The Irreducible Six-Dimensional Complex Representations of $\mathrm{Aut}(F_2)$ That Are Nontrivial on F_2

Dragomir Ž. Đoković

In memory of David Doniz (1967–1997)

CONTENTS

Denote by Φ_2 the automorphism group of the free group F_2 on two generators. We classify the irreducible 6-dimensional complex representations of Φ_2 whose restriction to F_2 is nontrivial. J. Dyer, E. Formanek, and E. Grossman have shown how the Bürau representation of the braid group B_4 gives rise to a one-parameter family of irreducible 6-dimensional representations of Φ_2. The faithfulness question for these and some other closely related representations of Φ_2 is open. Our classification shows that all other 6-dimensional representations of Φ_2 are not faithful.

1. INTRODUCTION AND PRELIMINARIES

Let F_2 be the free group of rank 2 on the generators x and y. We set $\Phi_2 = \mathrm{Aut}(F_2)$. To conform with the well established practise, we consider Φ_2 as the group of right automorphisms of F_2. The classical generators P, σ, U and presentation of Φ_2 (due to B. H. Neumann) are as follows [Magnus et al. 1966, Problem 2, p. 169]:

$$P : x \to y,\ y \to x;$$
$$\sigma : x \to x^{-1},\ y \to y;$$
$$U : x \to xy,\ y \to y;$$
$$\begin{aligned}\Phi_2 = \langle P, \sigma, U : P^2 &= \sigma^2 = (P\sigma)^4 = (P\sigma P U)^2 \\ &= (UP\sigma)^3 = [U, \sigma U \sigma] = 1.\rangle\end{aligned}$$

We prefer to replace U with the involution $\varepsilon : x \to y^{-1}x,\ y \to y^{-1}$. These generators are related by $\varepsilon = (P\sigma)^2 U\sigma$ and $U = (P\sigma)^2\varepsilon\sigma$. In terms of the generators P, σ, and ε, the defining relations are

$$P^2 = \sigma^2 = \varepsilon^2 = (P\sigma)^4 = (P\varepsilon)^3 = ((P\sigma)^2\varepsilon\sigma\varepsilon)^2 = 1.$$

Supported in part by the NSERC Grant A-5285.

AMS Subject Classification: Primary 20C15, 20C40; Secondary 20E05, 20F36

Keywords: free group on two generators, braid group on four strings, irreducible finite dimensional representations, Gröbner basis routine

1058-6458/2000 \$0.50 per page

The subgroup $\langle P, \sigma \rangle$, a dihedral group of order 8, will play an important role and we denote it by D_4.

We say that a group is *linear* if it has a faithful finite-dimensional linear representation over some field. The problem of linearity of Φ_2 is still unresolved. It is of interest to examine its complex linear representations of low degree. For this see [Đoković and Doniz 2000] and the references quoted there. That paper describes all indecomposable complex representations in degrees less than six. In this note we describe the computational results that we have obtained for 6-dimensional irreducible complex representations of Φ_2.

We start with the natural short exact sequence

$$1 \to F_2 \to \Phi_2 \to \mathrm{GL}_2(\mathbb{Z}) \to 1,$$

where F_2 is viewed as the group of inner automorphisms of itself, and $\mathrm{GL}_2(\mathbb{Z})$ is identified with the automorphism group of the free abelian group F_2/F_2' of rank 2 (G' denotes the derived subgroup of a group G).

Remarks. 1. An anonymous referee pointed out that the above sequence does not split [Potapchik and Rapinchuk 2000, Corollary 2.4]. On the other hand it almost splits in the sense that there is a section whose domain is a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ of index 8. For this see [Đoković 1983].

2. Under the canonical embedding $F_2 \to \Phi_2$, we have

$$x = (\sigma P \varepsilon \sigma P)^2,$$
$$y = (P \sigma P \varepsilon \sigma)^2.$$

3. Let F_n be the free group on n generators $x_1, \ldots, x_n$, and let Ω_n be the subgroup of $\Phi_n = \mathrm{Aut}(F_n)$ which leaves invariant the set

$$\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}.$$

Ω_n, a finite group of order $2^n\, n!$, is called the extended symmetric group in [Magnus et al. 1966, p. 163]. In the case $n = 2$, we have $\Omega_2 = D_4 = \langle P, \sigma \rangle$, and there is a retraction $\Phi_2 \to \Omega_2$ given by $P \to P$, $\sigma \to \sigma$, and $\varepsilon \to P$. Consequently the representations of D_4 can be regarded as representations of Φ_2.

4. E. Formanek and C. Procesi [1992] have shown that for $n > 2$ the groups Φ_n are not linear.

Magnus and Tretkoff [1979] have shown that if ρ is a faithful finite-dimensional complex linear representation of Φ_2, then at least one irreducible constituent of ρ is faithful. This explains why we are interested in irreducible representations only (the other reason being that this simplifies the classification problem considerably).

We introduce another restriction: we assume that our representations ρ of Φ_2 do not factor through the natural homomorphism $\Phi_2 \to \mathrm{GL}_2(\mathbb{Z})$, i.e., that $\rho(F_2)$ is not the trivial group. Of course, this restriction is harmless in regard to the above linearity problem.

Another important concept in our classification is that of weak equivalence, which was introduced in [Đoković and Platonov 1996]. The abelianization Φ_2/Φ_2' is a four-group. Hence Φ_2 has, up to equivalence, exactly four one-dimensional representations. They are given in Table 1, where χ_1 is the trivial representation.

	P	σ	ε
χ_1	1	1	1
χ_2	1	−1	1
χ_3	−1	1	−1
χ_4	−1	−1	−1

TABLE 1. One-dimensional representations of Φ_2.

We say that two representations ρ and ρ' of Φ_2 are *weakly equivalent* if ρ' is equivalent to a tensor products $\rho \otimes \chi_i$ or its dual. This is clearly an equivalence relation. Each weak equivalence class consists of $1, 2, 4$, or 8 ordinary equivalence classes.

Our main result is the list of the representatives of the weak equivalence classes of irreducible six-dimensional complex representations ρ of Φ_2 such that $\rho(F_2) \neq 1$. The list is given in Section 4. It consists of nine 1-parameter families and a single 2-parameter family of representations (the last family in our list). The list is somewhat redundant as two of the 1-parameter families can be embedded (apart from a few exceptional values of the parameter) into the 2-parameter family. One of the 1-parameter families (the fourth family) is weakly equivalent to the Bürau representation of Φ_2. It is easy to verify that none of the other families can contain a faithful representation of Φ_2, a fact first proved by Tenekedzhi [1986, Theorem 2].

2. TYPES OF REPRESENTATIONS

If ρ is a representation of Φ_2 we define its *type* to be the equivalence class of the restriction of ρ to the dihedral subgroup $D_4 = \langle P, \sigma \rangle$. As in [Đoković and Doniz 2000], we denote by A, B, C, and D the restrictions to D_4 of the 1-dimensional representations χ_1, χ_2, χ_3, and χ_4, respectively. By E we denote the irreducible 2-dimensional representation of D_4. The symbol ADEE denotes the type which is a direct sum of the 1-dimensional representations A and D and two copies of the 2-dimensional representation E. Similar notation will be used for other types.

A representation ρ and its dual always have the same type. On the other hand, ρ and $\rho \otimes \chi_i$ may be of different types. For instance, if ρ has the type BCEE, then $\rho \otimes \chi_2$ has the type ADEE. It follows from [Đoković and Doniz 2000, Proposition 1] that two weakly equivalent representations are either both faithful or both not faithful.

1	AAEE	6	AAABE	11	AADDE
2	ABEE	7	AAACE	12	AABCE
3	ACEE	8	AAADE	13	AABDE
4	ADEE	9	AABBE	14	AACDE
5	AAAAE	10	AACCE	15	ABCDE

TABLE 2. Types up to weak equivalence.

Lemma 2.1. *Every 6-dimensional representation of Φ_2 that is nontrivial on F_2 is weakly equivalent to a representation having one of the* 15 *types listed in Table* 2. *Two 6-dimensional representations of Φ_2 whose types are different and belong to Table* 2 *are not weakly equivalent.*

Proof. Let ρ be a 6-dimensional representation of Φ_2 that is nontrivial on F_2. Its type must contain E as a constituent; otherwise $\rho((P\sigma)^2) = 1$. Its type cannot be EEE because in that case $\rho((P\sigma)^2)$ is a central involution of $\rho(\Phi_2)$. (By Remark 2, $y = [(P\sigma)^2, \sigma\varepsilon\sigma]$.) Hence the type of ρ contains either one or two E's. Assume that it contains two E's. Then by replacing ρ with a suitable tensor product $\rho \otimes \chi_i$, we may assume that its type contains at least one A. Hence the type of ρ is now one of the first four types in Table 2. Next assume that the type of ρ contains a single E. By tensoring with a suitable χ_i, we may further assume that the multiplicity of A exceeds or is equal to the multiplicity of each of the representations B, C, D. This gives the remaining eleven possibilities in Table 2. It is easy to verify the last assertion of the lemma. □

3. BÜRAU REPRESENTATIONS OF Φ_2

This section introduces the 1-parameter family of irreducible 6-dimensional representations of Φ_2 that we call Bürau representations of Φ_2. It is based entirely on [Dyer et al. 1982].

Recall that the braid group B_4 has the presentation

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 : [\sigma_1, \sigma_3] = 1, \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$

The reduced Bürau representation of B_4 is defined by

$$\sigma_1 \to \begin{bmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sigma_2 \to \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\sigma_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & -t \end{bmatrix}.$$

We shall view this representation as a 1-parameter family of complex representations of B_4 with nonzero complex parameter t. This representation is irreducible if and only if $(1+t)(1+t^2) \neq 0$ [Formanek 1996]. If t is a root of unity, this representation is clearly not faithful. On the other hand, if t is a transcendental number, then it is not known whether or not this representation is faithful.

The center Z_4 of B_4 is infinite cyclic with the generator $\zeta = (\sigma_1\sigma_2\sigma_3)^4$. In the above representation we have $\zeta \to t^4 I_3$, where I_3 is the identity matrix. We now modify this representation by following the recipe from [Dyer et al. 1982] to obtain the representation

$$\sigma_1 \to \frac{1}{t}\begin{bmatrix} -t^3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sigma_2 \to \frac{1}{t}\begin{bmatrix} 1 & 0 & 0 \\ t^3 & -t^3 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\sigma_3 \to \frac{1}{t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t^3 & -t^3 \end{bmatrix},$$

which maps ζ to I_3. Hence this is in fact a representation of the quotient group $B_4^* = B_4/Z_4$. This representation is irreducible if and only if $(1+t^3)\times(1+t^6)\neq 0$.

Let Φ_2^+ denote the subgroup of Φ_2 of index 2 which is the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the natural map $\Phi_2 \to \mathrm{GL}_2(\mathbb{Z})$. One can easily check that $\Phi_2^+ = \langle P\sigma, P\varepsilon\rangle = \langle P\sigma, U\rangle$. The next lemma is taken from [Dyer et al. 1982].

Lemma 3.1. *There is a short exact sequence*

$$1 \to Z_4 \to B_4 \to \Phi_2^+ \to 1,$$

where the homomorphism $B_4 \to \Phi_2^+$ sends

$$\sigma_1 \to PU^{-1}P, \quad \sigma_2 \to PU\sigma U^{-1}, \quad \sigma_3 \to P\sigma U\sigma P.$$

Thus Φ_2^+ is isomorphic to B_4^*. Consequently the above representation of B_4^* can be viewed as a representation of Φ_2^+. By inducing, we obtain a 1-parameter family of irreducible 6-dimensional representation of Φ_2 of type BCEE. We refer to the representations belonging to this 1-parameter family as the *Bürau representations* of Φ_2. By tensoring with χ_2, we obtain a 1-parameter family of representations of type ADEE. This last family is given by

$$P \to \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix},$$

$$\sigma \to \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & t^{-3} \\ 0 & 0 & 0 & 0 & -t^3 & 1 \\ 0 & 1 & -t^{-3} & 0 & 0 & 0 \\ -1 & 1 & -t^{-3} & 0 & 0 & 0 \\ -t^3 & t^3 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\varepsilon \to \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -t^{-1} \\ 0 & 0 & 0 & t^{-1} & 0 & -t^{-1} \\ 0 & 0 & 0 & t^2 & -t^2 & 0 \\ -t & t & 0 & 0 & 0 & 0 \\ -t & t & -t^{-2} & 0 & 0 & 0 \\ -t & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $t \neq 0$ is a complex parameter. (To obtain the Bürau representations, one has just to replace the matrix corresponding to σ by its negative.) This representation and the Bürau representations of Φ_2 are irreducible if and only if

$$(1+t^3)(1+t^6)\neq 0.$$

Remark. The generators $u = P\sigma$ and $v = \varepsilon P$ of Φ_2^+ satisfy the relations

$$u^4 = v^3 = [u^2, vuv] = 1.$$

In fact this is a presentation of Φ_2^+, which follows from the above lemma and a result of D. Cooper and D. D. Long [1997, Corollary 5.3].

4. LIST OF THE IRREDUCIBLE REPRESENTATIONS

For each of the 15 types listed in Table 2, we have carried out an exhaustive search for irreducible representations of Φ_2. (See the next two sections for more details.) Such representations exist only for the five types ABEE, ACEE, ADEE, AACCE, and ABCDE.

Fix the following matrix form of the representation E of D_4:

$$E:\ P \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma \to \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 1-dimensional representations of D_4 are

$$\begin{aligned} A &: P \to 1, & \sigma &\to 1, \\ B &: P \to 1, & \sigma &\to -1, \\ C &: P \to -1, & \sigma &\to 1, \\ D &: P \to -1, & \sigma &\to -1. \end{aligned}$$

When we say that a representation ρ of Φ_2 has a certain type, then we shall assume in this section that $\rho(P)$ and $\rho(\sigma)$ are direct sums (in the order specified by the type) of the matrix representations listed above. For example, if ρ has the type ADEE, then

$$\rho(P) = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix},$$

$$\rho(\sigma) = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & 0 & & \\ & & 0 & 1 & & \\ & & & & -1 & 0 \\ & & & & 0 & 1 \end{bmatrix}.$$

Consequently ρ is uniquely determined by its type and the matrix $\rho(\varepsilon)$.

The dual representation, $\rho^\vee$, of ρ is given by:

$$\rho^\vee(P) = \rho(P),\ \rho^\vee(\sigma) = \rho(\sigma),\ \rho^\vee(\varepsilon) = {}^t\rho(\varepsilon).$$

(By ${}^tX$ we denote the transpose of a matrix X.) In particular ρ and $\rho^\vee$ have the same type. If ρ and $\rho^\vee$ are equivalent, we say that ρ is *self-dual.*

By $v_1, \ldots, v_6$ we denote the standard basis vectors of the column space $\mathbb{C}^6$.

Main Theorem. *Each 6-dimensional irreducible representation of Φ_2 that is nontrivial on F_2 is weakly equivalent to a representation belonging to one of the families listed below.*

First family: Type ABEE; parameter $s \neq 0, 1$;

$$\varepsilon \to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1-1/s & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1-s & s & 0 & 0 & 0 & 0 \\ s & -s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\rho(F_2)$ is a four-group. If $s = 1$ the subspace $\langle v_1, v_5, v_6 \rangle$ is invariant. Each representation in this family is self-dual.

Second family: Type ACEE; parameter $s \neq 0, 1$;

$$\varepsilon \to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1-1/s & 0 \\ 0 & 0 & 1-2s & 0 & 0 & 2-2s \\ 1-s & s & 0 & 0 & 0 & 0 \\ s & -s & 0 & 0 & 0 & 0 \\ 0 & 0 & 2s & 0 & 0 & 2s-1 \end{bmatrix}.$$

$\rho(F_2)$ is a four-group. If $s = 1$ the subspace $\langle v_1, v_5, v_6 \rangle$ is invariant. Each representation in this family is self-dual.

Third family: Type ADEE; parameter $s \neq 0, 1$;

$$\varepsilon \to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1-1/s & 0 \\ 0 & 0 & 1-2s & 0 & 0 & 2-2s \\ 1-s & s & 0 & 0 & 0 & 0 \\ s & -s & 0 & 0 & 0 & 0 \\ 0 & 0 & 2s & 0 & 0 & 2s-1 \end{bmatrix}.$$

$\rho(F_2)$ is a four-group. If $s = 1$ the subspace $\langle v_1, v_5, v_6 \rangle$ is invariant. Each representation in this family is self-dual.

Fourth family: Type ADEE; parameter $s \neq 0, 1, -\frac{1}{3}$;

$$\varepsilon \to \frac{1}{(1-s)^2} \times \begin{bmatrix} 3s+s^2 & -s-3 & 0 & 3s+1 & 3s+1 & 0 \\ 3s+s^2 & -3s-s^2 & 0 & 3s+1 & 3s^2+s & 0 \\ -3s-s^2 & 3s+s^2 & s^2-1 & -3s-s^2 & -3s-s^2 & 2s^2-2s \\ -3s^2-s & 3s+1 & 0 & -3s-s^2 & -3s-s^2 & 0 \\ 3s+1 & -3s-1 & 0 & s+3 & 3s+s^2 & 0 \\ -3s-s^2 & s+3 & 2-2s & -s-3 & -3s-s^2 & 1-s^2 \end{bmatrix}.$$

This family is equivalent to the representation displayed at the end of Section 3. More precisely, if t is a nonzero complex number and

$$(1-t)(1+t^3)(1+t^6) \neq 0,$$

then by setting

$$s = \left(\frac{1-t}{1+t}\right)^2,$$

we obtain a representation which is equivalent to the representation in Section 3 mentioned above. Note that t and $1/t$ correspond to the same value of s. It is not known whether or not this family contains a faithful representation of Φ_2.

The matrix that transforms the representation in Section 3 to the above one is given at the top of the next page. Its determinant is

$$-32t^6\,\frac{(1-t^2)^3(1+t^6)^4}{(1+t^3)^2}.$$

If $s = 0$, the subspace $\langle v_1, v_5, v_6 \rangle$ is invariant. If $s = -\frac{1}{3}$ the subspace $\langle v_3, v_4, v_5, v_6 \rangle$ is invariant. If $s = -3$ the representation above is equivalent to the representation of the third family for $s = \frac{1}{4}$.

Fifth family: Type AACCE; parameter $s \neq \pm 1$;

$$\varepsilon \to \frac{1}{2} \begin{bmatrix} 0 & 1+s & 1 & 0 & 2 & 0 \\ 1 & s & 0 & 1 & 0 & 2 \\ 1+s & -s-s^2 & 2s & 1+s & 0 & -2-2s \\ -s & s^2-s+1 & 1-2s & -s & -2 & 2s \\ 1-s & 0 & -s & -1-s & 0 & 0 \\ 0 & 1-s^2 & s-1 & 0 & 0 & -2s \end{bmatrix}.$$

The commutator $[x^2, y^2]$ lies in the kernel of this representation. If $s = 1$ the subspace $\langle v_1 + v_2 \rangle$ is invariant. If $s = -1$ the subspace $\langle v_1, v_2, v_4, v_5, v_6 \rangle$ is invariant.

$$\begin{bmatrix} \dfrac{(1+t^6)(1-t)}{1+t^3} & \dfrac{-(1+t^6)}{1-t+t^2} & t^3(1+t) & 1+t & 1-t & t^3(t-1) \\ 0 & 0 & (1+t)(t^3-1) & (1+t)(1+t^3) & (1-t)(1+t^3) & (1-t)(1-t^3) \\ \dfrac{t^3(1+t^6)(t-1)}{1+t^3} & \dfrac{t^3(1+t^6)}{1-t+t^2} & t^6(1+t) & (1+t)t^3 & t^3-t^4 & t^6(t-1) \\ \dfrac{(1+t^6)(1-t)}{1+t^3} & \dfrac{1+t^6}{1-t+t^2} & 1+t & t^3(1+t) & t^3(t-1) & 1-t \\ 0 & 0 & (1+t)(1+t^3) & (1+t)(t^3-1) & (1-t)(1-t^3) & (1-t)(1+t^3) \\ \dfrac{t^3(1+t^6)(t-1)}{1+t^3} & \dfrac{-t^3(1+t^6)}{1-t+t^2} & t^3(1+t) & t^6(1+t) & t^6(t-1) & t^3(1-t) \end{bmatrix}$$

Matrix transforming the representations of Φ_2 of type ADEE given in Section 3 to those of the fourth family.

Sixth family: Type ABCDE; parameter $s \neq \pm 1$;

$$\varepsilon \to \frac{1}{2}\begin{bmatrix} -s & s-1 & -s & -1 & 0 & 2s-2 \\ -1 & 0 & 1 & 0 & 2 & 0 \\ -s & -s+1 & -s & 1 & 0 & 2s-2 \\ -s-1 & 0 & s+1 & 0 & 2s-2 & 0 \\ 0 & s+1 & 0 & -1 & 0 & 0 \\ -s-1 & 0 & -s-1 & 0 & 0 & 2s \end{bmatrix}.$$

The commutator $[x^2, y^2]$ lies in the kernel of this representation. If $s = 1$ the subspace $\langle v_2, v_5, v_6\rangle$ is invariant. If $s = -1$ the subspace $\langle v_1, v_2, v_3\rangle$ is invariant.

Seventh family: Type ABCDE; parameter $s \neq \pm 1$;

$$\varepsilon \to \frac{1}{2}\begin{bmatrix} 0 & s-1 & 0 & s-1 & 0 & 2 \\ -1 & -s & 1 & s & -2 & 0 \\ 0 & s+1 & 0 & s+1 & 0 & 2 \\ -1 & s & 1 & -s & 2 & 0 \\ 0 & s^2-1 & 0 & 1-s^2 & 2s & 0 \\ s+1 & 0 & 1-s & 0 & 0 & 0 \end{bmatrix}.$$

The commutator $[x^2, y^2]$ lies in the kernel of this representation. If $s = 1$ the subspace $\langle v_2, v_3, v_4\rangle$ is invariant. If $s = -1$ the subspace $\langle v_1, v_2, v_4\rangle$ is invariant.

Eighth family: Type ABCDE; parameter s (no restrictions);

$$\varepsilon \to \frac{1}{2}\begin{bmatrix} s & s^2-1 & -s & s^2-1 & 2s & 2 \\ -1 & -s & 1 & -s & -2 & 0 \\ -s & 1-s^2 & s & 1-s^2 & 2s & 2 \\ -1 & -s & 1 & -s & 2 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & s & 1 & -s & 0 & 0 \end{bmatrix}.$$

$\rho(F_2)$ is a four-group.

Ninth family: Type ABCDE; parameter s (no restrictions);

$$\varepsilon \to \frac{1}{2}\begin{bmatrix} s & 1 & -s & -1 & 2 & 0 \\ 1-s^2 & -s & s^2-1 & s & -2s & 2 \\ -s & -1 & s & 1 & 2 & 0 \\ s^2-1 & s & 1-s^2 & -s & -2s & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ s & 1 & s & 1 & 0 & 0 \end{bmatrix}.$$

$\rho(F_2)$ is a four-group.

Tenth family: Type ABCDE; parameters u and v such that $u + v \neq \pm\frac{1}{2}$; setting $w = 4(u+v)^2 - 1$,

$$\varepsilon \to \begin{bmatrix} -v & \frac{1}{4}-v^2 & v & v^2-\frac{1}{4} & \frac{1}{2}-2v^2-2uv & u \\ 1 & v & -1 & -v & 2(u+v) & 1 \\ v & v^2-\frac{1}{4} & -v & \frac{1}{4}-v^2 & \frac{1}{2}-2v^2-2uv & u \\ -1 & -v & 1 & v & 2(u+v) & 1 \\ -\frac{1}{w} & \frac{u}{w} & -\frac{1}{w} & \frac{u}{w} & 0 & 0 \\ \frac{2(u+v)}{w} & \frac{1}{2}-\frac{2u^2}{w} & \frac{2(u+v)}{w} & \frac{1}{2}-2\frac{u^2}{w} & 0 & 0 \end{bmatrix}.$$

$\rho(F_2)$ is a four-group. For $u = 0$ and $v = -s/2$ we obtain (up to equivalence) all representations of the ninth family except those when $s = \pm 1$. For $u = (s^2-1)/2s$ and $v = -s/2$ we obtain (up to equivalence) all representations of the eighth family except those when $s = 0, \pm 1$.

The following result, first proved by Tenekedzhi [1986], is an immediate consequence of the Main Theorem (page 461) and the results presented in the previous sections.

Corollary 4.1. *If ρ is a faithful complex representation of Φ_2 of degree at most 6, then ρ has degree 6, it is irreducible and weakly equivalent to a Bürau representation of Φ_2.*

5. COMMENTS ON THE PROOF OF THE MAIN THEOREM

The method that we used to compile the list of irreducible representations is essentially the same as the one described in [Đoković and Doniz 2000]. A minor difference is that the generator τ which was used in that paper has been replaced by ε. These generators are closely related since $\tau = \sigma\varepsilon\sigma$. If we fix the type of the representation ρ, say we take the type ADEE, then the matrices $\rho(P)$ and $\rho(\sigma)$ are known (they are displayed in the beginning of the previous section). It remains to find the possible matrices ε. Among the defining relations of Φ_2, only three of them involve ε. We write these three relations in the form

$$\varepsilon^2 = 1, \quad P\varepsilon P = \varepsilon P\varepsilon, \quad \varepsilon\sigma\varepsilon P\sigma P\sigma = P\sigma P\sigma\varepsilon\sigma\varepsilon.$$

For simplicity, we shall write just P, σ, ε instead of $\rho(P)$, $\rho(\sigma)$, and $\rho(\varepsilon)$, respectively. The resulting scalar equations are polynomial equations of degree 2 in the 36 indeterminates, the entries of the matrix ε. The brute force attempt to solve the resulting system of polynomial equations fails and one has to develop a strategy to simplify the computations.

Let Z denote the centralizer of the matrices P and σ in $\mathrm{GL}_6(\mathbb{C})$. If $z \in Z$ the representations defined by ε and $z^{-1}\varepsilon z$ are equivalent. This fact is used to simplify the matrix ε. The simplifications consist in assuming that certain entries of ε are 0, some nonzero entries are normalized, say set equal to 1, or some entries are expressed in terms of other entries. In most cases this allows us to reduce the number of variables. Another device that we employ is to occasionally replace the matrix ε by its transpose (which amounts to replacing the representation ρ by its dual $\rho^\vee$).

Our main computational tool was Maple, and in particular its *groebner* package. By using it, we were able to handle all the types except the very last type of Table 2, namely ABCDE. The main reason for this difficulty is of course the fact that the above mentioned centralizer Z, in this case, is of very low dimension: it consists of diagonal invertible matrices with the last two diagonal entries equal. This means that the only kind of simplification that is available to us is the normalization of certain entries of ε.

This last case was very challenging, and we had to use Singular [Greuel et al. 1998] in order to handle it. Several first attempts to find a Gröbner basis for the ideal generated by our equations failed because of insufficient memory. The subcases where at least one of the entries of ε located at the positions $(i, 6)$ with $1 \le i \le 4$ is 0, were not hard to resolve. The hard case, when all these four entries of ε are nonzero, could be normalized by setting all of them equal to 1. Still Singular was complaining of not having enough memory. Final modification, that led to success, was to introduce four more variables (the normalization had reduced the number of variables by four) and use them to force the first four entries of the last row of ε to be nonzero. This we could do because otherwise our representation ρ would be equivalent to the dual of a representation already found in the previous cases.

The computations were done on a Sun SPARCstation 10 with 96 MB of memory. The CPU time usage was relatively low; the main bottleneck was insufficient memory.

6. A TYPICAL COMPUTATION

In this section we give the details of the proof that Φ_2 has no irreducible 6-dimensional representations of type AAEE. We write $\rho(\varepsilon)$ in the form

$$\rho(\varepsilon) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{bmatrix}.$$

The centralizer Z of $\rho(P)$ and $\rho(\sigma)$ in $\mathrm{GL}_6(\mathbb{C})$ consists of all invertible matrices z having the form:

$$z = \begin{bmatrix} a & b & & & & \\ c & d & & & & \\ & & e & & f & \\ & & & e & & f \\ & & g & & h & \\ & & & g & & h \end{bmatrix}.$$

Thus Z is isomorphic to $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, and, for brevity, we shall write this matrix also as $z = (z_1, z_2)$ where

$$z_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad z_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

The conjugation $\rho(\varepsilon) \to z^{-1}\rho(\varepsilon)z$ transforms the submatrix of $\rho(\varepsilon)$ in the intersection of the first two rows and the fourth and sixth columns as follows

$$\begin{bmatrix} a_4 & a_6 \\ b_4 & b_6 \end{bmatrix} \to z_1^{-1} \begin{bmatrix} a_4 & a_6 \\ b_4 & b_6 \end{bmatrix} z_2.$$

Consequently, we may assume that this submatrix is one of the three matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6–1}$$

Consider the first case, where $a_4 = b_6 = 1$ and $a_6 = b_4 = 0$. In this case we can make one further simplification. Namely by taking z with $z_1 = z_2$ and by conjugating $\rho(\varepsilon)$ with such a z, the above mentioned submatrix will not be altered, but the 2 by 2 block in the upper left hand corner will undergo a similarity transformation by z_1. Consequently we may also assume that $a_2 = 0$.

The three defining relations of Φ_2 that involve ε produce a system of 88 polynomial equations. Let I denote the ideal generated by these 88 polynomials. We used Singular to find a Gröbner basis of I (this was accomplished on our workstation in 55 seconds). It consists of 31 polynomials. The fact that this ideal is not the unit ideal means that the representations of type AAEE indeed exist. By inspecting this Gröbner basis we deduce that the variables a_5, c_2, c_5, c_6, d_2, d_3, d_5, d_6, e_1, f_3, and f_5 belong to I. (For the variables c_2, c_5, and c_6 we had to use the function *reduce* to check this claim.) This means that these entries in $\rho(\varepsilon)$ have to be 0. Consequently the subspace of $\mathbb{C}^6$ spanned by the standard basis vectors v_2, v_5, and v_6 is Φ_2-invariant, i.e., the representation is reducible.

Next we consider the second case, where $a_4 = 1$ and $a_6 = b_4 = b_6 = 0$. Now Singular spent 271 seconds in computing a Gröbner basis of the ideal I. In this case the basis consists of 178 polynomials. Again this means that the representations, with the above four entries specified, exist. By examining this basis we conclude that the entries b_1, c_1, c_2, c_5, d_3, d_5, d_6, and e_1 must vanish. If the entries b_3 and b_5 are both 0, then the subspace spanned by all v_i's with $i \neq 2$ is Φ_2-invariant. Hence, we may assume that b_3 or b_5 is not 0. On the other hand we find that the products $b_3 f_5$, $b_5 f_5$, $b_3 e_6$, and $b_5 e_6$ are all in I. Hence these products must vanish, and we deduce that $f_5 = e_6 = 0$. We now encode this new information in our matrix $\rho(\varepsilon)$ and rerun the Singular *groebner* command. We now find that b_2, c_5, e_5, and f_3 vanish. Furthermore we obtain that $f_6 = 1$ and $d_4 = -a_1$. We again update the matrix $\rho(\varepsilon)$ and rerun the *groebner* command. This time we find that $c_3 = 1$ and $b_5 e_2 = 1$. By conjugating $\rho(\varepsilon)$ by a suitable diagonal matrix $z \in Z$, we may additionally assume that $b_5 = e_2 = 1$. Now the *groebner* command gives the relation $d_2 = -d_1 a_5$. We can now deduce that the subspace spanned by the vectors $a_5 v_1 + v_2$, v_5, and v_6 is Φ_2-invariant. Hence we are done with second case.

Finally we consider the third case, where $a_4 = a_6 = b_4 = b_6 = 0$. If at least one of the entries d_1, d_2, f_1, f_2 is not 0, then the dual representation belongs to one of the two previous cases. As we are carrying out the classification only up to weak equivalence, we may assume that $d_1 = d_2 = f_1 = f_2 = 0$. The 2 by 2 submatrix of $\rho(\varepsilon)$ in the intersection of the first two rows and the third and fifth columns can be transformed into one of the three matrices of (6–1). In fact the zero matrix can be ruled out because in that subcase the subspace spanned by v_3, v_4, v_5, and v_6 is Φ_2-invariant. The first two subcases can be treated jointly by using only the facts that $a_3 = 1$ and $a_5 = b_3 = 0$. The fact that b_5 may be assumed to be 0 or 1 will be used only later. We insert this information into our matrix $\rho(\varepsilon)$. Then the ideal I has 80 generators, and the *groebner* routine produces quickly a Gröbner basis of I consisting of 56 polynomials. We now find that the variables b_1, c_4, c_6, d_3, d_5, d_6, e_4, e_6, f_4 and f_5 must vanish. We also find that $b_5 f_5$, $b_5 f_4$, $b_5 e_6$, and $b_5 e_4$ belong to I. If $b_5 = 0$ the subspace spanned by the v_i's with $i \neq 2$ is Φ_2-invariant. Hence we may assume that $b_5 \neq 0$, and consequently we must have $f_5 = f_4 = e_6 = e_4 = 0$. We also obtain that $c_1 = d_4 = 1$. Once again we invoke the *groebner* command on the updated matrix $\rho(\varepsilon)$. We obtain that several other entries must vanish, in particular the entries a_2, c_2, and c_5. Therefore the subspace spanned by v_2, v_5, and v_6 is Φ_2-invariant.

ACKNOWLEDGEMENTS

We are grateful to a referee who informed us that the sequence $1 \to F_2 \to \Phi_2 \to \mathrm{GL}_2(\mathbb{Z}) \to 1$ does not split (see Remark 1 on page 458) and that Corollary 4.1 had been proved by Tenekedzhi.

NOTE ADDED IN PROOF

From [Zinno 2000] we learned that Daan Krammer has shown that B_4 is a linear group, and that this has been extended to all braid groups B_n by Stephen Bigelow. Consequently, Φ_2 is a linear group [Dyer et al. 1982].

J. Moody [1993] has shown that the Bürau representation of B_n is not faithful for $n \geq 10$. This was improved to $n \geq 6$ by D. D. Long and M. Paton [1993], and recently Bigelow [1999] proved the same assertion for $n = 5$. The case $n = 4$ is apparently still open.

REFERENCES

[Bigelow 1999] S. Bigelow, "The Burau representation is not faithful for $n = 5$", *Geom. Topol.* **3** (1999), 397–404.

[Cooper and Long 1997] D. Cooper and D. D. Long, "A presentation for the image of Burau(4) $\otimes \mathbb{Z}_2$", *Invent. Math.* **127**:3 (1997), 535–570.

[Đoković 1983] D. Ž. Đoković, "The structure of the automorphism group of a free group on two generators", *Proc. Amer. Math. Soc.* **88**:2 (1983), 218–220.

[Đoković and Doniz 2000] D. Ž. Đoković and D. Doniz, "The complex indecomposable representations of Aut(F_2) of degree less than six", *Internat. J. Algebra Comput.* **10**:2 (2000), 229–260.

[Đoković and Platonov 1996] D. Ž. Đoković and V. P. Platonov, "Low-dimensional representations of Aut(F_2)", *Manuscripta Math.* **89**:4 (1996), 475–509.

[Dyer et al. 1982] J. L. Dyer, E. Formanek, and E. K. Grossman, "On the linearity of automorphism groups of free groups", *Arch. Math.* (*Basel*) **38**:5 (1982), 404–409.

[Formanek 1996] E. Formanek, "Braid group representations of low degree", *Proc. London Math. Soc.* (3) **73**:2 (1996), 279–322.

[Formanek and Procesi 1992] E. Formanek and C. Procesi, "The automorphism group of a free group is not linear", *J. Algebra* **149**:2 (1992), 494–499.

[Greuel et al. 1998] G.-M. Greuel, G. Pfister, and H. Schoenemann, "Singular version 1.2 User Manual", Technical Report 21, Centre for Computer Algebra, University of Kaiserslautern, 1998. See http://www.mathematik.uni-kl.de/~zca/Singular.

[Long and Paton 1993] D. D. Long and M. Paton, "The Burau representation is not faithful for $n \geq 6$", *Topology* **32**:2 (1993), 439–447.

[Magnus and Tretkoff 1979] W. Magnus and C. Tretkoff, "Representations of automorphism groups of free groups", pp. 255–259 in *Word problems, II: The Oxford book* (Oxford, 1976), edited by S. I. Adian et al., Studies in Logic and the Foundations of Math. **95**, North-Holland, Amsterdam, 1979.

[Magnus et al. 1966] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience, New York, 1966.

[Moody 1993] J. Moody, "The faithfulness question for the Burau representation", *Proc. Amer. Math. Soc.* **119**:2 (1993), 671–679.

[Potapchik and Rapinchuk 2000] A. Potapchik and A. Rapinchuk, "Low-dimensional linear representations of Aut $\mathbb{F}_n$, $n \geq 3$", *Trans. Amer. Math. Soc.* **352**:3 (2000), 1437–1451.

[Tenekedzhi 1986] V. V. Tenekedzhi, "Matrix representations of degree 6 of automorphism groups of free groups", pp. 109–125 in *Nekotorye problemy differentsial'nykh uravneniĭ i diskretnoĭ matematiki*, edited by M. M. Lavrent'ev, Novosibirsk. Gos. Univ., Novosibirsk, 1986. In Russian.

[Zinno 2000] M. G. Zinno, "Krammer's representation of the braid group", arXiv e-print math/0002136, 2000. See http://arXiv.org/abs/math/.

Dragomir Ž. Đoković, Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1 (djokovic@uwaterloo.ca)

Received June 28, 1999; accepted in revised form January 14, 2000