Real Schubert Calculus: Polynomial Systems and a Conjecture of Shapiro and Shapiro

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CONTENTS

Boris and Michael Shapiro have a conjecture concerning the Schubert calculus and real enumerative geometry and which would give infinitely many families of zero-dimensional systems of real polynomials (including families of overdetermined systems) — all of whose solutions are real. It has connections to the pole placement problem in linear systems theory and to totally positive matrices. We give compelling computational evidence for its validity, prove it for infinitely many families of enumerative problems, show how a simple version implies more general versions, and present a counterexample to a general version of their conjecture.

1. INTRODUCTION

The determination of the number of real solutions to a system of polynomial equations is a challenging problem in symbolic and numeric computation [Gonzalez-Vega et al. 1999; Sturmfels 1994; 1998] with real world applications [Dietmaier 1998]. Related questions include when a problem of enumerative geometry can have all solutions real [Sottile 1997a] and when may a given physical system be controlled by real output feedback [Byrnes 1989; Rosenthal et al. 1995; Syrmos et al. 1997]. In May 1995, Boris Shapiro and Michael Shapiro communicated to the author a remarkable conjecture connecting these three lines of inquiry.

They conjectured a relation between topological invariants of the real and of the complex points in an intersection of Schubert cells in a flag manifold, if the cells are chosen according to a recipe they give. When the intersection is zero-dimensional, this asserts that all points are real. Their conjecture is false — we give full description and present a counterexample in Section 5. However, there is considerable evidence for their conjecture if the Schubert cells are in a Grassmann manifold. It is this variant

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which is related to the lines of inquiry above and which this paper is about.

Here is the simplest (but still very interesting and open) special case of this conjecture: Let $m, p > 1$ be integers and let X be a $p \times m$-matrix of indeterminates. Let $K(s)$ be the $m \times (m+p)$-matrix of polynomials in s whose (i,j)-th entry is

$$\binom{j-i}{i-1} s^{j-i}. \tag{1–1}$$

Set

$$\varphi_{m,p}(s;X) := \det \begin{bmatrix} K(s) \\ I_p \quad X \end{bmatrix},$$

where I_p is the $p \times p$ identity matrix.

Conjecture 1.1 (Shapiro and Shapiro). *For all integers $m, p > 1$, the polynomial system*

$$\varphi_{m,p}(1;X) = \varphi_{m,p}(2;X) = \cdots = \varphi_{m,p}(mp;X) = 0 \tag{1–2}$$

is zero-dimensional with

$$d_{m,p} := \frac{1!\,2!\,3!\cdots(p-2)!\,(p-1)!\cdot(mp)!}{m!\,(m+1)!\,(m+2)!\cdots(m+p-1)!} \tag{1–3}$$

solutions, and all of them are real.

(After the acceptance of this paper, A. Eremenko and A. Gabrielov [1999] announced a proof of this conjecture when either m or p is 2.)

It is a Theorem of Schubert [1886] that $d_{m,p}$ is a sharp bound for the number of isolated solutions. Conjecture 1.1 has been verified for all $1 < m \leq p$ with $mp \leq 12$. The case of $(m,p) = (3,4)$ (when $d_{m,p} = 462$) follows from a heroic calculation of Faugère, Rouillier, and Zimmermann [Faugère et al. 1998]; see Section 2D for a discussion.

Conjecture 1.1 is related to a question of Fulton [Fulton 1996, § 7.2], who asked how many solutions to a problem of enumerative geometry may be real, where that problem consists of counting figures of some kind having a given position with respect to some given (fixed) figures. For 2-planes having a given position with respect to fixed linear subspaces, the answer is that all may be real [Sottile 1997b]. This was also shown for the problem of 3264 plane conics tangent to five given conics [Ronga et al. 1997]. More examples, including that of 3-planes in $\mathbb{C}^6$ meeting 9 given 3-planes nontrivially, are found in [Sottile 1997a; 1997c]. The result in [Faugère et al. 1998] extends this to 3-planes in $\mathbb{C}^7$ meeting 12 given 4-planes nontrivially.

Only the simplest form of the conjecture of Shapiro and Shapiro has appeared in print [Huber et al. 1998; Rosenthal and Sottile 1998; Sottile 1997a]. While more general forms have circulated informally, there is no definitive source describing the conjectures or the compelling evidence that has accumulated (or a counterexample to the original conjecture). The primary aim of this paper is to rectify this situation and make these conjectures available to a wider audience.

Structure of the Article

In Section 2, we describe a version of the conjecture related to the pole placement problem of linear systems theory. For this, the integers $1, 2, \ldots, mp$ in the polynomial system (1–2) of Conjecture 1.1 are replaced by generic real numbers and all $d_{m,p}$ solutions are asserted to be real. We present evidence (computational and Theorems) in support of it. Subsequent sections describe the conjecture in greater generality — for enumerative problems arising from the Schubert calculus on Grassmannians in Section 3 and a newer extension involving totally positive matrices [Ando 1987] in Section 4. We describe and give evidence for each extension and show how the version of the conjecture in Section 2 implies more general versions involving Pieri-type enumerative problems. In Section 5, we present a counterexample to their original conjecture and discuss further questions.

A remark on the form of these conjectures is warranted. Conjecture 1.1 gives an infinite list of specific polynomial systems, and conjectures that each has only real solutions. The full conjectures are richer. For each collection of *Schubert data*, Shapiro and Shapiro give a continuous family of polynomial systems and conjecture that each of the resulting systems of polynomials has only real solutions. Conjecture 1.1 concerns one specific polynomial system in each family, for an infinite subset of Schubert data.

Results here were aided or are due to computations. Further documentation including Maple V.5 and Singular 1.2.1 [Greuel et al. 1998] scripts used are available at [Sottile 1999a].

2. LINEAR EQUATIONS IN PLÜCKER COORDINATES

2A. Some Enumerative Geometry

Consider the following problem in enumerative geometry: How many p-planes meet mp general m-planes in $\mathbb{C}^{m+p}$ nontrivially?

The set of p-planes in $\mathbb{C}^{m+p}$, Grass(p, $m+p$), is called the *Grassmannian of p-planes in $\mathbb{C}^{m+p}$*. This complex manifold of dimension mp is an algebraic subvariety of the projective space $\mathbb{P}^{\binom{m+p}{p}-1}$. To see this, represent a p-plane X in $\mathbb{C}^{m+p}$ as the row space of a $p\times(m+p)$-matrix, also written X. The maximal minors of X are its *Plücker coordinates* and determine a point in $\mathbb{P}^{\binom{m+p}{p}-1}$. This gives the Plücker embedding of Grass(p, $m+p$). If X is generic, then its first p columns are linearly independent, so we may assume they form a $p\times p$-identity matrix. The remaining mp entries determine X uniquely and give local coordinates for Grass(p, $m+p$), showing it has dimension mp.

Consider a m-plane K to be the row space of a $m\times(m+p)$-matrix, also written K. Then $K\cap X$ is nontrivial if and only if

$$\det\begin{bmatrix} K \\ X \end{bmatrix} = 0.$$

Laplace expansion along X gives a linear equation in the Plücker coordinates of X.

If $K_1,\dots,K_{mp}$ are m-planes in general position, the conditions that X meet each of the K_i nontrivially are mp linear equations in the Plücker coordinates of X, and these are independent by Kleiman's Transversality Theorem [Kleiman 1974]. Thus there are finitely many p-planes X that meet each K_i nontrivially and the number of such planes is the degree of Grass(p, $m+p$) in $\mathbb{P}^{\binom{m+p}{p}-1}$, which Schubert [1886] determined to be $d_{m,p}$.

2B. The Conjecture of Shapiro and Shapiro

Shapiro and Shapiro gave a recipe for selecting real m-planes $K_1,\dots,K_{mp}$. They conjecture that when these planes are in in general position, all $d_{m,p}$ p-planes meeting each K_i are real. The standard rational normal curve is the image of the map $\gamma:\mathbb{R}\to\mathbb{R}^{m+p}$ given by

$$\gamma : s \longmapsto (1, s, s^2, \dots, s^{m+p-1}). \tag{2–1}$$

Then the matrix $K(s)$ of (1–1) has rows

$$\gamma(s), \gamma'(s), \frac{\gamma''(s)}{2}, \dots, \frac{\gamma^{(m-1)}(s)}{(m-1)!},$$

where we take derivatives with respect to the parameter s. Thus the row space of $K(s)$ is the m-plane osculating the rational normal curve at $\gamma(s)$. Let X be a $p\times m$-matrix of indeterminates. Define

$$\varphi_{m,p}(s;X) := \det\begin{bmatrix} K(s) \\ I_pX \end{bmatrix}.$$

Conjecture 2.1 (Shapiro and Shapiro). *For all integers $m,p>1$ and almost all distinct real numbers $s_1,\dots,s_{mp}$, the system of mp equations*

$$\varphi_{m,p}(s_1;X) = \varphi_{m,p}(s_2;X) = \cdots = \varphi_{m,p}(s_{mp};X) = 0 \tag{2–2}$$

is zero-dimensional with $d_{m,p}$ real solutions.

Let $K(s)$ denote both the $m\times(m+p)$-matrix defined above and its row space, an m-plane. Conjecture 2.1 asserts that the m-planes $K(s_1),\dots,K(s_{mp})$ are in general position, and any p-plane meeting each $K(s_i)$ is real. The systems are zero-dimensional [Brockett and Byrnes 1981; Eisenbud and Harris 1983] and there are generically no multiplicities. We see that Conjecture 1.1 is the special case $s_i = i$.

Example 2.2. We establish Conjecture 2.1 when $m = p = 2$. Then

$$\varphi_{2,2}(s;X) = \det\begin{bmatrix} 1 & s & s^2 & s^3 \\ 0 & 1 & 2s & 3s^2 \\ 1 & 0 & x_{11} & x_{12} \\ 0 & 1 & x_{21} & x_{22} \end{bmatrix}$$

is

$$s^4 - 2s^3x_{21} + s^2x_{22} - 3s^2x_{11} + 2sx_{12} + x_{11}x_{22} - x_{12}x_{21}.$$

We show that if $s,t,u,v\in\mathbb{R}$ are distinct, the system of polynomial equations

$$\varphi_{2,2}(s) = \varphi_{2,2}(t) = \varphi_{2,2}(u) = \varphi_{2,2}(v) = 0 \tag{2–3}$$

has all $d_{2,2} = 2$ solutions real. Our method will be to solve (2–3) by elimination.

Let e_i be the i-th elementary symmetric polynomial in s,t,u,v. In the lexicographic term order with $x_{11} > x_{12} > x_{22} > x_{21}$ on the ring

$$\mathbb{Q}(s,t,u,v)[x_{11}, x_{12}, x_{22}, x_{21}],$$

the ideal

$$\langle \varphi_{2,2}(s), \varphi_{2,2}(t), \varphi_{2,2}(u), \varphi_{2,2}(v)\rangle$$

has a Gröbner basis consisting of the polynomials $2x_{21}-e_1$, $x_{22}-3x_{11}-e_2$, $2x_{12}+e_3$, and $12x_{11}^2+4e_2x_{11}+e_1e_3-4e_4$. Thus, for distinct s,t,u,v, the system (2–3) has 2 solutions and they are real if the discriminant of the last equation,

$$16e_2^2-48e_1e_3+192e_4,$$

is positive. Expanding this discriminant in the parameters s,t,u,v, we obtain

$$8\big((s-t)^2(u-v)^2 + (s-u)^2(t-v)^2+(s-v)^2(t-u)^2\big).$$

Hence all solutions are real, establishing Conjecture 2.1 when $m=p=2$. Theorem 2.3 proves Conjecture 2.1 when $(m,p)=(2,3)$.

2C. The Pole Placement Problem

Suppose we have a physical system (for example, a mechanical linkage) with inputs $u\in\mathbb{R}^m$ and outputs $y\in\mathbb{R}^p$ for which there are internal states $x\in\mathbb{R}^n$ such that the system evolves by the first order linear differential equation

$$\begin{aligned}\dot{x}&=Ax+Bu,\\ y&=Cx.\end{aligned} \tag{2–4}$$

(We assume n is the minimal number of internal states needed to obtain a first order equation.) If the input is controlled by constant output feedback, $u=Xy$, we obtain

$$\dot{x}=(A+BXC)x.$$

The natural frequencies of this controlled system are the roots $s_1,\dots,s_n$ of

$$\varphi(s):=\det(sI_n-A-BXC). \tag{2–5}$$

The pole assignment problem asks the inverse question: Given a system (2–4) and a polynomial $\varphi(s)$ of degree n, which feedback laws X satisfy (2–5)?

A coprime factorization of the transfer function is two matrices $N(s)$, $D(s)$ of polynomials with

$$\det D(s)=\det(sI_n-A)$$

and $N(s)D(s)^{-1}=C(sI_n-A)^{-1}B$. This always exists. A standard transformation (see [Byrnes 1989, § 2]) shows that, up to a sign of ±1,

$$\varphi(s)=\det\begin{bmatrix}N(s) & D(s)\\ I_p & X\end{bmatrix}. \tag{2–6}$$

If we set $K(s):=[N(s)D(s)]$, write $K(s)$ for the m-dimensional row space of this matrix, and let X be the p-plane $[I_p\ X]$, then (2–6) is equivalent to

$$X\cap K(s_i)\neq\{0\}\quad\text{for}\quad i=1,\dots,n, \tag{2–7}$$

where $s_1,\dots,s_n$ are the roots of $\varphi(s)$.

If the m-planes $K(s_1),\dots,K(s_n)$ are in general position, then $mp\geq n$ is necessary for there to be any feedback laws. These m-planes are not *a priori* in general position.

To see this, let $K:\mathbb{P}^1\to\mathrm{Grass}(m,\,m{+}p)$ be the extension of the map given by $s\mapsto K(s)$. Then K is a parameterized rational curve of degree n in $\mathrm{Grass}(m,\,m{+}p)$. The space of all such curves K with n distinguished points $\{K(s_1),\dots,K(s_n)\}$ has dimension [Strømme 1987]

$$mp+n(m+p)+n.$$

The space of all n-tuples of m-planes has dimension nmp. Therefore when

$$n>mp/(mp-m-p-1),$$

such n-tuples constitute a proper subvariety of all n-tuples of m-planes.

However, by the General Position Lemma [Byrnes 1989] (see also [Eisenbud and Harris 1983]), there is a Zariski open subset of the data A,B,C,φ such that the m-planes $K(s_1),\dots,K(s_n)$ are in general position in that the set of X satisfying (2–7) has dimension $mp-n$.

Since all rational curves $K:\mathbb{P}^1\to\mathrm{Grass}(p,\,m{+}p)$ of degree n with $K(\infty)=[0\ I_p]$ arise in this way [Martin and Hermann 1978], the polynomial systems of Conjecture 2.1 are instances of the pole placement problem. Interestingly, these very systems figure prominently in a proof of the General Position Lemma [Byrnes 1980].

An important question is whether a given real system may be controlled by *real* feedback [Byrnes 1983; Rosenthal et al. 1995; Rosenthal and Sottile 1998; Syrmos et al. 1997; Willems and Hesselink 1978]: If all roots of $\varphi(s)$ are real, are there any real feedback laws X satisfying (2–6)? Few specific examples have been computed [Byrnes and Stevens 1982; Morse et al. 1981; Rosenthal and Sottile 1998; Willems and Hesselink 1978]. In [Rosenthal and Sottile 1998] an attempt was made to gauge how likely it is for a real system to be controllable by real feedback and how many of the feedback laws are real — in the case of $(m, p) = (2, 4)$ so that $d_{m,p} = 14$. In all, 600 different curves $K(s)$ were generated, and each of these were combined with 25 polynomials $\varphi(s)$ having 8 real roots. Only 7 of the resulting 15,000 systems had all feedback laws real. This is in striking contrast to the systems given in Conjecture 2.1, where all the feedback laws are conjectured to be real.

2D. Computational Evidence

Consider (2–6) as a map $\text{Grass}(p, m+p) \to \mathbb{P}^{mp}$ in local coordinates which associates a p-plane X to a polynomial φ (modulo scalars) of degree at most mp. When $K(s)$ is the curve $K_{m,p}(s)$ of Conjecture 2.1, the inverse image of the polynomial 1 is the single real point $[0\ I_p]$. Rosenthal suggested that the fibre over a nearby polynomial may consist of $d_{m,p}$ real points.

With this in mind, Rosenthal and Sottile [1998] tested and verified several thousands of instances of Conjecture 2.1 when $(m, p) = (2, 4)$. Each was a specific choice of m, p, and mp distinct real numbers $s_1, \ldots, s_{mp}$ for which we showed all solutions to (2–2) are real. Any verified instance implies that all nearby instances in the space of parameters $s_1, \ldots, s_{mp}$ has all of its solutions real. In light of the computations described in Section 2C, we felt that this provided overwhelming evidence for the validity of Conjecture 2.1.

Our method was to solve the polynomial systems by elimination (see [Cox et al. 1998, § 2] for a discussion of methods to solve systems of polynomial equations). We first choose distinct integral values of the parameters s_i and generate the resulting system of integral polynomial equations. Since we are performing an exact symbolic computation, we necessarily work with integral polynomials. Next, we compute an eliminant, a univariate polynomial $g(x)$ with the property that its roots are the set of x-coordinates of solutions to our system. When $g(x)$ has $d = d_{m,p}$ roots (Schubert's bound), there is a lexicographic Gröbner basis satisfying the Shape Lemma, since this system is zero-dimensional [Eisenbud and Harris 1983]. It follows that the solutions are rational functions (quotients of integral polynomials) of the roots of $g(x)$. In some instances, the eliminant we calculated did not have d roots. For these we found a different eliminant with d roots. Lastly, we checked that these eliminants had only real roots.

Table 1 gives the number of instances we know to have been checked. By Lemma 3.7(ii), there is a bijection between instances of $(m, p) = (a, b)$ and $(m, p) = (b, a)$. Table 1 also lists the running time to compute a degree reverse lexicographic Gröbner basis for the systems of Conjecture 1.1, and the size of that basis. The timed calculations used Singular-1.2.1 [Greuel et al. 1998] on a K6-2 300MHz processor with 256M running Linux. The checked instances reported in the last 3 columns are not due the the author. A more complete account is found in [Sottile 1999a].

The computations of the last two columns stand out. The first is the case $(8, 2)$ (also one instance each of $(7, 2)$ and $(4, 3)$) computed by Jan Verschelde [2000] using his implementation of the SAGBI homotopy algorithm described in [Huber et al. 1998]. Since the polynomial system of Conjecture 2.1 was ill-conditioned, he used instead the equivalent system of Conjecture 2.1′ (see Section 2E below), where the $P_i(s)$ were the Chebyshev polynomials. These numerical calculations give approximate solutions whose condition numbers determine a neighborhood containing a solution. The solutions of this real system are stable under complex conjugation, so it sufficed to check that each neighbourhood and its complex conjugate were disjoint from all other neighborhoods. This computation took approximately

(m, p)	$(4, 2)$	$(5, 2)$	$(3, 3)$	$(6, 2)$	$(7, 2)$	$(4, 3)$	$(2, 8)$
$d_{m,p}$	14	42	42	132	429	462	1430
# checked	> 12000	1000	550	55	2	2	1
time (sec)	0.04	1.42	1.50	78.6	8175	–	–
size	1.4K	12.8K	18.6K	202K	4.58M	32M	–

TABLE 1. Instances checked.

25 hours on a 166MHz Pentium II processor with 64M running Linux. These algorithms are 'embarrassingly parallelizable', and in principle they can be used to check far larger polynomial systems.

The second is the case of $(m,p)=(3,4)$ of Conjecture 1.1 (also all smaller cases with $m \leq p$), computed by Faugère, Rouillier, and Zimmermann [Faugère et al. 1998]. They first used FGB (see [Faugère n.d.]) to calculate a degree reverse lexicographic Gröbner basis for the system (1–2) for $(m,p)=(3,4)$ with $s_i=i$. This yielded a Gröbner basis of size 32M. They then computed a rational univariate representation [Rouillier 1998] (a sophisticated substitute for an eliminant) in two ways. Once using a multi-modular implementation of the FGLM [Faugère et al. 1993] algorithm and a second time using RS, an improvement of the RealSolving software [Rouillier n.d.] under development. The eliminant had degree 462 and size 3M, thus its general coefficient had 2,000 digits. Using an early implementation of Uspensky's algorithm, they verified that all of its zeroes were real, proving Conjecture 1.1 for $(m,p)=(3,4)$. In the course of this calculation, they found it necessary to rewrite their software.

2E. Equivalent Systems

The extension of the map (2–1) to $\mathbb{P}^1$

$$\gamma : [t,s] \longmapsto [t^{m+p-1}, st^{m+p-2}, \dots, s^{m+p-2}t, s^{m+p-1}]$$

is a parameterization of the standard real rational normal curve in $\mathbb{P}^{m+p-1}$ and $K(s)$ is the m-plane osculating this curve at the point $\gamma[1,s]$. In general, a parameterized real rational normal curve is a map $\gamma : \mathbb{P}^1 \to \mathbb{P}^{m+p-1}$ of the form

$$[t,s] \longmapsto [P_1(s,t), P_2(s,t), \dots, P_{m+p}(s,t)]$$

where $P_1(s,t),\dots,P_{m+p}(s,t)$ form a basis for the space of real homogeneous polynomials in s,t of degree $m+p-1$. All parameterized real rational normal curves are conjugate by a real projective transformation of $\mathbb{P}^{m+p-1}$. Conjecture 2.1 has a geometric formulation.

Conjecture 2.1 (Geometric form). *For all integers $m>1$ and $p>1$ and almost all choices of mp m-planes $K_1,\dots,K_{mp}$ osculating a real rational normal curve at distinct real points, there are exactly $d_{m,p}$ p-planes X satisfying*

$$X \cap K_i \neq \{0\} \quad \textit{for} \quad i=1,\dots,mp$$

and all of these p-planes X are real.

Thus Conjecture 2.1 is equivalent to a conjecture concerning a much richer class of polynomial systems.

Conjecture 2.1′. *Suppose $m,p>1$ are integers and $P_1(s),\dots,P_{m+p}(s)$ are a basis of the space of real polynomials of degree at most $m+p-1$. Let $K(s)$ be the $m\times(m+p)$ matrix of polynomials whose (i,j)-th entry is $P_j^{(i-1)}(s)$. Set*

$$\varphi(s;X) := \det \begin{bmatrix} K(s) \\ I_pX \end{bmatrix}.$$

Then, for almost all choices of distinct real numbers $s_1,\dots,s_{mp}$, the system

$$\varphi(s_1;X)=\varphi(s_2;X)=\cdots=\varphi(s_{mp};X)=0$$

has exactly $d_{m,p}$ solutions, and all of them are real.

The polynomial matrix $K(s)$ of Conjecture 2.1′ differs from that of Conjecture 2.1 by right multiplication by an invertible $(m+p)\times(m+p)$-matrix. Thus the resulting polynomial systems differ primarily by choice of local coordinates for the Grassmannian. In linear systems theory, two physical systems are output feedback-equivalent if their matrices of coprime factors $[N(s)D(s)]$ differ in this manner [Ravi et al. 1997].

We give an equivalent conjecture concerning a simpler system of polynomials with two fewer equations and unknowns. We may reparameterize the curve $K(s)$ of Conjecture 2.1 and assume $s_{mp-1}=0$ and $s_{mp}=\infty$. Observe that $K(0)=[I_p\ 0]$ and $K(\infty)=[0\ I_p]$. The collection of all p-planes X satisfying

$$X \cap [I_p\ 0] \neq \{0\} \quad \text{and} \quad X \cap [0\ I_p] \neq \{0\} \tag{2–8}$$

is an irreducible rational variety of dimension $mp-2$.

Let $\mathcal{X}$ be the set of all $p\times(m+p)$-matrices X whose entries $x_{i,j}$ satisfy

$$x_{i,j}=1 \quad \text{if} \quad \mathrm{j}=i<p \text{ or } (i,j)=(p,p+1),$$

$$x_{i,j}=0 \quad \text{if} \quad \begin{cases} i=1 \text{ and } j\geq m, \\ 1<i<p \text{ and } j<i \text{ or } j>i+m, \\ i=p \text{ and } j\leq p. \end{cases}$$

The remaining $mp-2$ entries are unconstrained and give coordinates for $\mathcal{X}$. The row space of a matrix X is a p-plane X satisfying (2–8) and almost all such p-planes arise in this fashion. Thus $\mathcal{X}$ parameterizes a dense subset of the subvariety of p-planes X satisfying (2–8).

For example, if $(m,p) = (4,3)$, then $\mathcal{X}$ is the set of all matrices of the form

$$\begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} & 0 & 0 & 0 \\ 0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} & 0 \\ 0 & 0 & 0 & 1 & x_{35} & x_{36} & x_{37} \end{bmatrix}.$$

Since the $(1,m)$-th entry of a matrix X in $\mathcal{X}$ vanishes,

$$\det \begin{bmatrix} K(s) \\ X \end{bmatrix}$$

factors as $s \cdot \psi(s;X)$.

Conjecture 2.1″. *Let $m,p>1$ be integers. Then, for almost all choices of nonzero real numbers $s_1,\ldots,s_{mp-2}$, the system of equations*

$$\psi(s_1;X)=\psi(s_2;X)=\cdots=\psi(s_{mp-2};X)=0 \qquad (2\text{–}9)$$

is zero-dimensional with $d_{m,p}$ solutions, and all of them are real.

The systems of Conjecture 2.1 and the variations given here are deficient: They have fewer solutions than standard combinatorial bounds. For example, if $p<m$, then the system (2–9) consists of $mp-2$ equations of degree p, thus its Bézout number is p^{mp-2}. A better combinatorial bound is the normalized volume of the Newton polytope $\mathcal{A}_{m,p}$ of the polynomial ψ [Kushnirenko 1975]. Table 2 compares these combinatorial bounds with $d_{m,p}$, for some values of m,p. The volumes of $\mathcal{A}_{m,p}$ were computed using PHC [Verschelde 1999], a software package for performing general polyhedral homotopy continuation. Note the striking difference between the equivalent systems m,p and p,m.

2F. Proof in the Case m = 2 and p = 3

Theorem 2.3. *Conjecture* 2.1 *holds for* $(m,p)=(2,3)$.

L. Gonzalez-Vega has also obtained this using resultants and Sturm–Habicht sequences.

Proof. We will prove the equivalent Conjecture 2.1″. Let $X := \{x_{12}, x_{23}, x_{24}, x_{35}\}$ be indeterminates. Set

$$\psi(s;X) = \det \begin{bmatrix} 1 & s & s^2 & s^3 & s^4 \\ 0 & 1 & 2s & 3s^2 & 4s^3 \\ 1 & x_{12} & 0 & 0 & 0 \\ 0 & 1 & x_{23} & x_{24} & 0 \\ 0 & 0 & 0 & 1 & x_{35} \end{bmatrix}.$$

We solve the system of polynomials

$$\psi(s;X) = \psi(t;X) = \psi(u;X) = \psi(v;X) = 0 \qquad (2\text{–}10)$$

by elimination.

The ideal $\langle \psi(s), \psi(t), \psi(u), \psi(v) \rangle$ in the ring

$$\mathbb{Q}(s,t,u,v)[x_{12}, x_{23}, x_{24}, x_{35}]$$

has degree $5 = d_{2,3}$ and the lexicographic Gröbner basis with $x_{12} < x_{23} < x_{24} < x_{35}$ contains the following univariate polynomial g, which is the universal eliminant for this family of systems:

$$x_{35}^5 - 4e_1x_{35}^4 + (4e_1^2 + 6e_2)x_{35}^3 - (12e_1e_2 + 4e_3)x_{35}^2 \\ + (9e_2^2 + 8e_1e_3 - 4e_4)x_{35} - (12e_2e_3 - 8e_1e_4)$$

Here e_i is the i-th elementary symmetric polynomial in s,t,u,v. We show that g has 5 distinct real roots for every choice of distinct parameters s,t,u,v. The discriminant Δ of g has degree 20 in the variables

(m,p)	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$	$(7,2)$	$(8,2)$	$(2,3)$	$(3,3)$	$(4,3)$	$(2,4)$	$(3,4)$
$d_{m,p}$	2	5	14	42	132	429	1430	5	42	462	14	462
vol $\mathcal{A}_{m,p}$	2	5	18	67	248	919	3426	5	130	3004	42	7156
p^{mp-2}	4	16	64	256	1024	4096	16384	81	2187	59049	4096	1048576

TABLE 2. Combinatorial bounds versus $d_{m,p}$.

s, t, u, v and 711 terms:

$$
\begin{aligned}
&9e_3^4e_2^2e_1^4 - 54e_3^4e_2^3e_1^2 + 81e_3^4e_2^4 - 32e_3^5e_1^5 + 204e_3^5e_2e_1^3 \\
&-324e_3^5e_2^2e_1 - 108e_3^6e_1^2 + 324e_3^6e_2 + 81e_4^2e_2^4e_1^4 - 486e_4^2e_2^5e_1^2 \\
&+ 729e_4^2e_2^6 - 54e_4e_3^2e_2^3e_1^4 + 324e_4e_3^2e_2^4e_1^2 - 486e_4e_3^2e_2^5 \\
&+204e_4e_3^3e_2e_1^5 - 1296e_4e_3^3e_2^2e_1^3 + 2052e_4e_3^3e_2^3e_1 - 8e_4e_3^4e_1^4 \\
&+738e_4e_3^4e_2e_1^2 - 2106e_4e_3^4e_2^2 - 108e_4e_3^5e_1 - 324e_4^2e_3e_2^2e_1^5 \\
&+ 2052e_4^2e_3e_2^3e_1^3 - 3240e_4^2e_3e_2^4e_1 - 108e_4^2e_3^2e_1^6 \\
&+738e_4^2e_3^2e_2e_1^4 - 2592e_4^2e_3^2e_2^2e_1^2 + 3834e_4^2e_3^2e_2^3 - 368e_4^2e_3^3e_1^3 \\
&+ 1800e_4^2e_3^3e_2e_1 - 27e_4^2e_3^4 + 324e_4^3e_2e_1^6 - 2106e_4^3e_2^2e_1^4 \\
&+ 3834e_4^3e_2^3e_1^2 - 972e_4^3e_2^4 - 108e_4^3e_3e_1^5 + 1800e_4^3e_3e_2e_1^3 \\
&- 5544e_4^3e_3e_2^2e_1 - 634e_4^3e_3^2e_1^2 + 984e_4^3e_3^2e_2 - 27e_4^4e_1^4 \\
&+ 984e_4^4e_2e_1^2 + 432e_4^4e_2^2 - 352e_4^4e_3e_1 - 64e_4^5.
\end{aligned}
$$

This vanishes when g has a double root. Thus the number of real roots of g is constant on each connected component (in $\mathbb{R}^4$) of the locus $\Delta \neq 0$. We show there is only one connected component, and so the number of real roots of g (and thus the original system) does not depend upon the choice of real parameters. Since the roots of g evaluated at $(s, t, u, v) = (1, 2, 3, 4)$ are

$$8,\ 8 \pm \sqrt{19},\ 8 \pm \sqrt{11},$$

it follows that there are always five real roots of g, and thus the system (2–10) has $d_{2,3} = 5$ real solutions whenever s, t, u, v are real and distinct.

We complete the proof. For $w \in \mathbb{Z}^{10}_{\geq 0}$, consider the polynomial

$$
\begin{aligned}
s^{w_1}t^{w_2}u^{w_3}v^{w_4}&(s-t)^{w_5}(s-u)^{w_6}(s-v)^{w_7} \\
&\times (t-u)^{w_8}(t-v)^{w_9}(u-v)^{w_{10}}. \qquad (2\text{–}11)
\end{aligned}
$$

Let A_w be the primitive part of the symmetrization of this polynomial. Thus, when all componentes of w are even, A_w is a sum of squares, none of which vanish on the locus where s, t, u, v are distinct. Then Δ is

$$
\begin{aligned}
&\tfrac{1}{2}(7A_{2220222224} + 3A_{2222402204} + 6A_{4222022222} \\
&+ 7A_{4220222222} + 2A_{4420022222} + 2A_{2222440022} \\
&+ 2A_{0222442022} + A_{4420202222} + 2A_{4222420022} \\
&+ A_{4220022422} + A_{0222442202} + A_{2202024422} \\
&+ 6A_{2222420024} + 10A_{4220022242} + 3A_{2222222222}).
\end{aligned}
$$

The term $7A_{4220222222}$ does not vanish when a single parameter is zero. Similarly, the term $3A_{2222402204}$ does not vanish when $s = u$ and $t = v$ (but $u \neq t$). Thus the locus where $\Delta = 0$ has dimension 2 and so its complement is connected. □

We have a Maple program that performs the computations described in this proof and runs in approximately 15 seconds on a K6-2 300MHz processor.

A *positive semidefinite* polynomial is a real polynomial that takes only nonnegative values. In the proof we showed Δ is positive semidefinite by exhibiting it as a sum of squares. Not all positive semidefinite polynomials are sums of squares of polynomials. There exist positive semidefinite polynomials of degree l in k variables which are not sums of squares of polynomials if $\min(k, l) > 2$ and $(k, l) \neq (3, 4)$ [Hilbert 1888]. For Δ, $(k, l) = (4, 20)$.

The form of the squares we used (2–11) for the discriminant Δ, while motivated by the observation that no two parameters $(0, s, t, u, v, \infty)$ should coincide, is justified by the observation that any real zero of Δ must also be a zero of all the squares, if Δ is a sum of squares. (See [Choi et al. 1987] for other applications of this idea.)

Each of the polynomials A_w is a sum of squares, the number given by the orbit of the symmetric group on its index w. Since 6 have trivial stabilizer, 7 are stabilized by a transposition, one by the dihedral group D_8, and one is invariant, there are $6 \cdot 24 + 7 \cdot 12 + 3 + 1 = 232$ squares in all. This is not the best possible. Choi, Lam, and Reznick [Choi et al. 1995] show, for degree l homogeneous polynomials in k variables that are a sum of squares of polynomials, at most

$$\Lambda(k, l) := \left\lfloor \frac{1}{2}\left(\sqrt{1 + 8\tbinom{k+l-1}{l}} - 1\right)\right\rfloor$$

squares are needed. Note that $\Lambda(4, 20) = 59$.

3. SCHUBERT CONDITIONS ON A GRASSMANNIAN

3A. The Schubert Calculus on Grass(p, m+p)

The enumerative problems of Section 2 are special cases of more general problems given by Schubert conditions on $\mathrm{Grass}(p, m{+}p)$. A *Schubert condition* on $\mathrm{Grass}(p, m{+}p)$ is an increasing sequence of integers

$$\alpha : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq m + p.$$

Let $\binom{[m+p]}{p}$ be the set of all such sequences. A *Schubert variety* $\Omega_\alpha K_\bullet$ is given by a Schubert condition

α and a complete flag $K_\bullet$ in $\mathbb{C}^{m+p}$, a sequence of subspaces

$$K_\bullet : K_1 \subset K_2 \subset \cdots \subset K_{m+p} = \mathbb{C}^{m+p}$$

where $\dim K_i = i$. Then the Schubert variety $\Omega_\alpha K_\bullet$ is the set of all p-planes X satisfying

$$\dim X \cap K_{m+p+1-\alpha_i} \geq p+1-i \tag{3–1}$$

for each $i = 1, 2, \ldots, p$. This irreducible subvariety of Grass(p, m+p) has codimension $|\alpha| := \sum_i (\alpha_i - i)$.

A sequence $\alpha^\bullet = \alpha^1, \ldots, \alpha^n$, with

$$\alpha^j \in \binom{[m+p]}{p} \quad \text{and} \quad \textstyle\sum_j |\alpha^j| = mp,$$

is *Schubert data* for Grass(p, m+p). Given Schubert data $\alpha^\bullet$ and flags $K^1_\bullet, \ldots, K^n_\bullet$ in general position, there are finitely many (complex) p-planes X which lie in the intersection of the Schubert varieties $\Omega_{\alpha^j} K^j_\bullet$ for $j = 1, \ldots, n$. The classical Schubert calculus [Kleiman and Laksov 1972] gives the following recipe for computing this number $d = d(m, p; \alpha^\bullet)$. Let $h_1, \ldots, h_m$ be indeterminates with $\deg h_i = i$. For each integer sequence $\beta_1 < \beta_2 < \cdots < \beta_r$ define the following polynomial

$$S_\beta := \det(h_{\beta_i - j})_{1 \leq i,j \leq r}.$$

Here $h_0 := 1$ and $h_i := 0$ if $i < 0$ or $i > m$. Let $\mathcal{I}$ be the ideal in $\mathbb{Q}[h_1, \ldots, h_m]$ generated by those S_β with $r = p+1$, $1 < \beta_1$, and $\beta_{p+1} \leq m+p$. The quotient ring $\mathcal{A}_{m,p} := \mathbb{Q}[h_1, \ldots, h_m]/\mathcal{I}$ is isomorphic to the cohomology ring of Grass(p, m+p). It is Artinian with one-dimensional socle in degree mp. In the socle we have the relation

$$d \cdot (h_m)^p - S_{\alpha^1} S_{\alpha^2} \cdots S_{\alpha^n} \quad \in \quad \mathcal{I}.$$

We can compute the number d by normal form reduction modulo any Gröbner basis for $\mathcal{I}$.

If γ is a rational normal curve, then the flag of subspaces osculating γ at a point is the *osculating flag* to γ at that point.

Conjecture 3.1 (Shapiro and Shapiro). *Let $m, p > 1$ and $\alpha^\bullet$ be Schubert data for* Grass(p, m+p). *For almost all choices of flags $K^1_\bullet, \ldots, K^n_\bullet$ osculating a fixed rational normal curve at real points, there are exactly $d(m, p; \alpha^\bullet)$ p-planes X in the intersection of Schubert varieties*

$$\Omega_{\alpha^1} K^1_\bullet \cap \Omega_{\alpha^2} K^2_\bullet \cap \cdots \cap \Omega_{\alpha^n} K^n_\bullet, \tag{3–2}$$

and each of these p-planes is real.

As with Conjecture 2.1, the intersection is zero-dimensional if the points of osculation are distinct [Eisenbud and Harris 1983], and there are no multiplicities for the important class of Pieri Schubert data, (described below) which includes the case of Conjecture 2.1.

If $\alpha_i = 1 + \alpha_{i-1}$, then condition (3–1) for $i-1$ implies (3–1) for i. Thus only those conditions (3–1) with $\alpha_i - \alpha_{i-1} > 1$ (or $\alpha_1 > 1$) are *essential*, and so only the subspaces $K_{m+p+1-\alpha_i}$ corresponding to essential conditions need be specified in a flag. If $\alpha := (1, 2, \ldots, p-1, p+1)$, then only the last condition is essential, thus the Schubert variety $\Omega_\alpha K_\bullet$ consists of those X with $\dim X \cap K_m \geq 1$. This shows Conjecture 2.1 is a special case of Conjecture 3.1.

3B. Systems of Polynomials

A complete flag $K_\bullet$ is represented by a nonsingular matrix also written $K_\bullet$: The i-plane K_i is the row space of K_i, the first i rows of $K_\bullet$. The condition that $\dim X \cap K_{m+p+1-\alpha_i} \geq p+1-i$ is given by

$$(m+p+1+i-\alpha_i)\text{-minors of } \begin{bmatrix} K_{m+p+1-\alpha_i} \\ X \end{bmatrix} = 0.$$

The flag $K_\bullet(s)$ that osculates the rational normal curve γ with the parameterization (2–1) at $\gamma(s)$ is represented by the $(m+p) \times (m+p)$-matrix whose (i,j)-th entry is $\binom{j-i}{i-1} s^{j-i}$.

Conjecture 3.1′. *Let $m, p > 1$ and $\alpha^\bullet$ be Schubert data for* Grass(p, m+p). *For almost all n-tuples of distinct real numbers $s_1, \ldots, s_n$, the system of polynomials*

$$(m+p+1+i-\alpha^j_i)\text{-}\textit{minors of } \begin{bmatrix} K_{m+p+1-\alpha^j_i}(s_j) \\ I_p \; X \end{bmatrix} = 0$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, n$ has $d(m, p; \alpha^\bullet)$ solutions, and each is real.

For any Schubert conditions α, β with $\alpha_i + \beta_{p+1-i} \leq m+p$ for $i = 1, \ldots, p$, let $\mathcal{X}_{\alpha,\beta}$ be the collection of all $p \times (m+p)$-matrices X whose entries x_{ij} satisfy

$$\begin{aligned} x_{i,\alpha_i} &= 1 && \text{for } i = 1, \ldots, p, \\ x_{i,j} &= 0 && \text{if } j < \alpha_i \text{ or } j > m+p+1-\beta_{p+1-i}. \end{aligned}$$

If $X \in \mathcal{X}_{\alpha,\beta}$, then the row space of X is a p-plane in the intersection $\Omega_\alpha K_\bullet(\infty) \cap \Omega_\beta K_\bullet(0)$. In this way, $\mathcal{X}_{\alpha,\beta}$ parameterizes a Zariski open subset of the set of all such p-planes. This parameterization can be used to obtain a system of equations simpler than, but equivalent to, the system of Conjecture 3.1′.

The map $\mathcal{X}_{\alpha,\beta} \to \mathrm{Grass}(p,\, m{+}p)$ is not injective. For example, $\mathcal{X}_{123,134}$ consists of all 3×7-matrices of the form

$$\begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} & 0 & 0 & 0 \\ 0 & 1 & x_{23} & x_{24} & x_{25} & 0 & 0 \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} & x_{37} \end{bmatrix}.$$

Let r_1, r_2, r_3 be the rows of such a matrix. If $x_{36} = x_{37} = 0$, then for each $a \in \mathbb{C}$, the matrix with rows $r_1, r_2 + ar_3, r_3$ is in $\mathcal{X}_{123,134}$, and these all have the same row space. Similarly, if $x_{25} = 0$, then the same is true of the matrices with rows $r_1 + ar_2, r_2, r_3$.

Let $\mathcal{X}^\circ_{\alpha,\beta} \subset \mathcal{X}_{\alpha,\beta}$ be the set of those matrices whose entries further satisfy the following condition:

For each $i = 2, \dots, p$, at least one x_{ij} is nonzero, for j satisfying

$$\beta_{p+1-i} \le m + p + 1 - j < \beta_{p+2-i}.$$

For $\mathcal{H}_{123,134}$ this condition is that $x_{25} \neq 0$ and $(x_{36}, x_{37}) \neq (0, 0)$. We made this definition so that the map $\mathcal{X}^\circ_{\alpha,\beta} \to \mathrm{Grass}(p,\, m{+}p)$ is injective.

3C. Pieri Schubert Conditions

If $\alpha \in \binom{[m+p]}{p}$ has $\alpha_{p-1} = p-1$ and $\alpha_p = p+a$, then the Schubert variety $\Omega_\alpha K_\bullet$ is

$$\{X \mid X \cap K_{m+1-a} \neq \{0\}\}.$$

We call such a Schubert condition a *Pieri condition* and denote it by J_a. *Pieri Schubert data* are Schubert data $\alpha^1, \dots, \alpha^n$ were at most 2 of the conditions α^i are *not* Pieri conditions. These include the Schubert data of Conjecture 2.1.

Proposition 3.2 [Eisenbud and Harris 1983, Theorem 9.1]. *If $\alpha^\bullet$ are Pieri Schubert data and the flags $K^1_\bullet, \dots, K^n_\bullet$ osculate a rational normal curve at general points, then the intersection of Schubert varieties*

$$\Omega_{\alpha^1} K^1_\bullet \cap \Omega_{\alpha^2} K^2_\bullet \cap \cdots \cap \Omega_{\alpha^n} K^n_\bullet$$

is transverse. In particular, there are no multiplicities.

Here is the main theorem of this section.

Theorem 3.3. *Let $a, b > 1$ and suppose that Conjecture* 2.1 *holds for this $(m,p) = (a,b)$. Then Conjecture* 3.1 *holds for any Pieri Schubert data for* $\mathrm{Grass}(p,\, m{+}p)$ *with $(p,m) \le (a,b)$ or $(p,m) \le (b,a)$, in each coordinate.*

We deduce Theorem 3.3 after Lemma 3.7, which shows some simple dependencies between Conjecture 3.1 for different collections of Schubert data.

Remark 3.4. If the conclusion of Proposition 3.2 held for all Schubert data, then the proof we give of Theorem 3.3 would imply its conclusion for all Schubert data as well. David Eisenbud pointed out that our proof shows that in the absence of this strengthening of Proposition 3.2, we may still deduce that all points in the intersection (3–2) of Conjecture 3.1 are real, although there may in general be multiplicities.

Pieri conditions are special because of Pieri's formula. For $\alpha, \beta \in \binom{[m+p]}{p}$ and $a > 0$, we write $\alpha <_a \beta$ if $|\alpha| + a = |\beta|$ and

$$\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < \cdots < \alpha_p \le \beta_p.$$

Proposition 3.5 (Pieri's Formula). *Let $J_a := 1 < 2 < \cdots < p{-}1 < p{+}a \in \binom{[m+p]}{p}$.*

(i) *In the cohomology ring $\mathcal{A}_{m,p}$ of* $\mathrm{Grass}(p,\, m{+}p)$, *$S_{J_a} = h_a$ and*

$$S_\alpha \cdot S_{J_a} = \sum_{\alpha <_a \beta} S_\beta.$$

(ii) *If $K_\bullet(s)$ and $K_\bullet(t)$ are flags osculating a rational normal curve at points s and t, then*

$$\lim_{s \to t} \left(\Omega_\alpha K_\bullet(t) \cap \Omega_{J_a} K_\bullet(s)\right) = \sum_{\alpha <_a \beta} \Omega_\beta K_\bullet(t).$$

Here, the limit is taken as cycles. By this we mean that the sum is the fundamental cycle of the limit of the schemes $\Omega_\alpha K_\bullet(t) \cap \Omega_{J_a} K_\bullet(s)$ as s approaches t along the rational normal curve.

(iii) *Suppose that $\alpha^\bullet = \alpha^1, J_a, \alpha^2, \dots, \alpha^n$ are Schubert data. Then*

$$d(m,p;\alpha^\bullet) = \sum_{\alpha^1 <_a \beta} d(m,p;\beta,\alpha^2,\dots,\alpha^n).$$

Statement (i) is the usual statement of Pieri's formula [Fulton 1997; Hodge and Pedoe 1952]. Statement (ii) is Theorem 8.1 of [Eisenbud and Harris 1983], and (iii) is a direct consequence of (i).

Definition (3–1) implies that $\Omega_\beta K_\bullet \subset \Omega_\alpha K_\bullet$ if and only if $\alpha \le \beta$ coordinatewise. In fact, $\Omega_\beta K_\bullet \cap \Omega_\alpha K_\bullet = \Omega_{\beta \vee \alpha} K_\bullet$, where $\beta \vee \alpha$ is the coordinatewise maximum of α and β. We make some definitions needed for the statement of Lemma 3.7.

Definition 3.6. Let $m, p > 1$ be integers.

1. For $\alpha \in \binom{[m+p]}{p}$ define $\alpha^\perp \in \binom{[m+p]}{m}$ to be the increasing sequence obtained from the numbers $\{1, 2, \ldots, m+p\} \setminus \{\alpha_1, \ldots, \alpha_p\}$. Given Schubert data $\alpha^\bullet$ for Grass(p, m+p), set $\alpha^{\bullet\perp}$ to be
$$(\alpha^1)^\perp, \ldots, (\alpha^n)^\perp.$$
2. Suppose $p > 2$. For $\alpha \in \binom{[m+p-1]}{p-1}$ define $\alpha^+ \in \binom{[m+p]}{p}$ to be $1 < 1+\alpha_1 < \cdots < 1+\alpha_{p-1}$. Given Schubert data $\alpha^\bullet$ for Grass(p, m+p), set $\alpha^{\bullet+}$ to be $(\alpha^1)^+, \ldots, (\alpha^n)^+$.
3. Let $\preceq$ be the partial order on Pieri Schubert data where we say that $\beta^\bullet$ covers $\alpha^\bullet = \alpha^1, \ldots, \alpha^n$ if either
$$\beta^\bullet = \beta, \alpha^3, \ldots, \alpha^n \text{ with } \alpha^2 = J_a \text{ and } \alpha^1 <_a \beta,$$
or
$$\beta^\bullet = \alpha^1, \ldots, \alpha^{n-2}, \beta \text{ with } \alpha^{n-1} = J_a \text{ and } \alpha^n <_a \beta.$$

Lemma 3.7. *Let $m, p > 1$ be integers.*

(i) *If $\alpha^\bullet$ is Schubert data for* Grass(p, m+p), *then $\alpha^{\bullet\perp}$ is Schubert data for* Grass(m, m+p). *Moreover, Conjecture* 3.1 *holds for $m, p, \alpha^\bullet$ if and only if it holds for $p, m, \alpha^{\bullet\perp}$.*

(ii) *Suppose $p > 2$ and let*
$$J_m := 1 < 2 < \cdots < p-1 < p+m.$$
If $\alpha^\bullet$ is Schubert data for Grass(p−1, m+p−1), *then $\beta^\bullet := \alpha^{\bullet+}, J_m$ is Schubert data for*
$$\text{Grass}(m,\ m{+}p).$$
Moreover, Conjecture 3.1 *holds for m, p−1, $\alpha^\bullet$ if and only if it holds for $m, p, \beta^\bullet$.*

(iii) *Suppose that $\alpha^\bullet, \beta^\bullet$ are Pieri Schubert data for* Grass(p, m+p) *with $\alpha^\bullet \preceq \beta^\bullet$. If Conjecture* 3.1 *holds for $\alpha^\bullet$ for* Grass(p, m+p), *then it holds for $\beta^\bullet$.*

Proof of Theorem 3.3.. First note that Conjecture 3.1 holds for Schubert data $\alpha^\bullet$ for Grass(p, m+p) if and only if it holds for any rearrangement of the data $\alpha^\bullet$. Suppose Conjecture 2.1 holds for Grass(b, a+b). Let $\alpha^\bullet$ be Pieri Schubert data for Grass(p, m+p) where $(m,p) \le (a,b)$ or $(m,p) \le (b,a)$ coordinatewise. Since $J_1^\perp = J_1$, Conjecture 2.1 holds also for $(m,p) = (b,a)$, by Lemma 3.7(i). Thus we may assume that $(m,p) \le (a,b)$. By Lemma 3.7(ii), there exist Pieri Schubert data $\beta^\bullet$ for Grass(b, $a+b$) such that Conjecture 3.1 holds for $\alpha^\bullet$ if and only if it holds for $\beta^\bullet$. Finally, Theorem 3.3 follows from (iii) by noting that the Schubert data of Conjecture 2.1, namely $\alpha^1 = \cdots = \alpha^{ab} = J_1$, is minimal among all Pieri Schubert data for Grass(b, $a+b$). □

Proof of Lemma 3.7. For (i), fix a real inner inner product on $\mathbb{C}^{m+p}$. Then the map
$$X \mapsto X^\perp$$
gives an isomorphism between Grass(p, m+p) and Grass(m, m+p). Given a flag $K_\bullet$ and an increasing sequence α, let $K_\bullet^\perp$ be the flag of annihilators of the subspaces of $K_\bullet$. Then we have
$$X \in \Omega_\alpha K_\bullet \iff X^\perp \in \Omega_{\alpha^\perp} K_\bullet^\perp.$$
Furthermore, if $K_\bullet(s)$ is the flag of subspaces osculating a rational normal curve γ at a point $\gamma(s)$, then $(K_{m+p-1}(s))^\perp$ is a rational normal curve with $K_\bullet^\perp(s)$ its osculating flag. Thus Conjecture 3.1 for Schubert data $\alpha^\bullet$ for Grass(p, m+p) is equivalent to Conjecture 3.1 for Schubert data $\alpha^{\bullet\perp}$ for Grass(m, m+p).

For (ii), let γ be the rational curve (2–1) with $K_\bullet(s)$ as before. Then $X \in \Omega_{J_m} K_\bullet(\infty)$ if and only if $\langle\gamma(\infty)\rangle = K_1(\infty) \subset X$. Consider the projection $\pi : \mathbb{C}^{m+p} \twoheadrightarrow \mathbb{C}^{m+p-1}$ from the last coordinate $\gamma(\infty)$. If $X \in \Omega_{J_m} K_\bullet(\infty)$, then $X' := \pi X$ is a (p−1)-plane. This induces an isomorphism $\pi : \Omega_{J_m} K_\bullet(\infty) \xrightarrow{\sim}$ Grass(p−1, m+p−1). The inverse map is given by $X' \mapsto K_1(\infty) + X'$.

The projection $\pi \circ \gamma$ is the standard rational normal curve γ' in $\mathbb{C}^{m+p-1}$. Similarly, the flag $K_\bullet{}'(s)$ osculating γ' at $\gamma'(s)$ is $\pi K_\bullet(s)$. Note that if L is a linear subspace of $\mathbb{C}^{m+p}$ with $\gamma(\infty) \notin L$, then $\dim X \cap L = \dim \pi X \cap \pi L$. In particular, if $X \in \Omega_{J_m} K_\bullet(\infty)$, $s \neq \infty$, and $\alpha \in \binom{[m+p-1]}{p-1}$, then
$$\dim X' \cap K'_{(m+p-1)+1-\alpha_i} \ge (p-1) + 1 - i$$
if and only if
$$\dim X \cap K_{m+p+1-(1+\alpha_i)} \ge p + 1 - (i+1).$$
Thus
$$X \in \Omega_{J_m} K_\bullet(\infty) \cap \Omega_{\alpha^+} K_\bullet(s) \iff X' \in \Omega_\alpha K_\bullet{}'(s).$$
In fact, this induces an isomorphism of schemes.

This gives a strong equivalence between enumerative problems: If $\alpha^1, \ldots, \alpha^n$ are in $\binom{[m+p-1]}{p-1}$ and

$s_1, \ldots, s_n$ any complex numbers, then the map π induces an isomorphism between the schemes

$$\Omega_{J_m} K_\bullet(\infty) \cap \bigcap_{i=1}^{n} \Omega_{(\alpha^i)^+} K_\bullet(s_i) \quad \text{and} \quad \bigcap_{i=1}^{n} \Omega_{\alpha^i} K_\bullet{}'(s_i).$$

Part (ii) follows by noting that any real reparameterization of the rational normal curve γ induces an isomorphism of polynomial systems, thus preserves real solutions. Hence given $s_0, s_1, \ldots, s_n \in \mathbb{P}^1_{\mathbb{R}}$, there is an equivalent system with $s_0 = \infty$.

It suffices to prove (iii) when $\beta^\bullet$ covers $\alpha^\bullet$ in the partial order $\preceq$ defined on Pieri Schubert data. Suppose Conjecture 3.1 fails for $\beta^\bullet$ and $\beta^\bullet$ covers $\alpha^\bullet$ with $\alpha^1 <_a \beta$ and $\alpha^2 = J_a$ as in Definition 3.6 (iii). Then there exist distinct real numbers $s_1, s_3, \ldots, s_n$ such that

$$\Omega_\beta K_\bullet(s_1) \cap \Omega_{\alpha^3} K_\bullet(s_3) \cap \cdots \cap \Omega_{\alpha^n} K_\bullet(s_n) \tag{3–3}$$

is transverse with some complex p-planes in the intersection. We may assume without any loss that $s_1 = 0$. Then there is an open subset $\mathcal{O}$ of the set of $(n-1)$-tuples of real numbers $s_3, \ldots, s_n$ such that (3–3) is transverse and contains a complex p-plane X.

By the dimensional transversality results of [Eisenbud and Harris 1983], we may assume further that for $\beta' \in \binom{[m+p]}{p}$ and $(s_2, \ldots, s_n) \in \mathcal{O}$, the intersection

$$\Omega_{\beta'} K_\bullet(0) \cap \bigcap_{i=3}^{n} \Omega_{\alpha^i} K_\bullet(s_i)$$

has the expected dimension and is transverse if 0-dimensional. This is empty if $|\beta'| > a + |\alpha^1|$, for dimension reasons. Thus

$$\left(\sum_{\alpha <_a \beta'} \Omega_{\beta'} K_\bullet(0) \right) \cap \bigcap_{i=3}^{n} \Omega_{\alpha^i} K_\bullet(s_i)$$

is transverse for $(s_3, \ldots, s_n) \in \mathcal{O}$.

Fix $(s_3, \ldots, s_n) \in \mathcal{O}$. By Proposition 3.5(i), there is an $\varepsilon > 0$ such that for $|t| < \varepsilon$

$$\Omega_\alpha K_\bullet(0) \cap \Omega_{J_a} K_\bullet(t) \cap \bigcap_{i=3}^{n} \Omega_{\alpha^i} K_\bullet(s_i)$$

is transverse. Here, when $t = 0$, replace $\Omega_\alpha K_\bullet(0) \cap \Omega_{J_a} K_\bullet(t)$ by $\sum_{\alpha <_a \beta} \Omega_{\beta'} K_\bullet(0)$. Since at $t = 0$ not all points in the intersection are real, the same holds for $0 < t < \varepsilon$. But then Conjecture 3.1 fails for the Schubert data $\alpha^\bullet$, which completes the proof of Lemma 3.7. □

3D. An Infinite Family

We show that Conjecture 3.1 holds for an infinite family of nontrivial Schubert data.

Theorem 3.8. *Conjecture* 3.1 *holds for any m with $p = 2$ and Pieri Schubert data where one condition is J_{m-1}.*

Proof. By Lemma 3.7(iii), it suffices to show this for $\alpha^1 = \cdots = \alpha^{m+1} = J_1$ and $\alpha^{m+1} = J_{m-1}$. Geometrically, we are looking for the 2-planes which meet a 2-plane and $m+1$ general m-planes nontrivially. We first show there are m such 2-planes. Let $L = K_2(\infty) = [0\ I_2]$ and $M = K_m(0) = [I_m\ 0]$, and let $N_i = K_m(s_i)$, where $s_1, \ldots, s_m$ are distinct nonzero real numbers. For each one-dimensional subspace λ of L and each $1 \leq i \leq m$, the composition

$$M \hookrightarrow L \oplus M \simeq \mathbb{C}^{m+2} \twoheadrightarrow L \oplus M/(\lambda + N_i) \simeq \mathbb{C}$$

defines a linear form $\psi_{i,\lambda}$ on M. Each one-dimensional subspace μ of its kernel gives a 2-plane $\lambda \oplus \mu$ containing λ and meeting both M and N_i nontrivially.

Thus if X is a 2-plane meeting L, M, and each N_i nontrivially, then $H \cap L = \lambda$ and $H \cap M = \mu$ are lines with μ in the kernel of each form $\psi_{i,\lambda}$. Hence the forms are dependent. Similarly, if λ is a line in L such that the forms $\psi_{i,\lambda}$ are dependent, then any line μ they collectively annihilate gives a 2-plane $\lambda \oplus \mu$ meeting L, M, and each N_i nontrivially. It follows that the number of such 2-planes is the degree of the determinant of the forms $\psi_{i,\lambda}$, a polynomial in $\lambda \in \mathbb{P}(L) \simeq \mathbb{P}^1$. Since each form $\psi_{i,\lambda}$ is a linear function of λ, the determinant has degree m, so there are m 2-planes X meeting L, M, and each N_i nontrivially.

We compute this determinant and show it has only real roots. Let $\lambda = \lambda(x)$ be the span of the vector

$$(0, \ldots, 0, 1, (m+1)x).$$

Let the rational normal curve γ have the parameterization

$$\gamma : s \longmapsto (1, -s, s^2, \ldots, (-1)^{m+1} s^{m+1}).$$

Then $K_m(s)$, the osculating m-plane to γ at $\gamma(s)$, is the kernel of the matrix

$$\begin{bmatrix} s^m & ms^{m-1} & \dots & \binom{m}{j}s^{m-j} & \dots & ms & 1 & 0 \\ 0 & s^m & \dots & \binom{m}{j-1}s^{m-j+1} & \dots & \binom{m}{2}s^2 & ms & 1 \end{bmatrix}.$$

If $R_j(s)$ is the linear form given by the j-th row of this matrix, then

$$((m+1)x + ms_i)R_1(s_i) - R_2(s_i)$$

vanishes on $\lambda(x)$ and its restriction to M gives the form $\psi_{i,\lambda(x)}$. This restriction is represented by the vector $\Lambda(s_i, x)$ whose j-th coordinate for $j = 0, \dots, m-1$ is

$$\tbinom{m+1}{j}\left((m-j+1)xs_i^{m-j} + (m-j)s_i^{m-j+1}\right).$$

We seek the determinant of the matrix

$$\begin{bmatrix} \Lambda(s_1, x) \\ \vdots \\ \Lambda(s_m, x) \end{bmatrix}.$$

This factors as $A \cdot B$, where A is the bidiagonal $m \times (m+1)$-matrix

$$\begin{bmatrix} m & (m+1)x & & 0 & & & \\ & m^2-1 & m(m+1)x & & & & \\ & & \ddots & \ddots & & & \\ & & & \binom{m+1}{j}(m-j) & \binom{m+1}{j}(m-j+1)x & & \\ & & & & \ddots & \ddots & \\ & & & & & \binom{m+1}{2} & \binom{m+1}{2}2x \end{bmatrix}$$

and B is the $(m+1) \times m$-matrix whose (i, j)-th entry is s_j^{m+2-i}. Numbering the rows of A and the columns of B from 0 to m, we see that

$$\det(A(x) \cdot B) = \sum_{i=0}^{m} (-1)^i \det A_i(x) \det B_i,$$

where A_i is the matrix A with its i-th column removed and B_i is the matrix B with its i-th row removed. We find that

$$\det A_i = m!(m+1-i)x^{m-i} \prod_{j=1}^{m} \tbinom{m+1}{j},$$

$$\det B_i = e_i(s_1, \dots, s_m)s_1 s_2 \cdots s_m \cdot \prod_{j<k}(s_j - s_k),$$

and so $\det(A \cdot B)$ is

$$m! \prod_{j<k}(s_j - s_k) \prod_{j=1}^{m} s_j \tbinom{m+1}{j} \cdot \left(\sum_{i=0}^{m} (-1)^i (m-i+1)x^{m-i} e_i(s_1, \dots, s_m) \right).$$

Thus the coordinate x of the line λ satisfies the polynomial

$$P_m(s_1, \dots, s_m; x) := \sum_{i=0}^{m} (-1)^i (m-i+1)x^{m-i} e_i(s_1, \dots, s_m).$$

Since we have $e_i(s_1, \dots, s_m) = e_i(s_1, \dots, s_{m-1}) + s_m e_{i-1}(s_1, \dots, s_{m-1})$, we see that

$$P_m(s_1, \dots, s_m; x) = (x - s_m)P_{m-1}(s_1, \dots, s_{m-1}; x) + x \prod_{i=1}^{m-1}(x - s_i).$$

To complete the proof, we use induction to show:

If $0 < s_1 < \dots < s_m$, the roots $r_1, \dots, r_m$ of P_m satisfy

$$0 < r_1 < s_1 < r_2 < s_2 < \dots < r_m < s_m. \qquad (*)$$

This suffices, if we can assume $0 < s_1 < \dots < s_m$. But we may assume this: Given a set of distinct real numbers $s_1, \dots, s_m, s_{m+1}, s_{m+2}$, we may assume $s_{m+2} = \infty$ and $s_{m+1} < s_1 < \dots < s_m$ and then apply the automorphism $s \mapsto s - s_{m+1}$ of $\mathbb{P}^1(\mathbb{R})$ which fixes $\infty = s_{m+2}$.

The case $m = 1$ of $(*)$ holds as $P_1(s_1; x) = 2x - s_1$. Suppose P_{m-1} satisfies $(*)$. Then the roots of $(x - s_m)P_{m-1}$ are $r_1 < r_2 < \dots < r_{m-1} < s_m$ and those of $x\prod_{i=1}^{m-1}(x - s_i)$ are $0 < s_1 < \dots < s_{m-1}$. Moreover the leading coefficients of both polynomials are positive. The result follows by the Intermediate Value Theorem: If $P(x)$ and $Q(x)$ are polynomials of degree n with positive leading coefficients and real interlaced roots p_i of P and q_i of Q

$$p_1 < q_1 < p_2 < q_2 < \dots < p_n < q_n,$$

then $P(x) + Q(x)$ has real roots r_i satisfying $p_i < r_i < q_i$, for $i = 1, \dots, n$. $\square$

3E. Computational Evidence

We have proven Conjecture 3.1 in a number of cases besides those of Theorem 3.8. We also have done many computations along the lines of those in Section 2D. To describe these, we use the following compact notation. If a Schubert condition α is repeated k times in some Schubert data, we abbreviate that by α^k. Thus, the conditions of Conjecture 2.1 are written as J_1^{mp}.

Theorem 3.9. *Conjecture* 3.1 *holds for the following Schubert data.*

(i) $(m,p)=(4,2)$, $\alpha^\bullet = J_2^4$. *Here,*

$$d(4,2;J_2^4)=3.$$

(ii) $(m,p)=(3,3)$, $\alpha^\bullet = J_2^4, J_1$. *Here,*

$$d(3,3;J_2^4,J_1)=3.$$

(iii) $(m,p)=(3,3)$, $\alpha^\bullet=(135)^2, J_1^3$. *Here,*

$$d(3,3;(135)^2,J_1^3)=6.$$

(iv) $(m,p)=(4,3)$, $\alpha^\bullet=(135)^4$. *Here,*

$$d(4,3;(135)^4)=8.$$

Proof. We consider a polynomial system with parameters, give a universal eliminant, and show the eliminant has only real roots for distinct values of the parameters. We work in the local parameterization $\mathcal{X}_{\alpha^1,\alpha^2}$ of Section 3B.

(i) Let $(m,p)=(4,2)$ and $\alpha^\bullet = J_2^4$. The equations are

$$\text{maximal minors}\begin{bmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 \\ 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \\ 0 & 0 & 1 & 3s & 6s^2 & 10s^3 \\ 1 & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{25} & x_{26} \end{bmatrix} = 0$$

and the same equations with t replacing s. The ideal of these polynomials contains the following univariate polynomial g of degree $3 = d(4,2,J_2^4)$

$$25x_{12}^3 - 25x_{12}^2(s+t) + x_{12}(19st+6s^2+6t^2) - 3(s^2t+st^2).$$

Its discriminant is a sum of squares with primitive part

$$9(s-t)^6 + 23s^2t^2(s-t)^2 + 9(s^6+t^6).$$

Since $g(x_{12};1,2)$ has roots

$$1 \quad\text{and}\quad 1 \pm \tfrac{1}{5}\sqrt{7},$$

we have shown that g always has real roots, when s and t are distinct.

(ii) Let $m=p=3$ and $\alpha^\bullet = J_2^4, J_1$. Here, $\mathcal{X}_{J_2,J_2}$ consists of all matrics X of the form

$$\begin{bmatrix} 1 & x_{12} & 0 & 0 & 0 & 0 \\ 0 & 1 & x_{23} & x_{24} & x_{25} & 0 \\ 0 & 0 & 0 & 0 & 1 & x_{36} \end{bmatrix}$$

and our equations are

$$\det\begin{bmatrix} K_3(s) \\ X \end{bmatrix} = \text{maximal minors} \begin{bmatrix} K_2(t) \\ X \end{bmatrix} = 0,$$

and the same equations with u replacing t. The ideal of these polynomials contains the following univariate polynomial g, here $e_1 = t+u$ and $e_2 = tu$.

$$\begin{aligned} x_{36}^3 - x_{36}(3s+4e_1) + x_{36}(4e_1^2+3e_2+10se_1) \\ -(6e_1e_2+8se_1^2+se_2) \\ = (x_{36}-2e_1)(x_{36}^2-2e_1x_{36}+3e_2) \\ - s(x_{36}^2-10e_1x_{36}+8e_1^2+e_2). \end{aligned}$$

These last two polynomials have roots

$$e_1 \pm \sqrt{e_1^2-3e_2}, \quad 2e_1, \quad \tfrac{5}{3}e_1 \pm \tfrac{1}{3}\sqrt{e_1^2-3e_2},$$

which are interlaced. For example, if $e_1>0$, then

$$\begin{aligned} e_1 - \sqrt{e_1^2-3e_2} &< \tfrac{5}{3}e_1 - \tfrac{1}{3}\sqrt{e_1^2-3e_2} \\ &< e_1 + \sqrt{e_1^2-3e_2} \\ &< \tfrac{5}{3}e_1 + \tfrac{1}{3}\sqrt{e_1^2-3e_2} < 2e_1. \end{aligned}$$

When s,t,u are distinct and different from 0, g always has 3 real roots, by the Intermediate Value Theorem. We could also note that the discriminant of g

$$\begin{aligned} s^2(t-u)^4 + t^4(s-u)^2 + u^4(s-t)^2 + s^2t^2(s-t)^2 \\ + s^2u^2(s-u)^2 + (s-t)^2(s-u)^2(t-u)^2 \\ + \tfrac{7}{2}\big(s^4(t-u)^2 + t^2(s-u)^4 + u^2(s-t)^4 + t^2u^2(t-u)^2\big) \end{aligned}$$

is a sum of squares and $g(x_{36};1,2,3)$ has approximate roots

$$4.736,\ 7.756,\ 10.508.$$

(iii) Let $(m,p)=(3,3)$ and $\alpha^\bullet = (135)^2, J_1^3$. Here, $\mathcal{X}_{135,135}$ consists of all matrics X of the form

$$\begin{bmatrix} 1 & x_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_{36} \end{bmatrix}$$

and our equations are

$$\det\begin{bmatrix} K_3(s) \\ X \end{bmatrix} = \det\begin{bmatrix} K_3(t) \\ X \end{bmatrix} = \det\begin{bmatrix} K_3(u) \\ X \end{bmatrix} = 0.$$

We write the universal eliminant, $g(x_{36})$, in terms of the elementary symmetric polynomials in s,t,u

$$9x_{36}^6 - 48e_1x_{36}^5 + (64e_1^2 + 108e_2)x_{36}^4 - (288e_1e_2 - 198e_3)x_{36}^3 + (320e_2^2 + 540e_1e_3)x_{36}^2 - 1200e_2e_3x_{36} + 1125e_3^2.$$

Evaluating the parameters (s,t,u) at $(1,2,3)$, we see that $g(x_{36};1,2,3)$ has approximate roots

$$1.491,\ 1.683,\ 3.210,\ 5.630,\ 9.213,\ 10.773.$$

The discriminant of g is a sum of squares. The primitive part of the discriminant is

$$e_3^4(4e_2^2e_1^2 - 15e_3e_1^3 - 15e_2^3 + 63e_3e_2e_1 - 81e_3^2) \times(256e_2^2e_1^2 - 768e_3e_1^3 - 768e_2^3 + 2592e_3e_2e_1 - 2187e_3^2)^2.$$

The second factor is the sum of squares

$$\tfrac{7}{2}(s-t)^2(s-u)^2(t-u)^2 + \tfrac{1}{2}s^2\big((t-u)^4 + t^2(s-u)^4 + u^2(s-t)^4\big).$$

Interestingly, the last (squared) factor is itself a sum of squares

$$112(s-t)^2(u^4+s^2t^2) + 112(t-u)^2(s^4+t^2u^2) + 112(u-s)^2(t^4+s^2u^2) + 16(s-t)^2(s-u)^2(t-u)^2 + 309s^2t^2u^2 + 16\big(s^4(t^2+u^2) + t^4(s^2+u^2) + u^4(t^2+u^2)\big).$$

(iv) Let $(m,p) = (4,3)$ and $\alpha^\bullet = (135)^4$. Here, $\mathcal{X}_{135,135}$ consists of all matrics X of the form

$$\begin{bmatrix} 1 & x_{12} & x_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_{24} & x_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_{36} & x_{37} \end{bmatrix}$$

and our equations are

$$\text{maximal minors}\begin{bmatrix} K_3(s) \\ X \end{bmatrix} = \text{maximal minors}\begin{bmatrix} K_5(s) \\ X \end{bmatrix} = 0,$$

and the same equations with t replacing s. In this case, the universal eliminant has 4 quadratic factors:

$$36x_{12}^2 - x_{12}(12t + 30s) + 6st + 5s^2,$$
$$36x_{12}^2 - x_{12}(12s + 30t) + 6st + 5t^2,$$
$$3x_{12}^2 - 2x_{12}(s+t) + st,$$
$$36x_{12}^2 - 30x_{12}(s+t) + 5t^2 + 14st + 5s^2.$$

When $s \neq t$ and neither is zero, we see that each has 2 real roots. □

Observe that in all 4 cases, the discriminant was a sum of squares and the eliminant has the correct number of real roots for distinct values of the parameters. Of particular note is that the system in (ii) is not symmetric in the parameters and the Schubert data of (iv) is not Pieri Schubert data.

There are several other cases for which these methods may work. There are 6 2-planes in $\mathbb{C}^2$ which meet 5 general 4-planes nontrivially, as $d(5,2;J_2^5) = 6$. Using the 6-dimensional system of local coordinates $\mathcal{X}_{14,14}$, we can compute a degree 6 eliminant in the variable x_{25}, and parameters s,t,u of the points of osculation of three flags. The discriminant has 388 terms and degree 30 in the parameters s,t,u. By the calculations in the first column of Table 3 below, Conjecture 3.1 would hold for these Schubert data, if this discriminant is a sum of squares or more generally, if it is positive semidefinite.

Another case is when $(m,p) = (4,2)$ and the Schubert data is J_2^2, J_1^4. Here $d(4,2;J_2^2,J_1^4) = 6$. Using the 4-dimensional system of local coordinates $\mathcal{X}_{14,14}$, we compute a degree 6 universal eliminant in the variable x_{25} and parameters s,t,u,v as before. The discriminant has 3 factors, 2 are the same cubic form, while the third has 1289 terms and degree 24 in the parameters s,t,u,v. We also check that there are 6 real roots of the eliminant for parameter values $1,2,3,4$, so Conjecture 3.1 would hold for these Schubert data, if this discriminant is a sum of squares.

Table 3 gives the number of instances of Conjecture 3.1 we have checked.

4. TOTAL POSITIVITY

Previous sections have dealt with Schubert conditions given by flags osculating a real rational normal curve. Recently, Shapiro and Shapiro have conjectured that a generalization of this choice involving totally positive real matrices would also give only real solutions. We describe that here, prove the first nontrivial instance, and present some computational evidence in support of this generalization.

A real upper triangular matrix g with 1's on its diagonal is *totally positive* if every minor of g is positive, except those minors which vanish on all upper triangular matrices. Let $\mathcal{TP}$ be the set of all totally

$\alpha^\bullet$	$(J_2)^5$	$(J_2)^6$	$(J_2)^7$	$(J_2)^6$	$(J_3)^5$	$(135)^5$	$(135)^2, (J_1)^6$
(m,p)	$(5,2)$	$(6,2)$	$(7,2)$	$(5,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$d(m,p;\alpha^\bullet)$	6	15	36	6	16	32	61
# checked	10000	2821	504	10160	2002	400	294

TABLE 3. General Schubert data tested.

positive, a multiplicative semigroup. Define a partial order on real flags $F_\bullet$ by $F_\bullet < gF_\bullet$ if $g \in \mathcal{TP}$.

Conjecture 4.1 (Shapiro and Shapiro). *For any $m, p > 1$, let $\alpha^\bullet$ be Schubert data for* Grass$(p, m+p)$. *If $F_\bullet{}^1 < \cdots < F_\bullet{}^n$ are real flags, then the Schubert varieties $\Omega_{\alpha^1} F_\bullet{}^1, \ldots, \Omega_{\alpha^n} F_\bullet{}^n$ intersect transversally, with all points of intersection real.*

We will prove Conjecture 4.1 in the first nontrivial case of $m = p = 2$. First, we relate Conjecture 4.1 to Conjecture 3.1. Let $K_\bullet(s)$ be the square matrix of size $(m+p)$ whose (i,j)-th entry is $\binom{j-i}{i-1} s^{j-i}$ (compare (1–1)). If $s > 0$, then $K_\bullet(s)$ is totally positive and for any s, t we have $K_\bullet(s) \cdot K_\bullet(t) = K_\bullet(s+t)$. To see this, first recall that $\mathcal{TP}$ is generated as a semigroup by $\exp(E_{i,i+1})$, where $E_{i,i+1}$ is the elementary matrix whose only nonzero entry is in position $i, i+1$ [Loewner 1955]. These assertions follow from the observation that

$$K_\bullet(s) = \exp(sN),$$

where N is the nilpotent matrix whose only nonzero entries are $(1, 2, \ldots, m+p-1)$ lying just above its main diagonal.

Theorem 3.3 holds in this new setting. For this, we alter the notion of Pieri Schubert data $\alpha^\bullet$ to Schubert data $\alpha^1, \ldots, \alpha^n$ where all except possibly α^1 and α^n are Pieri conditions.

Theorem 4.2. *Let $a, b > 1$ and suppose that Conjecture* 4.1 *holds for $(m,p) = (a,b)$ and Schubert data $\alpha^\bullet = (J_1)^{mp}$. Then Conjecture* 4.1 *holds for any Pieri Schubert data for* Grass$(p, m+p)$ *where $(m,p) \le (a,b)$ or (b,a) coordinatewise.*

Proof. The arguments used to prove Theorem 3.3 work here with minor adjustments.

We first remark that total positivity, and hence our order $<$ on real flags, is defined with respect to a choice of ordered basis for $\mathbb{R}^{m+p}$. Suppose that $e_1, \ldots, e_{m+p}$ is the basis we used to define this order. Then $F_\bullet < G_\bullet$ is and only if $G_\bullet <' F_\bullet$, where $<'$ is defined with respect to the basis $e_1, -e_2, e_3, -e_4, \ldots$. Similarly, if we have an inner product on $\mathbb{R}^{m+p}$ so that the basis $e_1, \ldots, e_{m+p}$ is orthonormal, then $F_\bullet < G_\bullet$ if and only if $F_\bullet^\perp <'' G_\bullet^\perp$, where $<''$ is defined with respect to the basis in reverse order $e_{m+p}, \ldots, e_2, e_1$. Thus

$$F_\bullet^1 < F_\bullet^2 < \cdots < F_\bullet^n \iff F_\bullet^n <' \cdots <' F_\bullet^2 <' F_\bullet^1$$

so that Conjecture 4.1 holds for Schubert data $\alpha^\bullet$ if and only if it holds for the data in reverse order. (This is the only rearrangment we used in the proof of Lemma 3.7.) Similarly, the analogue of Lemma 3.7(i) holds. For the analogue of Lemma 3.7(ii), permute the last two Schubert conditions, so that $\beta^\bullet$ is still Pieri Schubert data, in our new, restricted definition.

Finally, in the proof of Lemma 3.7(iii), replace $s_3, \ldots, s_n$ in defining the set $\mathcal{O}$ by fixing $F_\bullet^1$ to be the standard flag represented by the matrix I_{m+p} and let $\mathcal{O}$ be the set of all

$$F_\bullet^1 < \cdots < F_\bullet^n$$

for which the appropriate transversality conditions hold. Since $\mathcal{TP}$ is open, it follows that there exists $\varepsilon > 0$ and totally positive matrix M (which stabilizes $F_\bullet^1$) such that if $0 < s < \varepsilon$, then $F_\bullet^1 < M \cdot K_\bullet(s) \cdot F_\bullet^1 < F_\bullet^2$. Then the same arguments used to prove Theorem 3.3 suffice. In particular, the analog of Proposition 3.2 also holds in this setting. □

Totally positive matrices have a useful description. Let $\mathcal{U}$ be the group of real unipotent (upper triangular) matrices. Then $\mathcal{TP}$ is a connected component of the complement of a hypersurface H $\mathcal{U}$ defined by the vanishing of all minors consisting of the first i rows and last i columns [Shapiro and Shapiro 1995]. This has a geometric description.

Associating a matrix to a flag as in Section 3B, we may identify $\mathcal{U}$ with a Zariski open subset of the real flag manifold. Then the hypersurface H is the union of all positive codimension Schubert varieties defined by the flag determined by the identity matrix.

Given a matrix $M \in \mathcal{U}$, the translate $\mathcal{TP}.M$ is a component of the complement of all Schubert varieties of positive codimension defined by the flag given by M. Similarly, given a totally positive matrix M, the set of upper triangular matrices N for which there exists a totally positive g with $gN = M$ is the component of this complement containing the identity matrix.

Let $F_\bullet^1 < \cdots < F_\bullet^n$ be real flags. Using a real automorphism of the flag manifold, we may assume that $F_\bullet^1 = K_\bullet(0) = I_{m+p}$. Then $F_\bullet^2, \ldots, F_\bullet^n \in \mathcal{TP}$, since they are all translates of the identity by totally positive matrices. Also, $F_\bullet^1, \ldots, F_\bullet^{n-1}$ are in the same component of the complement of all positive dimensional Schubert cells defined by $F_\bullet^n$. If we now consider a real coordinate transformation fixing $F_\bullet^1$, but with $F_\bullet^n$ becoming $K_\bullet(\infty)$, then this complement becomes $\mathcal{TP}$, in these new coordinates.

Thus we may work in the local coordinates $\mathcal{X} := \mathcal{X}_{\alpha^1,\alpha^n}$. We do this in our proof of the following theorem and in subsequent calculations.

Theorem 4.3. *Conjecture* 4.1 *holds for $m = p = 2$ and Schubert data $(J_1)^4$.*

Proof. Let $F, G \in \mathcal{TP}$ be totally positive matrices and set $H = G \cdot F$. When $m = p = 2$, $\mathcal{X} = \mathcal{X}_{J_1,J_1}$ is the set of matrices

$$\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}.$$

For a matrix L, let L_{ij} denote the 2×2-minor of L given by the first two rows and columns i and j. Then the equations for a 2-plane in $\mathcal{X}$ to meet the flags given by F and H are

$$f := F_{24} - bF_{23} - aF_{14} + abF_{13},$$
$$h := H_{24} - bH_{23} - aH_{14} + abH_{13}.$$

The lexicographic Gröbner basis for this (assuming $a < b$) is

$$H_{13}f - F_{13}h = J_{14} - bJ_{24} - aJ_{34},$$
$$(H_{14} - bH_{13})f - (F_{14} - bF_{13})h$$
$$= J_{13} - b(J_{23} + J_{14}) + b^2 J_{24},$$

where J_{ij} is the (i,j)-th minor of the matrix

$$\begin{bmatrix} F_{24} & F_{23} & F_{14} & F_{13} \\ H_{24} & H_{23} & H_{14} & H_{13} \end{bmatrix}.$$

We may write the the discriminant of the quadratic equation for b as follows

$$(J_{23} + J_{14})^2 - 4J_{13}J_{24} = (L_{23} + L_{14})^2 - 4L_{13}L_{24},$$

where L is the matrix

$$\begin{bmatrix} F_{13} & F_{14} & H_{13} & H_{14} \\ F_{23} & F_{24} & H_{23} & H_{24} \end{bmatrix}.$$

Thus we will have two real roots for our original system if and only if

$$\Lambda(B) := L_{13} - B(L_{23} + L_{14}) + B^2 L_{24} = 0$$

has 2 real solutions. Painstaking calculations reveal that $\Lambda(1) = -G_{12}G_{34} < 0$. Since $L_{24} = H_{13}H_{24} - H_{23}H_{14} = H_{12}H_{34}$ by the Plücker relations, we see that $L_{24} > 0$ and so $\Lambda(B) = 0$ will have 2 real solutions. □

Table 4 shows the number of instances of Conjecture 4.1 that we have verified.

5. FURTHER REMARKS

We present a counterexample to the original conjecture of Shapiro and Shapiro and close with a discussion of further questions.

5A. A Counterexample to the Original Conjecture

The original conjecture of Shapiro and Shapiro concerned the M-property for flag manifolds [Shapiro and Shapiro 1992]. An algebraic set X defined over $\mathbb{R}$ has the M-property if the sum of the $\mathbb{Z}/2\mathbb{Z}$-Betti numbers of $X(\mathbb{R})$ and of $X(\mathbb{C})$ are equal. Shapiro and Shapiro conjectured that an intersection of Schubert cells in a flag manifold has the M-property, if the cells are defined by flags osculating the rational

$\alpha^\bullet$	$(J_1)^6$	$(J_2)^5$	$(135)^4$	$(J_1)^8$	$(J_2)^6$	$(135)(136)(J_1)^5$
(m,p)	$(3,2)$	$(5,2)$	$(4,3)$	$(4,2)$	$(6,2)$	$(4,3)$
d	5	6	8	14	15	25
# checked	12000	4000	4000	1500	300	150

TABLE 4. Instances checked.

normal curve at real points. When such an intersection is zero-dimensional all of its points are real. It is this consequence we have been studying.

While there is much evidence in support of this conjecture for zero dimensional intersections in a Grassmannian (Conjectures 2.1, 3.1, and 4.1), it does not hold for more general flag manifolds. In fact, we give a counter example in the simplest enumerative problem in a flag manifold that does not reduce to an enumerative problem in a Grassmannian.

Counterexample 5.1. Consider the manifold $\mathbb{F}(2,3;5)$ consisting of partial flags

$$X \subset Y$$

in $\mathbb{C}^5$ with $\dim X = 2$ and $\dim Y = 3$. This manifold has dimension 8; the projection to $\mathrm{Grass}(2,5)$ has fibre over a 2-plane X equal to

$$\mathbb{P}(\mathbb{C}^5/X) \simeq \mathbb{P}^2.$$

Given general 2-planes a, b, c and general 3-planes A, B, C, there are 4 flags $X \subset Y$ satisfying the following conditions:

1. X meets a, B, and C nontrivially, and
2. $\dim Y \cap A \geq 2$ and Y meets b and c nontrivially.

That this number is 4 may be verified using the Schubert calculus for a flag manifold [Fulton 1997] or the equations we give below.

Let $K_\bullet(s)$ be the flag of subspaces osculating the standard rational normal curve. Set

$$\begin{aligned} a &:= K_2(4), & A &:= K_3(0) \\ b &:= K_2(1), & B &:= K_3(3) \\ c &:= K_2(-5), & C &:= K_3(-1) \end{aligned}$$

We claim that of the 4 flags $X \subset Y$ satisfying conditions 1 and 2 above for this choice of a, b, c, A, B, C, 2 are real and 2 are complex.

We outline the computation. Choose local coordinates for $\mathbb{F}(2,3;5)$ as follows. Let Y be the row space of the 3×5-matrix

$$\begin{bmatrix} 0 & 0 & 1 & x_{14} & x_{15} \\ 1 & 0 & x_{23} & x_{24} & x_{25} \\ 0 & 1 & x_{33} & x_{34} & x_{35} \end{bmatrix}$$

and let X be the row space of its last 2 rows. We seek the solutions to the overdetermined system of polynomials

$$\begin{aligned} \det \begin{bmatrix} K_2(1) \\ Y \end{bmatrix} &= \det \begin{bmatrix} K_2(-5) \\ Y \end{bmatrix} \\ &= \det \begin{bmatrix} K_3(3) \\ X \end{bmatrix} = \det \begin{bmatrix} K_3(-1) \\ X \end{bmatrix} \\ &= \text{maximal minors } \begin{bmatrix} K_2(4) \\ X \end{bmatrix} \\ &= \text{maximal minors } \begin{bmatrix} K_3(0) \\ Y \end{bmatrix} = 0. \end{aligned}$$

These polynomials generate a zero-dimensional ideal containing the univariate polynomial

$$27063 - 117556x_{14} - 5952x_{14}^2 - 10416x_{14}^3 + 32400x_{14}^4,$$

which is part of a lexicographic Gröbner basis satisfying the Shape Lemma. This polynomial has approximate roots

$$-.736 \pm 1.30\sqrt{-1},\ .227,\ 1.62.$$

Thus 2 of the flags are complex.

5B. Further Questions

While Counterexample 5.1 shows that we cannot guarantee all points of intersection real when the Schubert varieties are given by flags osculating a real rational normal curve, a number of questions remain (besides the resolution of the conjectures of the previous sections). There remains the original question of Fulton.

Question 1. Given Schubert data for a flag manifold, do there exist real flags in general position whose corresponding Schubert varieties have *only* real points of intersection?

In every case we know, this does happen. For instance, if we change the 3-plane B to $K_3(2)$ in Counterexample 5.1, then all 4 solution flags are real. There is also the following result, showing this holds in infinitely many cases. A Grassmannian Schubert condition is a Schubert condition on a flag which only imposes conditions on one of the subspaces. We likewise define Grassmannian Schubert data. For example, Counterexample 5.1 involves Grassmannian Schubert data. Let $\mathbb{F}(2,\, n{-}2;\, n)$ be the manifold of flags $X \subset Y$ in $\mathbb{C}^n$ where $\dim X = 2$ and $\dim Y = n - 2$.

Proposition 5.2 [Sottile 1997c, Theorem 13]. *Given any Grassmannian Schubert data for $\mathbb{F}(2, n-2; n)$, there exist real flags whose corresponding Schubert varieties meet transversally with all points of intersection real.*

The beauty of the conjectures of Shapiro and Shapiro is that they give a simple algorithm for selecting the flags defining the Schubert varieties.

Question 2. Can the choice of flags in Question 1 (or Proposition 5.2) be made effective? In particular, is there an algorithm for selecting these flags?

While computing the examples described here, we have made a number of observations which deserve further scrutiny. These concern eliminant polynomials in the ideals defining the intersections of Schubert varieties in the local coordinates we have been using.

Suppose we have Schubert data $\alpha^\bullet$, and have chosen local coordinates either for the Grassmannian or are working in $\mathcal{X}_{\alpha^n,\alpha^{n-1}}$. Conjecture 3.1 or 4.1 may be formulated in terms of a parameterized system of polynomials with parameters either $s_1,\ldots,s_n$ in the case of Conjecture 3.1 or $(n-1)$-tuples of totally positive matrices (or in terms of some parameterization of $\mathcal{TP}$ [Berenstein et al. 1996]). For each of the coordinates, the ideal of this system contains a universal eliminant, which is the minimal univariate polynomial in that coordinate with coefficients rational functions in the parameters.

We ask the following questions about the eliminant.

Question 3. Does the universal eliminant have degree equal to the generic number of solutions? That is, do generic solutions satisfy the shape lemma?

Question 4. Let Δ be the discriminant of the polynomial system, a polynomial in the parameters which vanishes when there are solutions with multiplicities.

(a) Is the locus $\Delta \neq 0$ connected?
(b) In the case of Conjecture 3.1, where Δ is a polynomial in the parameters $s_1,\ldots,s_n$, is Δ always a sum of squares of polynomials?
(c) If so, are these polynomials monomials in the s_i and their differences $(s_i - s_j)$? This would imply that the polynomial systems are always multiplicity-free for distinct real values of the parameters, and hence the stronger version of Theorem 3.3 mentioned in Remark 3.4.

The discriminants we have computed for instances of the conjectures for the Grassmannian (including the discriminant for system of Theorem 4.3) are always nonnegative when the parameters are distinct. For the case of Counterexample 5.1, we computed a discriminant for a simpler, but equivalent system, in the spirit of sections 2E and 3B. This polynomial in parameters s_1, s_2, t_1, t_2 is symmetric in the s's and in the t's separately (and in the transformation $s_i \leftrightarrow t_i$) and has degree 24. It has three factors, the first of degree 20 with 857 terms, and the square

$$\big(2s_1s_2 + 2t_1t_2 - (s_1+s_2)(t_1+t_2)\big)^2.$$

While this factor will not prevent the discriminant from being a sum of squares, this factor shows that there is a choice of distinct parameters for which the discriminant vanishes. Indeed, if we set $s_1 = 3, s_2 = 6, t_1 = 9$, and $t_2 = 5$, then this factor vanishes, and the resulting system has a root of multiplicity 2. This also explains why different values of the parameters in Counterexample 5.1 give different numbers of real and complex solutions.

Question 5. When the universal eliminant factors over $\mathbb{Z}$, it reflects either some underlying geometry or some interesting arithmetic. More generally, one might ask about the Galois group of these enumerative problems [Harris 1979], or the Galois group of the universal eliminant. For instance, is it the full symmetric group? That is not always the case, as the example of Theorem 3.9(iv) shows.

Question 6. In many cases with the substitution of $s_i = i$, the eliminant factors over the integers. This happens in Conjecture 1.1, Theorem 2.3, Theorem 3.9(i) and (iv), and in other cases. Table 5 lists the degrees of the factors in the case of Conjecture 1.1. Why does this choice of $s_i = i$ induce a factorization? Is there any special geometry or interesting arithmetic here? If 2 parameters are allowed to come together, then the resulting ideal factors in a way respecting the product of Schubert classes, by the Corollary to Theorem 1 in [Eisenbud and Harris 1987]. From the Schubert calculus, we would expect factors of 9 and 5 for $(m, p) = (2, 4)$, 14 and 28 for

(m,p)	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$	$(7,2)$	$(3,3)$	$(3,4)$
$d_{m,p}$	5	14	42	132	429	42	462
factors	$2,3$	$6,8$	$10,32$	$20,112$	—	$6,36$	$16,30,416$

TABLE 5. Factorization of the eliminant.

$(m,p) = (2,5)$, and 21 and 21 for $(m,p) = (3,3)$, but these do not appear in Table 5.

5C. Further Developments

Since this paper was written, we have found further evidence in support of these conjectures of Shapiro and Shapiro, and also more examples of enumerative problems that are known that may have all their solutions real. In [Sottile 1999b], we show there is a choice of $s_1, \ldots, s_{mp}$ in Conjecture 2.1 for which all $d_{m,p}$ p-planes are real. More generally, the main result of that paper is that for Pieri Schubert data in Conjecture 3.1, there is a choice of $s_1, \ldots, s_n$ for which all p-planes in the transverse intersection (3–2) are real.

We have also answered Question 1 affirmatively for Grassmannian Schubert data where each condition comes from a Pieri Schubert condition on a Grassmannian [Sottile 2000b]. Similarly, a large class of enumerative problems arising in the quantum cohomology of flag manifolds (and related to systems theory) may have all their solutions be real [Sottile 2000a]. The method of proof in these cases is related to the methods used to establish Theorem 3.3 and also to the homotopy continuation algorithms of [Huber et al. 1998]. In a related development, Dietmaier has shown that all 40 positions of the Stewart platform in robotics may be real [Dietmaier 1998].

A consequence of [Sottile 1999b] is that Conjecture 3.1 follows from the stronger version of Proposition 3.2 mentioned in Remark 3.4. While all this bolsters our conviction that these conjectures are true, they are still open. All of these results, and the evidence for these conjectures of Shapiro and Shapiro presented here, do show that there should be a broader theory of real enumerative geometry to explain these phenomena.

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