A Chaotic Cousin of Conway's Recursive Sequence

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CONTENTS

I introduce the recurrence D(n) = D(D(n−1))+D(n−1−D(n−2)), D(1) = D(2) = 1, and study it by means of computer experiments. The definition of D(n) has some similarity to that of Conway's sequence defined by a(n) = a(a(n−1)) + a(n − a(n−1)), a(1) = a(2) = 1. However, unlike the completely regular and predictable behaviour of a(n), the D-numbers exhibit chaotic patterns. In its statistical properties, the D-sequence shows striking similarities with Hofstadter's Q(n)-sequence, given by Q(n) = Q(n − Q(n−1)) + Q(n − Q(n−2)), Q(1) = Q(2) = 1. Compared to the Hofstadter sequence, D shows higher structural order. It is organized in well-defined "generations", separated by smooth and predictable regions. The article is complemented by a study of two further recurrence relations with definitions similar to those of the Q-numbers. There is some evidence that the different sequences studied share a universality class.

INTRODUCTION

The recursion relation

$$Q(n)=Q(n-Q(n-1))+Q(n-Q(n-2)) \text{ for } n>2,$$
$$Q(1)=Q(2)=1,$$

introduced by D. R. Hofstadter [1979], is a challenge [Guy 1981, Problem E31]. Its apparently chaotic behaviour (see Figure 1) is far from being understood. There appear to be no rigorous results about the behaviour of $Q(n)$.

In [Pinn 1999] I reported a number of empirical observations on the Q-numbers. The main conclusions were:

- The sequence shows signs of order. It is organized in "generations", making up for a Fibonacci-type structure on a logarithmic scale.
- The variance of fluctuations around $n/2$ grows like n^α, with $\alpha = 0.88 \pm 0.01$.
- $R(n) = (Q(n) - n/2)/n^\alpha$ has a strongly non-Gaussian probability density p^*.
- There is scaling: $x_m = R(n) - R(n-m)$ is distributed according to $\lambda_m p^*(x_m/\lambda_m)$. The rescaling factor λ_m converges exponentially fast to $\sqrt{2}$

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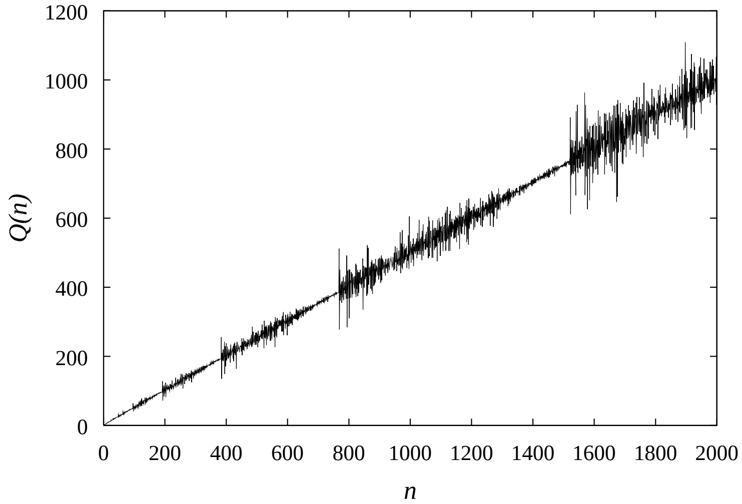

FIGURE 1. Graph of $Q(n)$.

for large m, i.e., $\lambda_m \sim \exp(-m/\xi)$, with a decay length $\xi = 3$.

It is an interesting question whether similar observations can be made on other integer recurrences. In this paper, I introduce and study the recurrence

$$D(n) = D(D(n-1)) + D(n-1-D(n-2)) \text{ for } n>2,$$
$$D(1) = D(2) = 1.$$

Its definition is not too different from that of Conway's sequence $a(n)$, defined by

$$a(n) = a(a(n-1)) + a(n-a(n-1)) \text{ for } n > 2,$$
$$a(1) = a(2) = 1.$$

The a-sequence has been investigated by Hofstadter, Conway and others at various times since about 1975 [Conolly 1989; Hofstadter 1988]. Conway discovered many of its properties. A cash prize that he offered for information about its asymptotic growth was won by Mallows [Mallows 1991]. See also [Kubo and Vakil 1996] for a detailed study of $a(n)$.

Conway's sequence has a lot of fascinating properties. However, it behaves in a regular and completely predictable way. In contrast, $D(n)$ develops chaotic and irregular patterns, separated by smooth and predictable regions. The latter property underlines its close relation to the $a(n)$ function.

In Section 1, I formulate a conjecture about the "genealogy" of the D-numbers. The statements of the conjecture have been confirmed with the help of a computer for the first 2^{26} terms of the sequence. A proof is still lacking. Section 2 is about some striking similarities in the behaviours of $a(n)$ and $D(n)$, allowing for a kind of "marriage" of the two sequences. Section 3 reports on empirical observations of mainly statistical properties of $D(n)$, like step size distribution, scaling properties, and frequency counting. Section 4 complements the study of the D-sequence by empirical investigations of two further chaotic recurrences that might be called chaotic cousins of the Hofstadter sequence. It appears that all sequences studied share various statistical properties. This suggests they could belong to a common universality class. Because of its clear structure the D-sequence seems to be a natural candidate for rigorous studies of this class.

1. CONJECTURE ABOUT THE GENEALOGY OF D(n)

Figure 2 shows the first 2048 terms of the a- and D-sequences. Both a and D are organized in "generations" of increasing length and stay in some neighbourhood of $n/2$. These facts become even more obvious when looking at $2a(n) - n$ and $2D(n) - n$; see Figure 3.

To make the "genealogy" more precise, we define a generation number $g(n)$ for each $n \geq 1$ by

$$g(n) = \begin{cases} 0 & \text{if } n = 1, \\ k & \text{if } 2^{k-1} < n \leq 2^k \text{ for } n > 1. \end{cases}$$

This can also be written as $g(n) = \lceil \log_2 n \rceil$, where $\log_2$ denotes the logarithm with respect to base 2, and $\lceil x \rceil$ is the smallest integer greater than or equal to x. As in [Pinn 1999], we interpret $D(n)$ as the sum of its *mother* at position $D(n_1)$ and its *father* at $D(n_2)$, with

$$n_1 = D(n-1),$$
$$n_2 = n-1-D(n-2).$$

Table 1 shows the structure of the generations and the genealogy. An inspection of an extended version of Table 1 suggests the following conjecture (confirmed with the help of a computer for $k \leq 26$):

Conjecture. For generation k, with $k \geq 5$, the following properties hold:

C1. For the first $k-2$ members the function D takes the value 2^{k-2}. The $(k-1)$th element is $2^{k-2}+1$. (The first $k-1$ members of a generation will be called the *head*.)

C2. For the last $k-2$ members the function D takes the value 2^{k-1}. The element just before the last

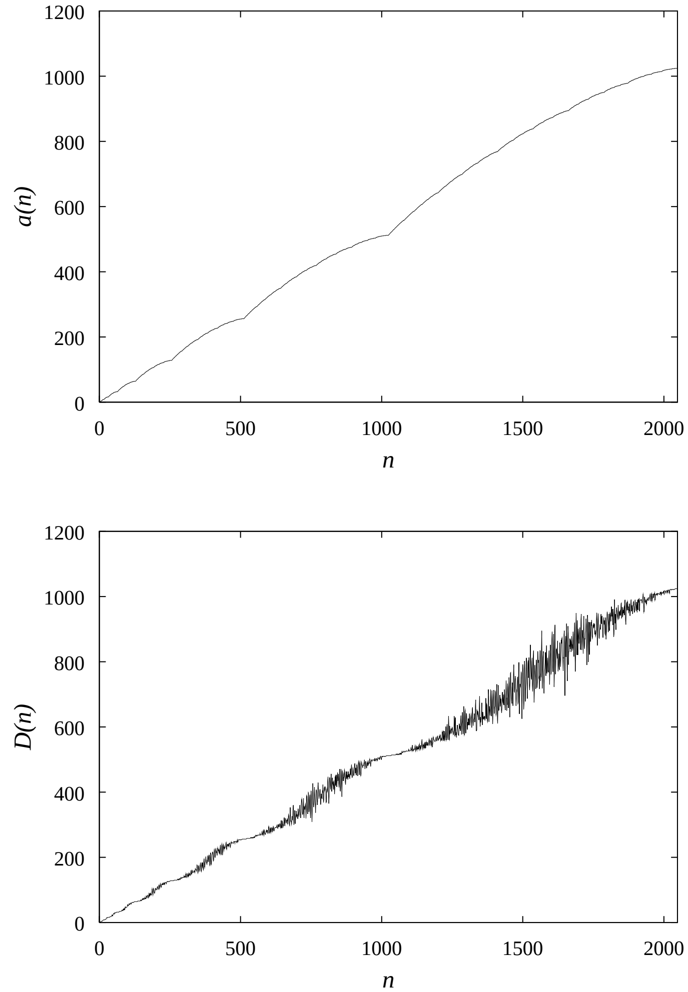

FIGURE 2. Graphs of $a(n)$ and $D(n)$.

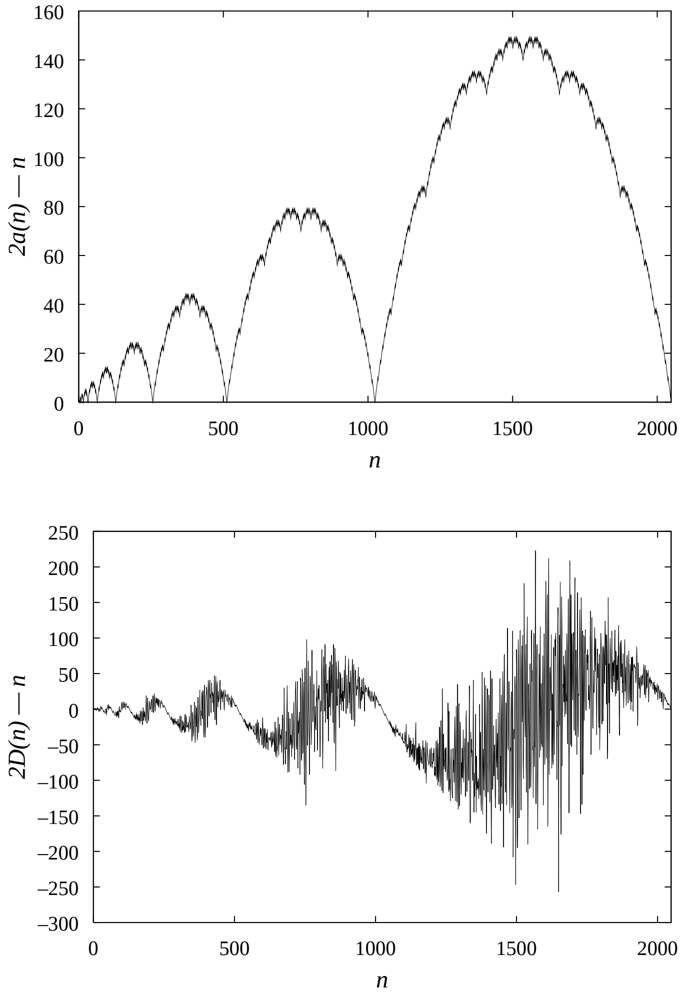

FIGURE 3. Graphs of $2a(n) - n$ and $2D(n) - n$.

$k-2$ members has value $2^{k-1}-1$. We will call the last $k-1$ members of a generation its *tail.*

C3. The last member of generation $k-2$ is simultaneously the mother of all head members of generation k and the father of the first head member of generation k. The fathers of the remaining head members are (in ascending order) the members of the head of generation $k-1$.

C4. The parents of the tail members are tail members of generation $k-1$.

C5. The values of $D(n)$ lie in the range $[2^{k-2}, 2^{k-1}]$.

2. MARRIAGE OF a(n) AND D(n)

An interesting observation can be made when one plots together $D(n)$ and $a(n)$. Figure 4 shows the function $2D(n)-n$, together with $\pm(2a(n)-n)$. The latter two functions nicely model the "outer" boundary of the fluctuating $D(n)$ in some neighbourhood of the generation boundaries.

The close relation of a and D is also underlined by the following experiment: Use the a-recurrence of Conway to generate the first k generations of numbers. Then continue with the recursion relation of the D-numbers. The resulting function $\beta_k(n)$ is thus given by

$$\begin{aligned} \beta_k(\beta_k(n-1)) + \beta_k(n-1-\beta_k(n-2)) \quad & \text{for } g(n) > k, \\ a(n) \quad & \text{for } g(n) \leq k. \end{aligned}$$

Graphs illustrating the behaviour of $2\beta_k(n) - n$ are shown in Figure 5, for k from 7 to 10. With increasing k the "chaotic" fluctuations get reduced and the function becomes very similar to $a(n)$. It seems that one can in this way generate a large family of sequences with different "levels of chaos".

k	n	n_1	$g(n_1)$	n_2	$g(n_2)$	$D(n)$
2	3	1	0	1	0	2
	4	2	1	2	1	2
3	5	2	1	2	1	2
	6	2	1	3	2	3
	7	3	2	4	2	4
	8	4	2	4	2	4
4	9	4	2	4	2	4
	10	4	2	5	3	4
	11	4	2	6	3	5
	12	5	3	7	3	6
	13	6	3	7	3	7
	14	7	3	7	3	8
	15	8	3	7	3	8
	16	8	3	7	3	8
5	17	8	3	8	3	8
	18	8	3	9	4	8
	19	8	3	10	4	8
	20	8	3	11	4	9
	21	9	4	12	4	10
	22	10	4	12	4	10
	23	10	4	12	4	10
	24	10	4	13	4	11
	25	11	4	14	4	13
	26	13	4	14	4	15
	27	15	4	13	4	15
	28	15	4	12	4	14
	29	14	4	13	4	15
	30	15	4	15	4	16
	31	16	4	15	4	16
	32	16	4	15	4	16

k	n	n_1	$g(n_1)$	n_2	$g(n_2)$	$D(n)$
6	33	16	4	16	4	16
	34	16	4	17	5	16
	35	16	4	18	5	16
	36	16	4	19	5	16
	37	16	4	20	5	17
	38	17	5	21	5	18
	39	18	5	21	5	18
	40	18	5	21	5	18
	41	18	5	22	5	18
	42	18	5	23	5	18
	43	18	5	24	5	19
	44	19	5	25	5	21
	45	21	5	25	5	23
	46	23	5	24	5	21
	47	21	5	23	5	20
	48	20	5	26	5	24
	49	24	5	28	5	25
	50	25	5	25	5	26
	51	26	5	25	5	28
	52	28	5	25	5	27
	53	27	5	24	5	26
	54	26	5	26	5	30
	55	30	5	28	5	30
	56	30	5	25	5	29
	57	29	5	26	5	30
	58	30	5	28	5	30
	59	30	5	28	5	30
	60	30	5	29	5	31
	61	31	5	30	5	32
	62	32	5	30	5	32
	63	32	5	30	5	32
	64	32	5	31	5	32

TABLE 1. Genealogy of the D-sequence. The head, body and tail of a generation are separated by horizontal lines. Note that tails are not defined for $k < 5$.

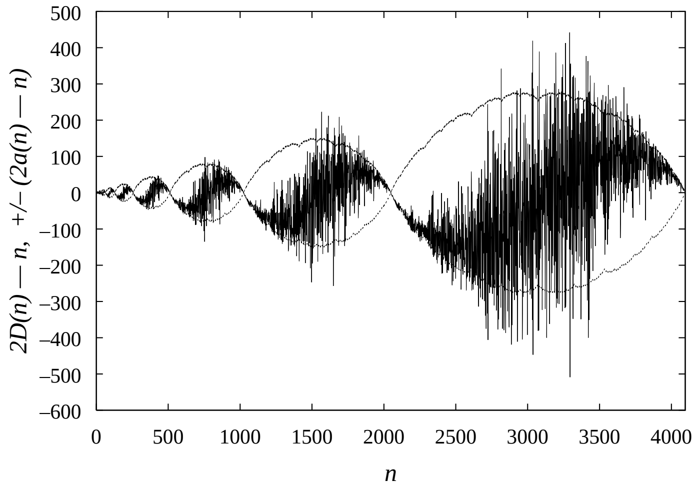

FIGURE 4. Graph of $2D(n) - n$ and $\pm(2a(n) - n)$.

3. EMPIRICAL INVESTIGATION OF STATISTICAL PROPERTIES

In generation k, that is, for $2^{k-1} < n \le 2^k$, the function $D(n)$ takes values in the range $2^{k-2} < n \le 2^{k-1}$. It seems natural to plot $y = D(n)/2^{k-1}$ in terms of

$$x = \frac{n - 2^{k-1}}{2^{k-1}}.$$

We have $0 < x \le 1$, and $y \le 0.5 \le 1$. Plots of this type for generations 6 to 13 are shown in Figure 6. The similarity of the graphs suggests that there could be some statistical properties becoming independent of k when k becomes large.

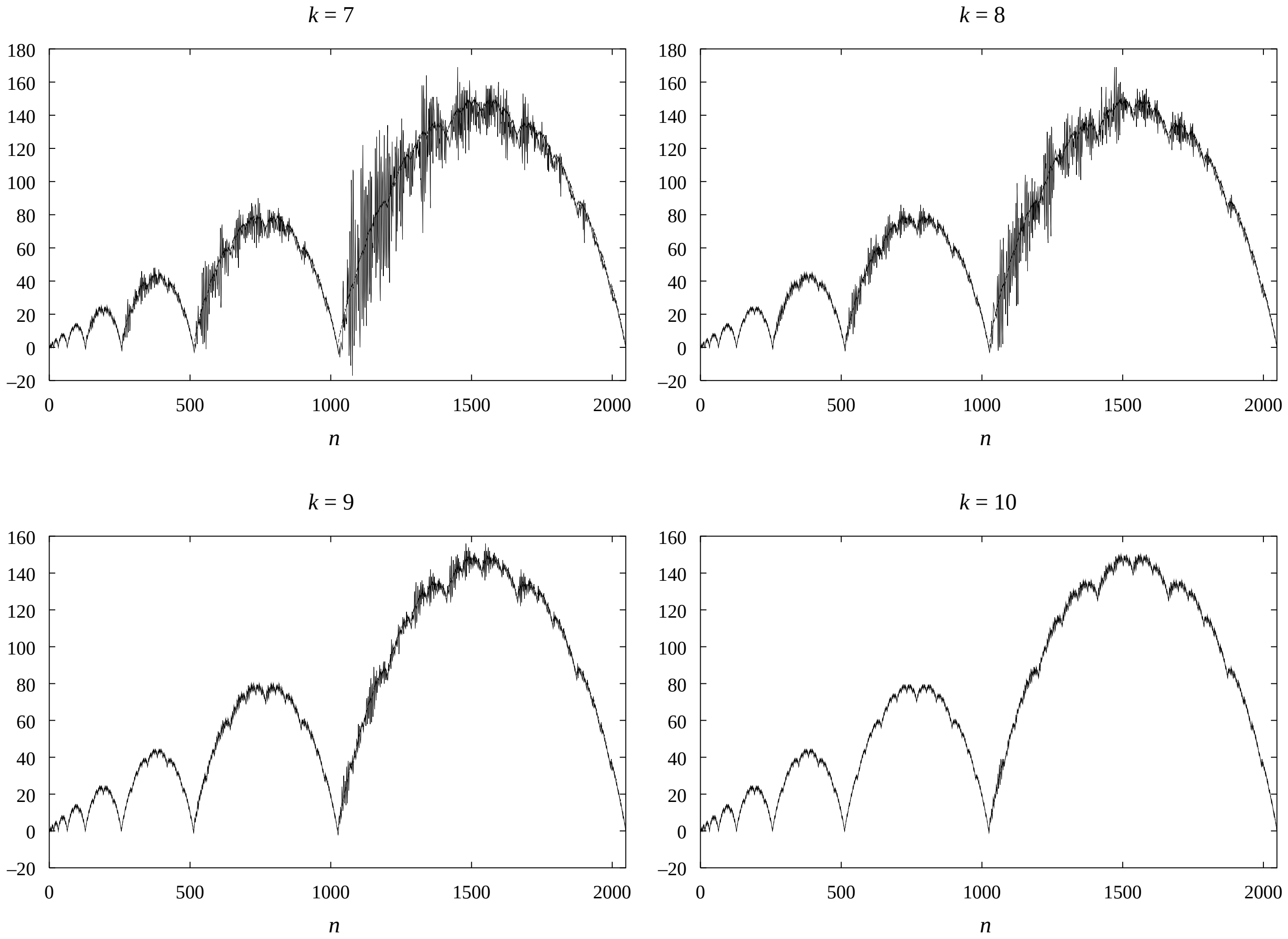

FIGURE 5. Graphs of $2\beta_k(n) - n$, for $k = 7, 8, 9, 10$.

3A. Step Size Statistics

The function inside a given generation may be considered representing a random walk of $2^{k-1}-1$ steps, starting from 2^{k-2} and arriving at 2^{k-1}. It is interesting to look at the distribution of the step sizes. Set

$$S(n) = D(n) - D(n-1)\,.$$

The square of the variance of this quantity is given by

$$M(k)^2 = \left\langle S(n)^2 \right\rangle_k - \left\langle S(n) \right\rangle_k^2,$$

where $\langle \cdot \rangle_k$ denotes the average over the k-th generation. Table 2 shows numerical results for $\log_2 M(k)$ for generations 13 to 25 and also the logarithmic ratios $\alpha_k = \log_2(M(k)/M(k-1))$. The results for the latter quantity converge to 0.88 ± 0.01. We conclude that

$$\frac{M(k)}{M(k-1)} \simeq 2^{\alpha},$$

k	$\log_2 M(k)$	α_k
13	6.857	0.949
15	8.683	0.910
17	10.498	0.896
19	12.291	0.888
21	14.071	0.888
22	14.961	0.890
23	15.845	0.884
24	16.726	0.882
25	17.598	0.872

TABLE 2. Variances $M(k)$ and logarithmic ratios $\alpha_k = \log_2(M(k)/M(k-1))$.

with $\alpha = 0.88 \pm 0.01$. This exponent is consistent with the one found for the Hofstadter $Q(n)$ [Pinn 1999].

Figure 7 shows a histogram p^* of the variable $x = S(n)/2^{0.88\,(k-1)}$, for $k = 24$ and $k = 25$, plotted on top of each other. The two histograms match nicely. The statistical distribution for $k = 25$ is

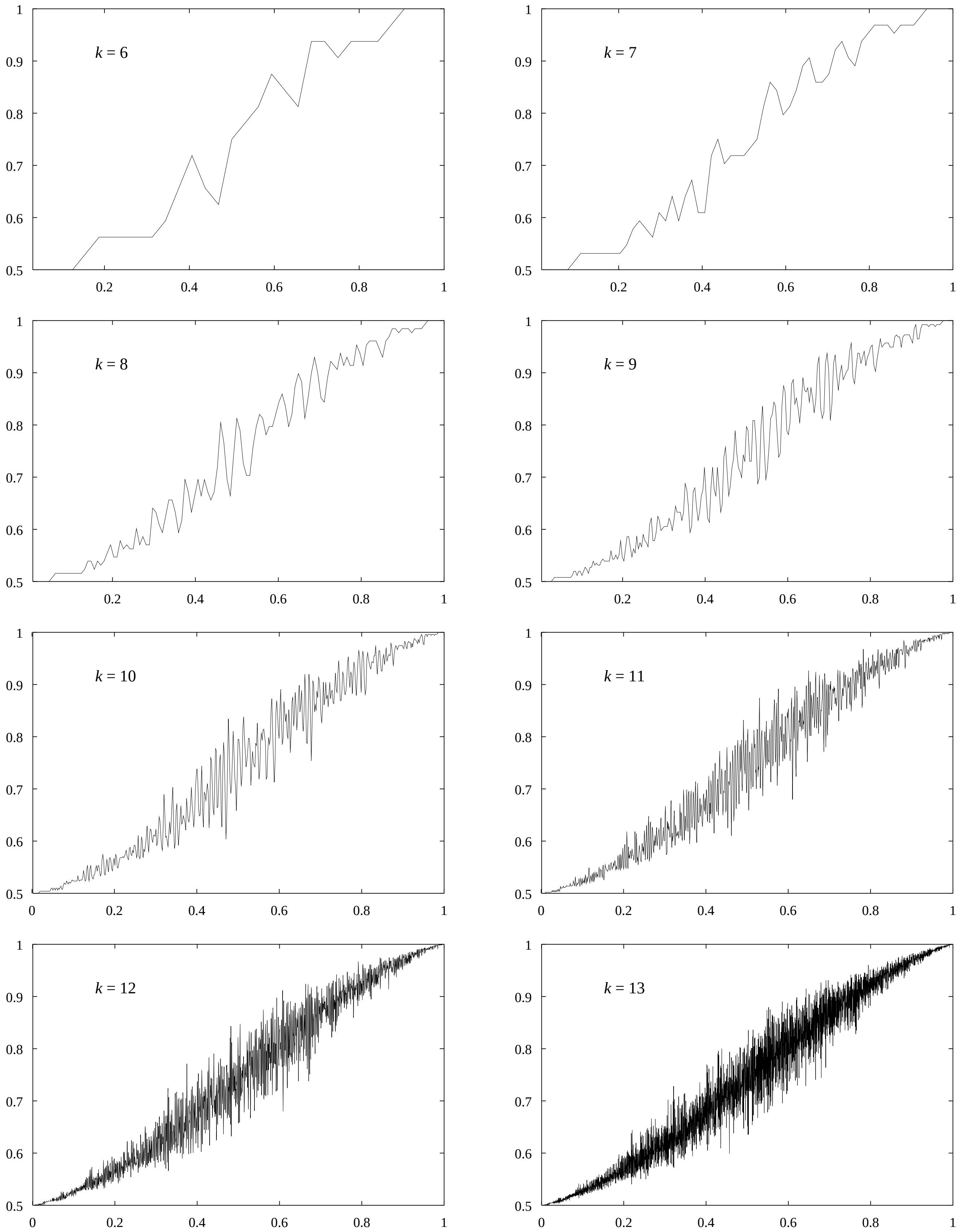

FIGURE 6. Rescaled graphs of $D(n)$, for generations $6 \leq k \leq 13$.

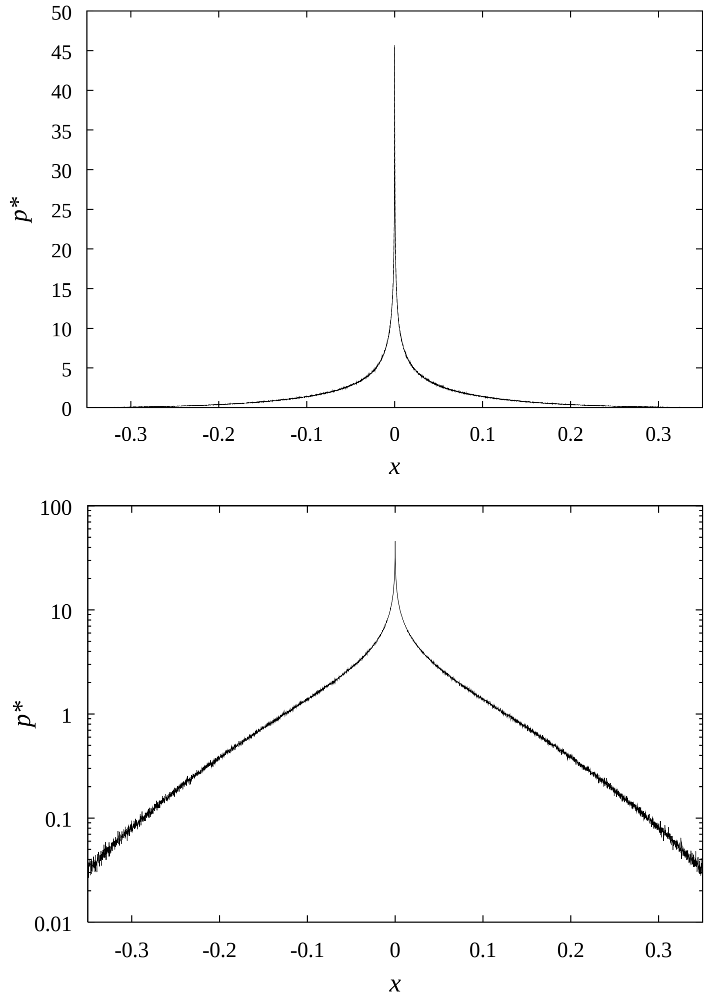

FIGURE 7. Top graph: statistical distribution of $x = (D(n) - D(n{-}1))/2^{0.88\,(k-1)}$, in generations $k = 24$ and $k = 25$. Bottom: same distribution on logarithmic scale for $k = 25$.

plotted on a logarithmic scale in the lower part of the figure. As was the case with the distribution function of suitable Q-number observables, the tails can be nicely fitted with a properly rescaled error function erfc, defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, \exp(-t^2).$$

It was observed in [Pinn 1999] that the probability density $p_m(x_m)$ of the rescaled difference $x_m = (Q(n) - Q(n-m))/n^\alpha$ was (up to a rescaling) with high precision identical with the distribution p^* of $(Q(n) - n/2)/n^\alpha$, i.e.,

$$p_m(x_m) = \lambda_m p^*(x_m/\lambda_m). \tag{3–1}$$

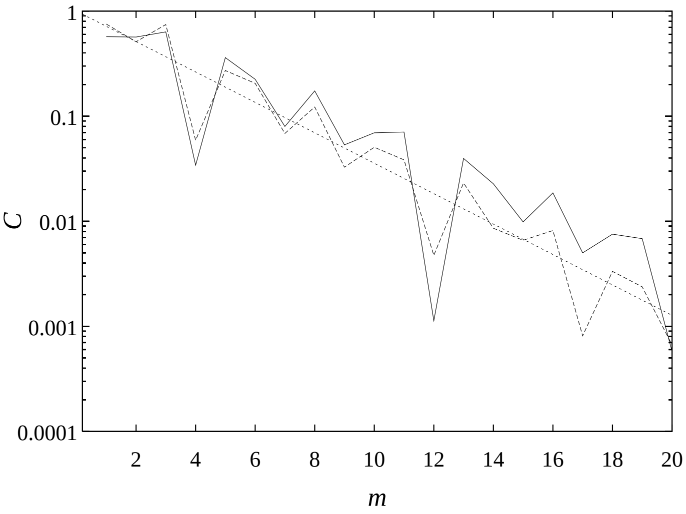

FIGURE 8. The functions $C_D = |\lambda_m^2{-}1.57|$ (full lines), $C_Q = |\lambda_m^2{-}2|$ (dotted lines), and $\exp(-m/3)$.

A similar type of scaling applies here. We define

$$x_m = (D(n) - D(n-m))/2^{\alpha\,(k-1)}.$$

Note that in the present case p^* is the distribution of x_1. One observes validity of Equation (3–1) with very good precision for $m \geq 2$. One can determine the λ_m from the second moments,

$$\lambda_m^2 = \frac{\langle x_m^2\rangle - \langle x_m\rangle^2}{\langle x_1^2\rangle - \langle x_1\rangle^2}.$$

They converge against $\lambda_\infty^2 \approx 1.57$. Looking at

$$C = |\lambda_m^2 - \lambda_\infty^2|$$

as function of m, we observe a striking similarity with the corresponding function for the Hofstadter sequence; see Figure 8. The ups and downs in both cases are very similar. The decay is approximated by $\exp(-m/3)$.

3B. Numbers Left Out and Frequency Counting

A. K. Yao [1997] has observed that the range of the Q-sequence seems to omit infinitely many positive integers.

The D-function maps generation k, i.e., the range $[2^{k-1}+1, 2^k]$, to the interval $I_k = [2^{k-2}, 2^{k-1}]$. We consider the question which fraction $r(M)$ of the $2^{k-2}+1$ numbers in I_k are generated exactly M times. It turns out that these fractions converge with increasing k. Table 3 shows $r(M)$, $M \leq 6$, for $k = 23$ and $k = 24$. The D-function omits some 14% of all numbers. The table also shows $r(M)$ for the sequences Q, F_{10}, and F_{11}. The latter two

M	$k=23$	$k=24$	Q $n<2^{21.5}$	F_{10} $n<2^{22}$	F_{11} $n<2^{22}$
0	0.1446	0.1443	0.1358	0.1358	0.1342
1	0.2728	0.2722	0.2709	0.2706	0.2697
2	0.2615	0.2624	0.2700	0.2703	0.2709
3	0.1730	0.1731	0.1803	0.1804	0.1810
4	0.0885	0.0886	0.0900	0.0903	0.0909
5	0.0379	0.0380	0.0362	0.0361	0.0365
6	0.0143	0.0141	0.0122	0.0120	0.0122

TABLE 3. Relative frequency $r(M)$ of numbers in I_k that are generated by D exactly M times. The last column gives estimates for the r-ratios of the sequences Q, F_{10}, and F_{11}. The latter two recurrences will be introduced in Section 4.

sequences are close relatives of the Hofstadter sequence and will be introduced in Section 4. There is a fair agreement of the ratios $r(M)$ for all the four sequences. Figure 9 shows $r(M)$ for $M \leq 16$ in generations 23 and 24. Only a small deviation between the two sets of numbers is seen for larger M.

Figure 10 shows a plot of the i-th left-out number in I_k, rescaled by a factor 2^{k-1}. The x-variable is i divided by the length of interval I_k. The graphs are for $k = 16$ and 17. The difference between the two curves is already small. Of course, such graphs can also be generated for $M \neq 0$. They look similar.

4. TWO COUSINS OF HOFSTADTER'S SEQUENCE

It is natural to generalize Hofstadter's recurrence (see beginning of this article) by introducing constant shifts i and j in the arguments on the right hand side:

$$\begin{aligned} F_{ij}(n) &= F_{ij}(n_1) + F_{ij}(n_2) \quad \text{for } n > 2, \\ F_{ij}(1) &= F_{ij}(2) = 1, \end{aligned} \tag{4–1}$$

with $n_1 = n-i-F_{ij}(n-1)$ and $n_2 = n-j-F_{ij}(n-2)$. Of course, one has to check whether the recursion (together with given initial conditions) leads to a well-defined sequence for all n. Ill-definition occurs if there exists an n such that either n_1 or n_2 is outside of $[1, n-1]$. It turns out that the recursion (4–1) is ill-defined except for the cases $ij = 00$, 01, 10, and 11, where I confirmed consistency for $n \leq 2^{26}$. Note that $F_{00} = Q$. The sequence with $ij = 01$ seems to have a simple regular structure, very similar to Tanny's sequence [1992]. (It might be interesting to compare also with the appearance of Golomb's recursions studied in [Barbeau and Tanny 1996; 1997].) The other two cousins, F_{10} and F_{11}, look chaotic. A graph of the first 2000 elements of F_{00}, F_{10}, and F_{11} is shown in Figure 11.

4A. Statistical Properties

We consider the sequences $\tilde{F}_{ij}(n) = F_{ij}(n) - n/2$. Again we study the variances $M(k)$, defined by

$$M(k)^2 = \langle \tilde{F}_{ij}(n)^2 \rangle_k - \langle \tilde{F}_{ij}(n) \rangle_k^2,$$

where $\langle \cdot \rangle_k$ denotes the average over intervals

$$[2^{k-1}+1,\ 2^k].$$

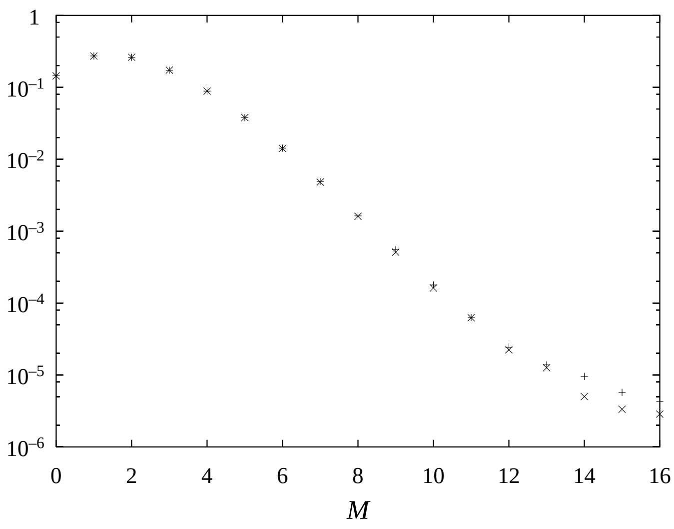

FIGURE 9. The function $r(M)$, for $k = 23$ (+) and $k = 24$ (×).

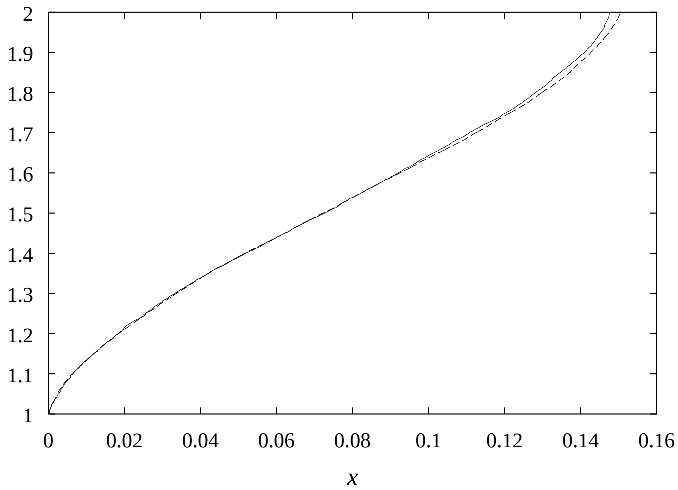

FIGURE 10. The i-th left-out number in I_k, rescaled by a factor 2^{k-1}, for $k = 16$ (full line) and $k = 17$ (dotted line). The x-variable is i divided by the length of interval I_k.

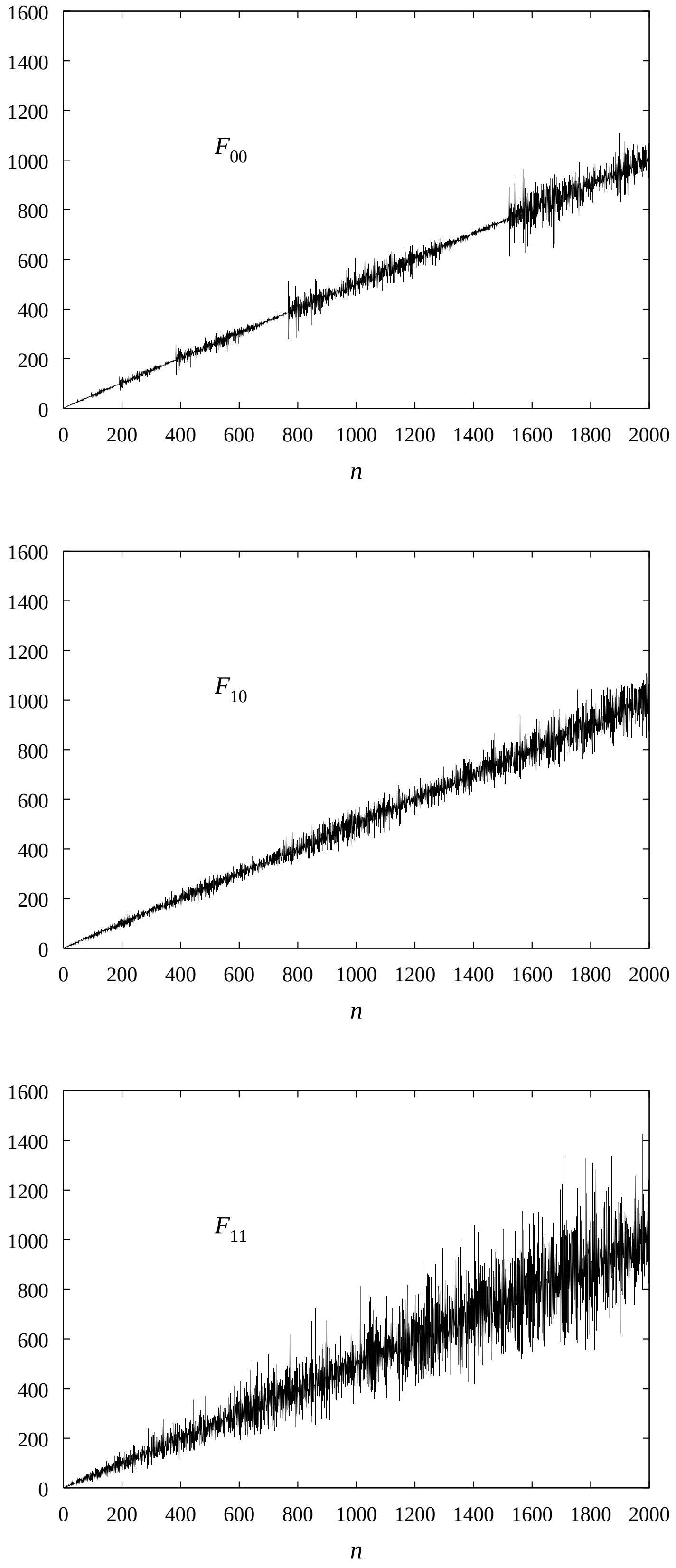

FIGURE 11. The sequences $F_{00} = Q$, F_{10}, and F_{11}.

Table 4 shows the results for

$$\alpha_k = \log_2(M(k)/M(k-1)),$$

where $k \le 25$. We estimate for $\alpha = \lim_{k\to\infty} \alpha_k$ and obtain 0.88 ± 0.01 for $ij = 00$ (Hofstadter sequence), 0.86 ± 0.01 for $ij = 10$, and 0.89 ± 0.01 for $ij = 11$. It seems that the exponent for F_{10} is smaller than that for the other sequences. There are still fluctuations in Table 4, and we cannot strictly rule out the possibility that the exponents of the three sequences agree.

k	00	10	11
13	0.849	0.852	0.867
14	0.885	0.864	0.925
15	0.879	0.869	0.904
16	0.879	0.862	0.883
17	0.870	0.863	0.895
18	0.882	0.865	0.889
19	0.881	0.859	0.895
20	0.882	0.857	0.886
21	0.882	0.859	0.891
22	0.880	0.864	0.890
23	0.882	0.861	0.887
24	0.880	0.857	0.884
25	0.876	0.851	0.878
α	0.88^*	0.86^*	0.89^*

TABLE 4. Logarithmic variance ratios α_k for $\tilde{F}_{00}$, $\tilde{F}_{10}$, and $\tilde{F}_{11}$. The * indicates ± 0.01, that is, an uncertainty of 0.01 on a 1 sigma level.

Figure 12 shows the statistical distribution functions of the quantities $\tilde{F}_{ij}(n)/n^\alpha$, where the α's are taken from the last line of Table 4. The binning was done over periods $[2^{k-1}, 2^k]$ for the 10 and 11 sequences. For F_{00} the generation structure requires intervals $[2^{k-1.5}, 2^{k-0.5}]$. The distributions for the different k's agree nicely. The plot shows the $k = 24$ results. The function with the highest peak belongs to $ij = 00$, the F_{11}-numbers have the broadest distribution. In contrast to the 00-distribution which (as the D-distribution) goes like $\exp(-cx^2)/x$ for large x, the 10- and 11-distributions can be fairly well approximated by Gaussians. It is an interesting question whether the various behaviours can be understood and modelled. It seems natural to try a fit with limiting distributions of random walks. Narrow non-Gaussian distribution can in principle be generated by sub-diffusive random walks [Bouchaud and Georges 1990]. The observed asymptotics $\sim$

$\exp(-cx^2)/x$, however, does not seem to be compatible with sub-diffusion.

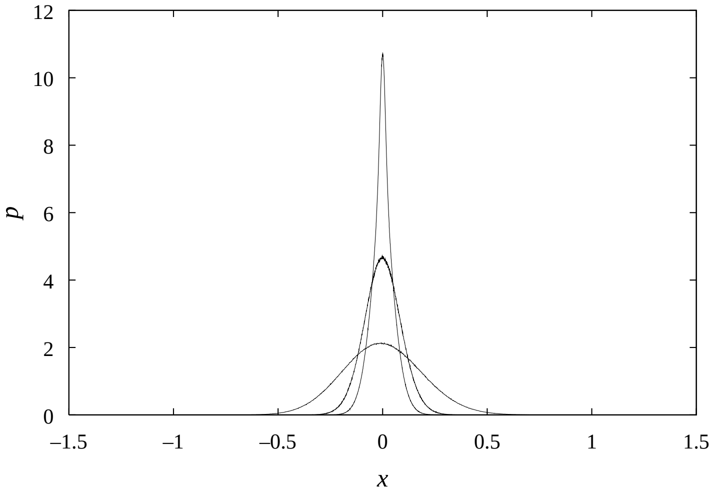

FIGURE 12. Statistical distributions of $\tilde{F}_{ij}(n)/n^{\alpha}$, for $ij = 00$ (highest peak), 10, and 11 (broadest).

Again we observe scaling, if we look at the distributions of

$$x_m = \tilde{F}_{ij}(n-m) - \tilde{F}_{ij}(n)\,.$$

More precisely, the probability density of x_m, $m > 2$ is up to a rescaling the same as that of x_{m-1}. For $ij = 11$ one can detect some small scaling violations for the first 2 values of m. The approach of the λ_m factors to their asymptotic value is the same for all three F-sequences, and very similar to that of the D-sequence. The convergence is again governed by a correlation length of 3.

4B. Correlation Functions

For all three F-sequences we define a variable σ_n by

$$\sigma_n = \begin{cases} +1 & \text{if } F(n) \geq n/2, \\ -1 & \text{else }. \end{cases}$$

Then we "measure" the 2-point correlator

$$G(m) = \langle \sigma_n \sigma_{n-m} \rangle - \langle \sigma_n \rangle^2$$

over the range $[2^{16}, 2^{24}]$. The results for $|G(m)|$ are shown in Figure 13. The lower part of the figure shows $|G(m)|$ on a logarithmic scale, together with the functions C_Q and C_D of Figure 8. The surprise is not only that the correlators of the three F-sequences seem to be identical. They also have a striking similarity with the functions describing the decay of the rescaling factors λ_m.

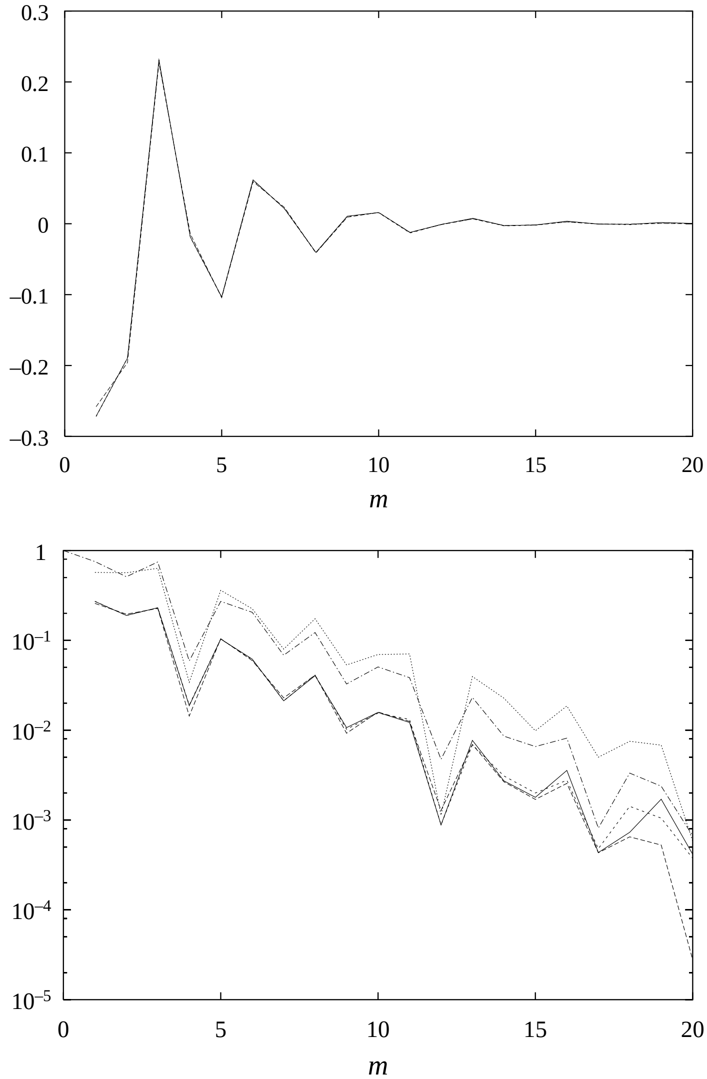

FIGURE 13. Top: $G(m)$ for $ij = 00$, 10, and 11. The lower plot shows $|G(m)|$ for the same three sequences on a logarithmic scale (lower three graphs) together with the functions C_D and C_Q of figure 8 (upper two graphs).

4C. Frequency Counting

Results for the relative frequencies of numbers n occurring M times in the F-sequences were already given in Table 3. They agree fairly well with those for D and Q.

SUMMARY AND CONCLUSIONS

In this paper, a chaotic cousin of Conway's sequence was introduced and studied empirically. Its statistical properties showed some intriguing similarities with the Hofstadter sequence Q and also with the two cousins F_{10} and F_{11}:

- All the four sequences studied have (to the given precision) the same exponent α, governing the increase of variance with increasing n or k. (The value for F_{10} seems to be a little bit lower, but agreement can however not be excluded.)
- The probability densities obey a scaling law. The rescaling parameter follows a characteristic convergence, governed by a correlation length 3.
- The correlation function $G(m)$ is identical for all three F-sequences. It also decays with correlation length 3, and in a way very similar to the behaviour of the λ_m factors.
- The relative frequencies of numbers occurring exactly M times in the sequence seems to be the same for all the four sequences.

The D-numbers and the three F-sequences have a lot of common structure. One might say that they share a universality class. A precise definition of such a class is, however, still lacking.

The D-sequence is unique insofar, as it has a regular generation structure with smooth interplays in-between. This could make it a candidate for studies aiming at some rigorous results about the chaotic recurrence relations.

It is presently an open question how much one can learn from the relation of the D-recurrence with the "solved" a-sequence. That there is some deep relation is suggested by the apparent similarity of the two sequences in the regions between the generations. The experiments with seeding the D-recurrence with k generations of a-numbers (Section 2) could be a first step towards a better understanding of this relation.

Whenever one observes the phenomenon of universality in a *model*, one is tempted to look for realizations of the same universality class in *nature*. It is an interesting question whether recurrences of the type studied in this article represent real physical processes or might be of use in the study of some dynamical system occurring in real life. A physical picture (e.g., in terms of random walks in some bizarre surrounding) could perhaps help to better understand some of the interesting properties of these sequences.

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