0$ is strictly increasing on $[-1, -0.92]$. From Lemma 7.1, f_n does not vanish on $[-0.92, 0]$ and it satisfies

$$f_n(0)f_n(-1) = -1.$$

So f_n has a unique root in $[-1, 0]$. □

We now study the property of the reciprocal polynomial f_n^* of f_n in order to determine the real roots of f_n outside the unit circle.

Lemma 9.7. *Let $f_n \in \mathcal{B}_0 \cup \mathcal{B}_1$. We can then write*

$$f_n(X) = \varepsilon_n Q(X) X^{n-\deg Q} + f_r(X),$$

with $r = n - \deg Q - 1$ and where $Q(X)$ is one of $f_{10}(X)$, $-f_{14}(X)$, $f_{18}(X)$, $-f_{22}(X)$.

Proof. For convenience, we first prove a similar result for $m \equiv 3 \bmod 4$.

If $m \equiv 3 \bmod 4$ then $f_m(X) = f_3(X)f_v(X^4)$. Without loss of generality we can assume that $\varepsilon_m = 1$. As $\psi(+-+-+)$ never occurs in the Thue–Morse word $\mathcal{E}$, we see that the tail of f_m must be one of the following words

$$\begin{aligned} {}_{m-11}\mathcal{E}_m &= \psi(-++), \\ {}_{m-15}\mathcal{E}_m &= \psi(+--+), \\ {}_{m-19}\mathcal{E}_m &= \psi(-++-+), \\ {}_{m-23}\mathcal{E}_m &= \psi(+--+--+). \end{aligned}$$

Thus $f_m(X)$ equals

$$P(X)X^{m-degP} + f_r(X),$$

where $P(X)$ is one of $-f_{11}(X)$, $f_{14}(X)$, $-f_{18}(X)$, or $f_{22}(X)$. If $n = m-1$ then

$$f_n(X) = Q(X)X^{n-degQ} + f_r(X),$$

where $Q(X)$ is one of $-f_{10}(X)$, $f_{14}(X)$, $-f_{18}(X)$, or $f_{22}(X)$. Finally, $\varepsilon_m = -\varepsilon_n$ and the proof is completed. □

Lemma 9.8. *Let $f_n \in \mathcal{B}_0 \cup \mathcal{B}_1$. Then*

$$\mathcal{R}(f_n) \cap [-1.3, -1] = \varnothing.$$

Proof. Let $f_n \in \mathcal{B}_0 \cup \mathcal{B}_1$. Using Lemma 9.7 we write

$$f_n(x) = \varepsilon_n Q(x)x^{n-\deg Q} + f_r(x),$$

with $Q \in \{-f_{10}, f_{14}, -f_{18}, f_{22}\}$ and

$$r = n - \deg Q - 1 \equiv 3 \bmod 4.$$

If f_n vanishes at x then $x^{n-\deg Q}|Q(x)| = |f_r(x)|$ and

$$\begin{aligned} |f_r(x)| &\leqslant \left|\frac{(x^2-1)(x^{n-\deg Q})}{x^2+1}\right| \leqslant |x^{n-\deg Q} - 1| \\ &= x^{n-\deg Q} - 1. \end{aligned}$$

Thus

$$|Q(x)| \leqslant \frac{x^{n-\deg Q} - 1}{x^{n-\deg Q}} \leqslant 1.$$

Therefore if $|Q(x)| > 1$ then $f_n(x) \neq 0$. Now the minimum of

$$-f_{10}(x),\ f_{14}(x),\ -f_{18}(x),\ f_{22}(x)$$

is > 1 on $[-1.3, -1[$. Since $f_n(-1) \neq 0$ we have $\mathcal{R}(f_n) \cap [-1.3, -1] = \varnothing$. □

Lemma 9.9. *If $f_n \in \mathcal{B}_0 \cup \mathcal{B}_1$ then f_n has a unique root in $[-2, -1]$.*

Proof. By Lemma 9.7,

$$f_n^*(x) - \varepsilon_n Q^*(x) = f_r^*(x)x^{\deg Q+1}$$

with $r = n - \deg Q - 1 \equiv 3 \bmod 4$. Therefore

$$f_r^*(x) = f_3(x)g(x^4),$$

where g is a unimodular polynomial. Thus

$$\left|(f_r^*(x)x^{\deg Q+1})'\right| \leqslant \sum_{k=0}^{v}\left|(x^{4k}f_3(x)x^{\deg Q+1})'\right|.$$

Now for n odd and x in $[-0.8, -0.5]$, $|(f_3(x)x^n)'| \leqslant 2.52(-0.8)^n - 0.81(-0.8)^n n$. As

$$\sum_{k=0}^{\infty} 2.52(-0.8)^{4k+11} - 0.81(-0.8)^{4k+11}(4k+11) < 1.3,$$

we deduce that on this interval $|(f_r^*(x)x^{\deg Q+1})'| < 1.3$ as soon as $\deg Q \geqslant 10$. The study of the derivatives of f_{10}^*, $-f_{14}^*$, f_{18}^* and $-f_{22}^*$, ensures that they are greater than $1.3671\ldots$ in modulus and that they keep the same sign on $[-0.8, -0.5]$. The derivative of f_n^* then keeps its sign on $[-0.8, -0.5]$. Now

$$f_n^*(0)f_n^*(-1) = -1$$

and $f_n^*(x)$ does not vanish on $[-1, -0.8]$ by Lemma 9.8. Thus f_n^* has a unique real root in $[-1, -0.5]$. This proves that f_n vanishes only once in $[-2, -1]$. □

We can now prove Theorems 6.1 and 6.2.

10. PROOF OF THE THEOREMS

Proof of Theorem 6.1. Let n be even. Lemmas 9.2, 9.3, 9.4 establish the theorem when $f_n \in \mathcal{A}_0$.

If $f_n \in \mathcal{A}_1$, Lemmas 9.2, 9.5 and 9.6 ensure the result.

If $f_n \in \mathcal{B}_0$, Lemmas 9.2, 9.3, 9.6 and 9.9 allow us to conclude.

If $f_n \in \mathcal{B}_1$, Lemmas 9.2, 9.4, 9.5 and 9.9 complete the proof for the case n even.

Let n be odd. If k is the 2-adic valuation of $n+1$ then we see that

$$f_n(X) = f_{2^k-1}(X)f_v(X^{2^k})$$

with v even. From what we have just proved f_v has a real positive root if and only if $\varepsilon_v = -1$. The relation $\varepsilon_{2^k-1} = (-1)^k$ and Lemma 8.2 complete the proof. □

Proof of Theorem 6.2. Let N be an integer. We investigate polynomials of even degree $n < N$. The relation $|f_{N-1}(1)| \leqslant 1$ implies that

$$\frac{N-1}{2} \leqslant \bigl|\{0 \leqslant i \leqslant N-1 \mid \varepsilon_i = 1\}\bigr| \leqslant \frac{N+1}{2}.$$

The same inequalities hold for

$$\bigl|\{0 \leqslant i \leqslant N-1 \mid \varepsilon_i = -1\}\bigr|.$$

Since $\varepsilon_i = 1$ is equivalent to $\varepsilon_{2i} = 1$ and $\varepsilon_i = 1$ is equivalent to $\varepsilon_{2i+1} = -1$, the four sets

$$\{0 \leqslant i \leqslant N-1 \mid \varepsilon_i = (-1)^r \text{ and } i \equiv s \bmod 4\},$$

for $r \in \{0,1\}$ and $s \in \{0,2\}$, have between $N/8 - C$ and $N/8 + C$ elements, for some constant C independent of N. So the Thue–Morse polynomials of even degree less than N contribute to at least $\frac{3N}{4} - 6C$ and to at most $\frac{3N}{4} + 6C$ real roots.

We now examine odd degrees. Let k be an integer. We consider the integers $i \in [0, N-1]$ such that the 2-adic valuation of $i+1$ is precisely k. These i's can be written generically $q2^{k+1} + 2^{k-1}$ with $q \in \mathbb{N}$ and $\varepsilon_i = \varepsilon_q(-1)^k$. So for $q \equiv 0 \bmod 4$ and $t = q, q{+}1, q{+}2, q{+}3$, the Thue–Morse polynomials of degree $t2^{k+1} + 2^{k-1}$ have $8k$ roots. Now for every N and k fixed, one can make $\lfloor N/2^{k+3}\rfloor$ such groups which give $8k\lfloor N/2^{k+3}\rfloor$ roots, and for each k we forget at most $8k$ roots in the sum.

Thus it is easy to see that

$$\begin{aligned} -\frac{6C}{N} &\leqslant \frac{1}{N}\sum_{n=0}^{N-1}\rho(f_n) - \frac{1}{N}\left(\sum_{k=0}^{\infty} 8k\left\lfloor\frac{N}{2^{k+3}}\right\rfloor + \frac{3N}{4}\right) \\ &\leqslant \frac{6C}{N} + \frac{1}{N}\sum_{k=0}^{\lfloor\log_2 N\rfloor} 8k, \end{aligned}$$

which ensures that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\rho(f_n) = \frac{11}{4}. \qquad \square$$

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ADDENDUM

After this article was submitted, a generalization of its methods allowed Doche to find families of $(+,-)$ sequences for which

$$\liminf_{n\to\infty}\rho(f_n)/\log n > 0.$$

See [Doche 1999] for details.

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